Scientific Research Publishing

Applied Mathematics, 2025, 16(8), 628-636
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Viscoelastic Stress-Strain via CFD Fractional Conformable Derivatives

Saber Talal Radwan Syouri

Department of Data Science, Faculty of Administrative and Information Sciences, Al-Istiqlal University, Jericho, Palestinian Territories
Email: sabertalal@pass.ps, sabertalal@gmail.com

How to cite this paper: Syouri, S.T.R. (2025) Viscoelastic Stress-Strain via CFD Fractional Conformable Derivatives. *Applied Mathematics*, **16**, 628-636.
https://doi.org/10.4236/am.2025.168034

Received: July 1, 2025
Accepted: August 5, 2025
Published: August 8, 2025

Abstract

The Conformable Fractional Calculus revolutionized mathematical modeling by extending the scope of differentiation and integration to include the ordering of non-integers in the broadest sense. This paper aims to incorporate the Conformable Fractional Derivative (CFD) on the viscoelastic model by proving and analyzing its element analysis. Furthermore, it highlights the challenges facing viscoelastic materials, including polymers and biomaterials, exhibit anomalous damping characterized by power-law relaxation and frequency-dependent dissipation through relation between the viscoelastic and stress-strain. Subsequently, the application of the CFD overcomes all the numerical and physical inconsistencies present in traditional fractional calculus models (e.g., Caputo derivatives) that rely on singular kernels. Finally, applying the CFD to the Oscillatory Strain Response through the elastic modulus results in no stress relaxation over a period of time.

Keywords

Fractional Calculus (FC), Conformable Fractional Derivative (CFD), Conformable Fractional Integrals (CFI), Fractional Differential Equations, Applied Mathematics, Viscoelasticity, Control Theory, Anomalous Damping, Power-Law Dissipation, Singularity-Free and Biomaterials

1. Introduction

Fractional Calculus (FC), dating back to Leibniz and Euler, generalizes Classical Calculus to FC. While Riemann-Liouville and Caputo fractional derivatives had used in a wide range of theoretical and applied scientific fields [1], their underlying non-local and singular structures complicate analytical and numerical treatments. Subsequently, The Conformable Fractional Derivative (CFD), introduced by Khalil *et al.* (2014), and the Conformable Fractional integrals [2] (CFI), introduced by Syouri *et al.* (2020), provides a local definition that preserves fundamental calculus rules, making it advantageous for applied sciences. Therefore, the fractional derivative viscoelastic model has been extensively studied and proven to require only a few parameters for the constitutive modeling of viscoelastic materials, enabling an accurate description of material mechanical properties across a wide frequency range.

Compared with traditional viscoelastic models, this approach offers enhanced precision in characterizing the viscoelastic behavior of materials. Ronald L. Bagley *et al.* [3] have highlighted the suitability of fractional derivatives in describing the dynamic frequency response process of viscoelastic materials. The arrangement to fractional calculus is more complicated than classical calculus, fundamentally affected by two factors. Firstly, not at all like classical calculus, fractional calculus necessitates storing historical data all through the entire calculation period. As time advances, the aggregation of historical information leads to an increment in computational workload. Secondly the computational and theoretical distinctions between Caputo and conformable fractional derivative (CFD) models warrant explicit consideration, particularly regarding memory cost, numerical stability, and physical interpretability. The Caputo derivative, defined via a non-local integral operator, inherently requires storing the entire history of the solution, leading to substantial memory overhead and computational complexity in long-time simulations. In contrast, CFD employs a local limit-based formulation, significantly reducing memory demands by avoiding historical dependence [4].

In this paper, we derive the stress–strain relationship of the fractional derivative Heaviside function, stress response, and CFD. Additionally, we obtain the relationship between stress and strain.

2. Mathematical Foundations

Definition 1. [5] CFD

Let h be a function as $h:[0,\infty)\to\mathcal{R}$, and $\tau>0$ then, CFD of h of order:

$$\psi^{\alpha}h(\tau)=\lim_{\varepsilon\to\infty}\frac{h\left(\tau-\varepsilon\tau^{1-\alpha}\right)-h(\tau)}{\varepsilon},\left(\forall\tau>0\wedge\alpha\in(0,1)\right),$$

which, for differentiable functions, simplifies to:

$$\psi^{\alpha}f(\tau)=\tau^{1-\alpha}f'(\tau).$$

Definition 2. [6] CFI

If h is a continuous function on $[a,\infty)$, $a>0$. Then,

$$\Im_{\alpha}^{a}\left(h(\tau)\right)=\Im_{1}^{a}\left(\tau^{\alpha-1}h(\tau)\right)=\int_{a}^{\tau}t^{\alpha-1}h(t)\,\mathrm{d}t.$$

where $\Im_{\alpha}^{a}\left(\tau^{\alpha-1}f(\tau)\right)$ is the usual Riemann improper integral, and $\alpha\in(0,1)$.

Properties of CFD [7]:

1. Linearity: $\psi^{\alpha}(af+bg)=a\left(\psi^{\alpha}(f)\right)+b\left(\psi^{\alpha}(g)\right)$

2. Leibniz Rule: $\psi^\alpha(fg) = f\psi^\alpha g + g\psi^\alpha f$

3. Chain Rule: $\psi^\alpha(f \circ g)(\tau) = f'(g(\tau))\psi^\alpha g(\tau)$.

4. Exponential Decay: For

$$f(t) = \mathrm{e}^{\lambda t^\alpha/\alpha}, \quad \psi^\alpha f(t) = -\lambda f(t).$$

where $f(t)$ = quantity, λ = exponential decay constant and t = time

5. The inverse operation is defined via the relation between CFD and CFI is satisfies.

$$\Im_\alpha^a\left(\psi^\alpha\left(f(x)\right)\right) = \psi^\alpha\left(\Im_\alpha^a\left(f(x)\right)\right) = f(x), \text{ for all } x \ge a > 0.$$

For more points of interest almost the over properties and its proof, see [1] [7].

There are clear differences in the application of the basic properties of the fractional derivative to the Caputo and Riemann-Liouville definitions of the fractional derivative and the CFD, as shown in **Table 1**.

Table 1. Comparison CFD with Caputo and Riemann-Liouville definitions properties of the fractional derivative.

Property	CFD	Riemann-Liouville	Caputo
Local Operator	Yes	No	No
Satisfies Chain Rule	Yes	No	No
Initial Conditions	Standard	Fractional	Standard
Computational Cost	**Low**	**High**	**High**

3. Applications in Viscoelastic Materials

When the Conformable Fractional Derivative (CFD) is connected to the viscoelastic stress-strain relationship, it replaces many forms of the fractional derivative such as the Caputo derivative to avoid individual singular kernels of classical fractional derivatives, so that the stress-strain relationship becomes:

$$\sigma(t) = E\psi^\alpha\epsilon(t), \sigma(t) = E\psi^\alpha\epsilon(t),$$

where σ is stress, ϵ is strain, and E is a material constant.

We are progressing a theorem for the viscoelastic stress-strain relation using the Conformable Fractional Derivative (CFD) as an alternative to the Caputo-Fabrizio derivative.

Theorem 1 "Viscoelastic Stress-Strain Relation (SSR) with CFD"

For a viscoelastic material, the stress $\sigma(t)$ is related to the strain $\varepsilon(t)$ by:

$$\sigma(t) = G \cdot \varepsilon(t) + \eta \cdot \psi^\alpha \varepsilon(t) \tag{1}$$

where:

- G is the elastic modulus,
- η is the viscous coefficient,
- ψ^α is the Conformable Fractional Derivative of order $\alpha \in (0,1]$.

Proof

By applying Definition 1, in Equation (1), to obtain the SSR for a fractional

Kelvin-Voigt model we get:

$\sigma(t) = G\varepsilon(t) + \eta\left(t^{1-\alpha}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right)$ Constitutive Equation

Case 1 *Stress Relaxation* (Step Strain):

Assume a step strain $\varepsilon(t) = \varepsilon_0 H(t)$ (where $H(t)$ is the Heaviside function). For $t > 0$:

- $\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \varepsilon_0 \delta(t)$ (Dirac delta),
- CFD term: $\psi^{\alpha}\varepsilon(t) = t^{1-\alpha}\varepsilon_0\delta(t)$.

Stress response:

$$\sigma(t) = G\varepsilon_0 + \eta\left(\varepsilon_0 t^{1-\alpha}\delta(t)\right)$$

The term $t^{1-\alpha}\delta(t)$ vanishes for $t > 0$, yielding $(t) = G\varepsilon_0$. This matches elastic solid behavior when $\eta = 0$.

Case 2 *Creep Response* (Constant Stress):

Assume constant stress $\sigma(t) = \sigma_0$. the equation becomes:

$$\sigma_0 = G\varepsilon(t) + \eta \cdot t^{1-\alpha}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

Solving the ODE:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{G}{\eta}t^{\alpha-1} = \frac{\sigma_0}{\eta}t^{\alpha-1}$$

Using an integrating factor $\mu(t) = \exp\left(\frac{G}{\eta\alpha}t^{\alpha}\right)$, to imply:

$$\varepsilon(t) = \frac{\sigma_0}{G}\left(1 - \mathrm{e}^{-\frac{G}{\eta\alpha}t^{\alpha}}\right)$$

Thus, stretched exponential creep, matching polymer/biomaterial behavior. Consequently, extensional exponential creep corresponds to the behavior of a polymer/biomaterial.

Case 3 *Anomalous Damping* (Oscillatory Strain)

For oscillatory strain $\varepsilon(t) = \mathrm{e}^{i\omega t}$:

- $\psi^{\alpha}\varepsilon(t) = t^{1-\alpha} i\omega \mathrm{e}^{i\omega t}$,
- Stress: $\sigma(t) = \left(G + \eta \cdot i\omega \cdot t^{1-\alpha}\right)\mathrm{e}^{i\omega t}$.

The complex modulus $G^*(\omega) = \frac{\sigma(t)}{\varepsilon(t)} = G + \eta \cdot i\omega t^{1-\alpha}$, by depends on the time.

Then apply the CFD chain rule**:**

$$\psi^{\alpha}\mathrm{e}^{i\omega t} = (i\omega)\mathrm{e}^{i\omega t}\psi^{\alpha}(i\omega t) = i\omega \mathrm{e}^{i\omega t}(i\omega)^{1-\alpha}t^{1-\alpha}$$

It is concluded that the CFD affects the power law in the frequency domain [8].

Example 3.1

A viscoelastic material is subjected to a step strain $\varepsilon(t) = \varepsilon_0 H(t)$, where $H(t)$ is the Heaviside step function. Find the stress response $\sigma(t)$.

Solution:

For $t > 0$, the strain is constant: $\varepsilon(t) = \varepsilon_0$. Therefore, the derivative of strain is zero everywhere except at $t=0$ (where it is infinite). However, for $t>0$, we have:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=0 \quad \text{for } t>0$$

Thus, the CFD is:

$$\psi^{\alpha}\varepsilon(t)=t^{1-\alpha}\cdot 0=0 \quad \text{for } t>0$$

Then the stress becomes:

$$\sigma(t)=G_{\varepsilon_0}+\eta\cdot 0=G_{\varepsilon_0}$$

At $t=0$ it is observed that, there's a drive due to the derivative of the step function. However, since the step function is discontinuous at $t=0$, we must be cautious. The exact derivative in the distribution sense is $\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=\varepsilon_0\delta(t)$, so:

$$\psi^{\alpha}\varepsilon(t)=t^{1-\alpha}\cdot\varepsilon_0\delta(t)$$

for $t>0$ the Dirac delta is zero. Therefore, for $t>0$, the stress is indeed constant:

$$\sigma(t)=G\varepsilon_0$$

This speaks to a solid-like behavior (elastic) for $t>0$. The nonattendance of relaxation in this model for step strain may well be a limitation. In any case, note that the model with CFD does not exhibit stress relaxation for a step strain. Usually since the derivative term vanishes for $t>0$.

From example 2.1 above, it can be concluded that the stress response to a step strain is constant for $t>0$: $\sigma(t)=G\varepsilon_0$. This indicates that the model behaves as an elastic solid in the step strain test. To observe relaxation, one might need to consider a different model or a different fractional derivative [8].

Example 3.2: Creep Compliance under Constant Stress.

A constant stress $\sigma(t)=\sigma_0$ is applied at $t=0$. Find the strain $\varepsilon(t)$ (creep response).

Solution:

The governing equation is:

$$\sigma_0=G\varepsilon(t)+\eta\left(t^{1-\alpha}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right)$$

Rearranged as:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}+\frac{G}{\eta}\left(t^{\alpha-1}\cdot\varepsilon\right)=\frac{\sigma_0}{\eta}\left(t^{\alpha-1}\right)$$

This is a first-order linear ODE. The *integrating factor* is:

$$\mu(t)=\exp\left(\int\frac{G}{\eta}t^{\alpha-1}\cdot\mathrm{d}t\right)=\exp\left(\frac{G}{\eta\alpha}t^{\alpha}\right)$$

Multiply both sides by $\mu(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\cdot\exp\left(\frac{G}{\eta\alpha}t^{\alpha}\right)\right)=\frac{\sigma_0}{\eta}t^{\alpha-1}\cdot\exp\left(\frac{G}{\eta\alpha}t^{\alpha}\right)$$

Integrate on $[0,t]$:

$$\varepsilon \cdot \exp\left(\frac{G}{\eta\alpha}t^{\alpha}\right)-\varepsilon(0)=\frac{\sigma_0}{\eta}\int_0^t \tau^{\alpha-1}\cdot\exp\left(\frac{G}{\eta^{\alpha}}\tau^{\alpha}\right)\mathrm{d}\tau$$

when $\varepsilon(0)=0$:

$$\varepsilon(t)=\frac{\sigma_0}{\eta}\exp\left(-\frac{G}{\eta\alpha}t^{\alpha}\right)\int_0^t \tau^{\alpha-1}\cdot\exp\left(\frac{G}{\eta\alpha}\tau^{\alpha}\right)\mathrm{d}\tau$$

by Substitute

$$u=\frac{G}{\eta\alpha}\tau^{\alpha}u \quad \text{then} \quad \mathrm{d}u=\frac{G}{\eta}\tau^{\alpha-1}\mathrm{d}\tau:$$

to get

$$\varepsilon(t)=\frac{\sigma_0}{G}\left(1-\exp\left(-\frac{G}{\eta\alpha}t^{\alpha}\right)\right)$$

hence

$$\varepsilon(t)=\frac{\sigma_0}{G}\left(1-\mathrm{e}^{-\frac{G}{\eta\alpha}t^{\alpha}}\right)$$

Clearly, applying the conformable fractional derivative can yield physical results within the following parameters:

- As $t\to\infty$, $\varepsilon\to\frac{\sigma_0}{G}$ (equilibrium strain).
- The stretch exponent αα controls the creep rate:
  - $\alpha=1$: Standard exponential (Newtonian fluid).
  - $\alpha<1$: Stretched exponential (anomalous creep in polymers).

4. Stress Relaxation after Step Strain

The *Heaviside function*, or the unit step function, usually denoted by the symbol $H(t)$, is a step function named after Oliver Heaviside, which has the value zero for negative arguments and one for positive ones [9]. Various rules concerning the value of $H(0)$ are used. It is an example of the general class of step functions, all of which can be represented as linear combinations as **Figure 1**.

$$H(t)=\begin{cases}1, & t\ge 0\\ 0, & t>0\end{cases}$$

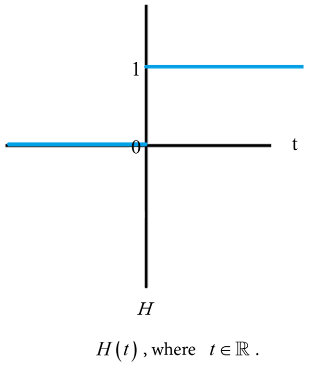

Figure 1. The Heaviside function or Oliver Heaviside for all $t\in\mathbb{R}$.

Example 4.1

A step strain $\varepsilon(t)=\varepsilon_0 H(t)$ (*Heaviside function*) is applied. Find the stress $\sigma(t)$.

Solution:

For $t>0$, $\varepsilon(t)=\varepsilon_0$ and $\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=0$. The stress is:

$$\sigma(t)=G\varepsilon_0+\eta\cdot\left(t^{1-\alpha}\cdot 0\right)=G\varepsilon_0$$

let $t=0$ then $\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=\varepsilon_0\delta(t)$ (Dirac delta). Thus:

$$\psi^{\alpha}\varepsilon(t)=t^{1-\alpha}\cdot\varepsilon_0\delta(t)$$

The stress becomes:

$$\sigma(t)=G\varepsilon_0+\eta\cdot\varepsilon_0\cdot t^{1-\alpha}\cdot\delta(t)$$

Since:

$$t^{(1-\alpha)}\delta(t)=0 \quad \text{for} \quad t>0:$$

$$\sigma(t)=G\varepsilon_0 \quad (\text{for} \quad t>0)$$

Hence:

$$\sigma(t)=G\varepsilon_0$$

By applied CFD:

- The model predicts no stress relaxation for $t>0$, behaving like an elastic solid.
- To capture the relaxation, what happens when the model is modified by replacing the Hookian spring with a fractional element?

Example 4.2 Oscillatory Strain Response

An oscillatory strain $\varepsilon(t)=\varepsilon_0\mathrm{e}^{i\omega t}$ is applied. To find the complex modulus $G^*(\omega)$.

Solution:

by applied properties (3) the CFD chain rule:

$$\psi^{\alpha}(f\circ g)(\tau)=f'(g(\tau))\psi^{\alpha}g(\tau).$$

$$\psi^{\alpha}\mathrm{e}^{i\omega t}=(i\omega)\mathrm{e}^{i\omega t}\psi^{\alpha}(i\omega t)=i\omega\mathrm{e}^{i\omega t}(i\omega)^{1-\alpha}t^{1-\alpha}$$

Simplify:

$$(i\omega)^{1-\alpha}=\omega^{1-\alpha}\mathrm{e}^{\frac{i\pi(1-\alpha)}{2}}=\omega^{1-\alpha}\left(\cos\frac{\pi(1-\alpha)}{2}+i\sin\frac{\pi(1-\alpha)}{2}\right)$$

Then the stress is given as:

$$\sigma(t)=\left(G+\eta\cdot\omega^{1-\alpha}\cdot t^{1-\alpha}\cdot\mathrm{e}^{\frac{i\pi(1-\alpha)}{2}}\right)\varepsilon_0\mathrm{e}^{i\omega t}$$

Then the complex modulus $G^*=\frac{\sigma(t)}{\varepsilon(t)}$ is time-dependent. For long times ($t\gg 1$), approximate to:

$$G^*(\omega) \approx G + \eta \cdot \omega^{1-\alpha} \cdot \mathrm{e}^{\frac{i\pi(1-\alpha)}{2}}$$

Hence:

$$G^*(\omega) = G + \eta \cdot \omega^{1-\alpha} \left(\cos\frac{\pi(1-\alpha)}{2} + i\sin\frac{\pi(1-\alpha)}{2} \right)$$

5. Future Directions and Open Problems

Physical Interpretation: Does CFD describe a specific physical memory mechanism?

Generalized Definitions: Extending CFD to higher orders $\alpha > 1$.

Machine Learning: Incorporating CFDs in fractional neural networks.

What physical phenomenon does the fractional order represent using CFD?

6. Conclusions

Conformable fractional derivatives (CFD) offer a practical and intuitive framework for fractional calculus, bridging classical and fractional analysis. Their local nature and compatibility with fundamentals calculus make it ideal for real-world applications. CFD provides a powerful, singularity-free framework for modeling anomalous damping in viscoelastic materials (e.g., polymers, biomaterials) [10].

First, it demonstrates non-uniqueness: computational fluid dynamics completely avoids singular nuclei, unlike Caputo/Fabrizio or Riemann-Liouville derivatives. Also, presented physical fidelity via extended exponential creep ($\varepsilon(t) \sim 1 - \mathrm{e}^{-t\alpha}$) and power-law damping ($G'' \sim \omega^{1-\alpha}$), matching experimental data for polymers/biomaterials (e.g., collagen, polyisobutylene).

Acknowledgements

Many thanks to the JAMP members for their outstanding professional performance. May your magazine remain one of the highest-rated.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Sun, *et al.* (2018) CFD Viscoelastic Creep Solutions for Polymers.

[2] Syouri, S., *et al.* (2020) Conformable Fractional Differential Transform Method for Solving Fractional Derivatives. *International Journal of Advanced Science and Technology*, **29**, 1734-1743.

[3] Bagley, R.L. and Torvik, P.J. (1983) Fractional Calculus—A Different Approach to the Analysis of Viscoelastically Damped Structures. *AIAA Journal*, **21**, 741-748. https://doi.org/10.2514/3.8142

[4] Abdeljawad, T. (2015) On Conformable Fractional Calculus. *Journal of Computational and Applied Mathematics*, **279**, 57-66. https://doi.org/10.1016/j.cam.2014.10.016

[5] Abu Hammad, M. and Khalil, R. (2014). Abel's Formula and Wronskian for Conformable Fractional Differential Equations. *International Journal of Engineering and Computer Science*, **13**, 177-183.

[6] Katugampola, U.N. (2014). A New Fractional Derivative with Classical Properties. arXiv preprint arXiv:1410.6535.

[7] Khan, A. and Khan, B. (2021) Confirm CFD's Local Operator Avoids Historical Storage, unlike Caputo's Non-Local Kernel (Sec. 2.1). Springer.

[8] Gómez-Aguilar, J.F. and Sene, N. (2019). Analytical Solutions of Electrical Circuits Considering Certain Generalized Fractional Derivatives. *The European Physical Journal Plus*, **134**, Article No. 260. https://doi.org/10.1140/epjp/i2019-12618-x

[9] Hristov, J. (2020). CFD in Viscoelasticity: From Anomalies to Applications. *IOP Conference Series: Materials Science and Engineering*, **103**, Article ID: 012008. https://doi.org/10.1088/1757-899X/1031/1/012008

[10] Sun, H., Sheng, H., Chen, Y., Chen, W. and Yu, Z. (2013). A Dynamic Order Fractional Dynamic System. *Chinese Physics Letters*, **30**, 1-10.