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Embedded Vector Solitons

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Abstract

In this article, we demonstrate that embedded vector solitons (EVSs) indeed exist. In other words, we prove that there exist systems of nonlinear partial differential equations (NLPDEs) which have soliton solutions (u,v), where each of the solitons, $u(z,t)$ and $v(z,t)$, is an *embedded soliton*. We present two systems of NLPDEs with these peculiar EVSs: a system of two cubic-quintic nonlinear Schrödinger (NLS) equations, and a system of two complex modified Korteweg-de Vries (cmKdV) equations. These two systems are the first systems of NLPDEs with EVSs known to date. Moreover, we demonstrate that these two systems also possess dark vector solitons and a highly unusual type of vector soliton: mixed vector solitons, which are composed of a *bright soliton* and a *dark* one.

Keywords

Embedded Solitons, Vector Solitons, Nonlinear Schrödinger Equation, Complex Modified Korteweg-de Vries Equation

1. Introduction

Vector solitons (VSs) are groups of solitary waves that travel together along different types of nonlinear systems, and they are the solutions of systems of nonlinear partial differential equations. The applications of VSs cover the design of secure communication systems, the increase of data transmission rates in optical fibers, the trapping and manipulation of atoms and nanoparticles, the control of magnetization in data storage devices, and many other fields. An indication of the interest in the study of VSs is that Scopus, the famous database, finds 952 articles on VSs, published between the year 2000 and 2025, and these articles have received 18,543 citations.

On the other hand, embedded optical solitons are solitary light pulses that can propagate along an optical fiber without resonating with the small-amplitude waves capable of propagating in the fiber, despite having wavenumbers that are contained (*embedded*) in the range of the linear dispersion relation of the system. The first example of an embedded soliton (ES) with an exact analytical form was found in 1997, while studying a higher-order nonlinear Schrödinger (NLS) equation containing fourth-order dispersion and a quintic nonlinearity [1]. A second example of these peculiar solitons was found in 1999, while studying a system of two generalized NLS equations [2], and it was in this second reference where the term *embedded soliton* was proposed. The ESs of these equations are isolated solutions, which have unique values for the amplitudes, widths, and wavenumbers. However, in 2000 a *family* of ESs was found in a numerical way [3]. And in 2003, a continuous family of ESs, with a well-defined analytical form, was also found while studying the propagation of light pulses in liquid crystals [4]. In this case, the ESs were the solutions of a complex modified Korteweg-de Vries (cmKdV) equation [5]-[8]. Later on it was found that *embedded lattice solitons* (ELSs) can exist in discrete systems [9]. The first ELS discovered was an isolated solution, but in 2006 it was found that continuous families of ELSs can exist [10]. The precise explanation for the mysterious radiationless character of the ESs was found in 2003: in Ref. [11] it was clearly explained why ESs do not resonate with the small-amplitude radiation waves whose wavenumbers coincide with those of the ESs. The stability of ESs is another topic where interesting results have been found. Isolated ES, existing for unique values of the amplitude, width, and wavenumber, are usually *semistable* solutions [3], as these pulses will not be able to continue being exact ESs if they are subject to any perturbation that diminishes their energies. On the other hand, with the discovery of ESs, which exist in families, the possibility of having truly *stable* ESs appeared. The application of the Vakhitov-Kolokolov (VK) criterion of stability [12] [13] confirmed that VK-stable ESs do indeed exist. And an interesting result in this area is that *the same equation* can have VK-stable ESs, VK-unstable ESs, VK-stable standard solitons (*i.e.*, not embedded) and VK-unstable standard solitons, depending on the values of the coefficients of the equation [14].

In this communication, we study a topic that has not been addressed in the past. In this paper we investigate whether *embedded vector solitons* can exist.

There exist many physical situations where the propagation of light pulses can be described by a system of two nonlinear partial differential equations (NLPDEs) similar to the NLS equation. The coupling between these equations may be *incoherent* when it only depends on the intensities of the two pulses [15]-[17], but we may also have *coherently* coupled systems [18]-[25]. The solution of such a system is defined by a pair of complex functions, which we shall call here u and v, and each of these functions describes a light pulse that propagates along the physical system under consideration. When these pulses are able to advance along the system without changing their forms, we may consider them as *solitons*, and the pair (u,v) may be referred to as a *vector soliton* (to use the terminology introduced by Christodoulides and Joseph [26] long ago). We should observe, however, that the use of the term "soliton" in these contexts does not imply that the NLPDEs under consideration are *integrable* by inverse scattering. Moreover, it should be noted that VSs defined by systems of more than two NLPDEs may also exist, and in such cases we speak of *multicomponent vector solitons* [27].

Vector solitons (VSs) are known to exist in several physical systems [28]-[38]. In some cases, the two pulses which conform a VS are produced with a single laser, as in a birefringent optical fiber, where u and v describe the two polarization components of the electric field [39]. But there also exist VSs (u,v) generated with lasers of different frequencies. And in such cases the existence of these VSs is particularly interesting, as optical solitons generated with lasers of different frequencies travel with different velocities along a nonlinear medium (as an optical fiber). More precisely, they travel at different speeds *if they travel alone*. However, when two light pulses with carrier waves of different frequencies travel simultaneously along an optical fiber, it is possible (under particular circumstances), that the nonlinear interaction between the pulses decreases the speed of the faster soliton, and increases the speed of the slower one, in such a way that both solitons may advance at the same speed.

We know, therefore, that the investigation on ESs and VSs has produced many results. However, the intersection of these two fields has not yet been studied. In particular, heretofore we don't know if *embedded vector solitons* (EVSs) may exist (*i.e.*, VSs whose two components, u and v, are ESs).

In this communication, we show that EVS do indeed exist, and we will present two different systems of NLPDEs that have EVSs. The structure of the article is the following. In Section 2 we present a higher-order NLS system which describes the propagation of two light pulses along a single-mode optical fiber. A change of variables permits us to simplify this system, thus obtaining a new reduced system where the first and third temporal derivatives have disappeared. It is shown that in certain cases, the new reduced system can be obtained from a suitable Lagrangian density. In Sec. 3 we show that this reduced NLS system has soliton solutions, (u,v) and it is proved that these solitons are *embedded*. Therefore, these results prove that *embedded vector solitons* do indeed exist. Applying the VK criterion we find that these ESs may be composed by a VK-stable soliton, accompanied by a VK-unstable one. In Sec. 4, a second system, now containing two coupled complex modified Korteweg-de Vries (cmKdV) equations, is presented. It is shown that this cmKdV system has a continuous family of EVSs, and the stability of these solitons is analyzed by means of the VK criterion. In Secs. 5 and 6 it is shown, respectively, that both systems (the NLS system studied in Sec. 3 and the cmKdV system studied in Sec. 4) also have *dark vector solitons.* Then, in Secs. 7 and 8, it is shown that in both systems it is also possible the propagation of *mixed vector solitons*, *i.e.*, vector solitons where one of the solitons is *bright*, and se second one is *dark*. Finally, in Sec. 9, we present a summary and the conclusions of this communication.

2. Higher-Order Nonlinear Schrödinger System

The simultaneous propagation of two light pulses (with different carrier frequencies ω_1 and ω_2) which advance along a single-mode optical fiber is usually described by a system of two coupled NLS equations containing first and second order dispersive terms, and the coupling entering in the nonlinear terms [17]. However, to describe the behavior of short pulses (approaching the femtosecond regime), intense enough to require taking into consideration saturable nonlinearities, it is necessary to consider a system of the form:

$$iR_z + i\varepsilon_1 R_t + \alpha_1' R_{tt} - i\lambda_1 R_{ttt} + \beta_1 R_{4t} + \gamma_1\left(|R|^2 + \sigma_1 |S|^2\right)R - \delta_1\left(|R|^2 + \sigma_2|S|^2\right)^2 R = 0, \tag{1}$$

$$iS_z + i\varepsilon_2 S_t + \alpha_2' S_{tt} - i\lambda_2 S_{ttt} + \beta_2 S_{4t} + \gamma_2\left(|S|^2 + \sigma_1 |R|^2\right)S - \delta_2\left(|S|^2 + \sigma_2|R|^2\right)^2 S = 0. \tag{2}$$

In these equations the coefficients of the dispersive terms ε_n, α_n', λ_n and β_n are real constants which depend on the carrier frequencies ω_n, and the coefficients of the nonlinear terms γ_n and δ_n depend on the carrier frequencies and the properties of the fiber. Moreover, R, S and z are normalized variables, $t = \left(T - Z/v_g\right)/T_0$ is a normalized *retarded time* (Z and T are standard space and time), and v_g has been defined as $v_1 v_2/(v_1 + v_2)$, where v_1 and v_2 are the group velocities corresponding to the two carrier frequencies. With this choice of v_g we will have $\varepsilon_1 = -\varepsilon_2$.

In this and the following section we will investigate if the system (1)-(2) permits the propagation of *embedded vector solitons*. The existence of such vector solitons is uncertain, as ESs are usually semi-stable solutions, which start emitting radiation when they are perturbed, as a consequence of a resonance between the solitons and the small-amplitude linear waves capable of propagating in the system. When this radiation is emitted, the solitons begin to lose energy continuously, and this energy loss may eventually destroy the solitons. In the case of two interacting ESs which advance simultaneously along a fiber, the interaction might play the role of a perturbation, and both ESs might start emitting radiation, with the consequent energy loss. Moreover, if an embedded vector soliton (EVS) could propagate as a solution of the system (1)-(2), it would also be uncertain if this EVS would be an isolated solution, only existing for unique values of the heights, widths and wavenumbers (as occurs with the ESs of Refs. [1, 11]), or EVSs of different heights and widths might exist (as occurs with the ESs of the cmKdV equation).

If the coefficients of the Equations (1)-(2) satisfy the conditions:

$$\varepsilon_n + \frac{\alpha_n' \lambda_n}{2\beta_n} - \frac{\lambda_n^3}{4\beta_n^2} = 0, \tag{3}$$

the system (1)-(2) can be simplified by introducing the following change of variables:

$$R(z,t) = P(z,t)\,e^{i(K_1 z - \Omega_1 t)}, \tag{4}$$

$$S(z,t) = Q(z,t)\,e^{i(K_2 z - \Omega_2 t)}, \tag{5}$$

where the parameters Ω_n and K_n are defined as follows:

$$\Omega_n = -\frac{\lambda_n}{4\beta_n}, \tag{6}$$

$$K_n = \varepsilon_n \Omega_n - \alpha_n' \Omega_n^2 + \beta_n \Omega_n^4 + \lambda_n \Omega_n^3. \tag{7}$$

Introducing these changes of variables in (1)-(2), and defining the new coefficients α_n in the following way:

$$\alpha_n = \alpha_n' - 6\beta_n \Omega_n^2 - 3\lambda_n \Omega_n \tag{8}$$

the system (1)-(2) can be transformed into the reduced system:

$$iP_z + \alpha_1 P_{tt} + \beta_1 P_{4t} + \gamma_1 \left(|P|^2 + \sigma_1 |Q|^2\right) P - \delta_1 \left(|P|^2 + \sigma_2 |Q|^2\right)^2 P = 0, \tag{9}$$

$$iQ_z + \alpha_2 Q_{tt} + \beta_2 Q_{4t} + \gamma_2 \left(|Q|^2 + \sigma_1 |P|^2\right) Q - \delta_2 \left(|Q|^2 + \sigma_2 |P|^2\right)^2 Q = 0. \tag{10}$$

This transformation is quite convenient, as the first and third order time derivatives no longer appear in the new system (9)-(10), and the absence of these derivatives opens the possibility that system (9)-(10) may have solitary wave solutions which do not move along the retarded time axis, thus simplifying the calculations (either analytical or numerical).

It might be worth observing that in the symmetric case when:

$$\gamma_1 = \gamma_2 \equiv \gamma, \tag{11}$$

$$\delta_1 = \delta_2 \equiv \delta, \tag{12}$$

and additionally:

$$\sigma_2 = 1, \tag{13}$$

the system (9)-(10) can be obtained from the following Lagrangian density:

$$\begin{aligned} \mathcal{L} = {} & \frac{i}{2}\left(P^* P_z - P P_z^*\right) - \alpha_1 P_t P_t^* + \beta_1 P_{tt} P_{tt}^* + \frac{\gamma}{2} P^2 \left(P^*\right)^2 - \frac{\delta}{3} P^3 \left(P^*\right)^3 \\ & + \frac{i}{2}\left(Q^* Q_z - Q Q_z^*\right) - \alpha_2 Q_t Q_t^* + \beta_2 Q_{tt} Q_{tt}^* + \frac{\gamma}{2} Q^2 \left(Q^*\right)^2 - \frac{\delta}{3} Q^3 \left(Q^*\right)^3 \\ & + \sigma_1 \gamma P P^* Q Q^* - \delta\left[P P^* Q^2 \left(Q^*\right)^2 + P^2 \left(P^*\right)^2 Q Q^*\right]. \end{aligned} \tag{14}$$

This "Lagrangian case" might have some theoretical interest, but it will not be studied in the present communication since (as we shall see further below) when $\sigma_2 = 1$ the system (9)-(10) does not have soliton solutions.

3. Embedded Vector Solitons of the NLS System (9)-(10)

In the first part of this section we will show that the NLS system (9)-(10) has EVSs, and the VK criterion will be used in the second part in order to estimate the stability of these EVSs.

3.1. Existence of the NLS Solitons

Let us investigate if the system (9)-(10) accepts solitary wave solutions of the form:

$$P = A_1 \operatorname{sech}\left(\frac{t-az}{w}\right) e^{i(k_1 z-\omega_1 t)} \tag{15}$$

$$Q = A_2 \operatorname{sech}\left(\frac{t-az}{w}\right) e^{i(k_2 z-\omega_2 t)} \tag{16}$$

Substituting (15)-(16) into Equation (9) we obtain an equation containing five types of hyperbolic functions: ch^{-1}, ch^{-3}, ch^{-5}, $\mathrm{sh}/\mathrm{ch}^2$ and $\mathrm{sh}/\mathrm{ch}^4$. For this equation to be satisfied, the coefficients of each of the hyperbolic functions must be equal to zero. Setting equal to zero the coefficients of $\mathrm{sh}/\mathrm{ch}^4$ and $\mathrm{sh}/\mathrm{ch}^2$ we obtain that:

$$\omega_1 = 0\,, \tag{17}$$

$$a = 0\,. \tag{18}$$

Taking into account these results, the equation obtained when (15) and (16) are substituted into Equation (9) takes the form:

$$\begin{aligned}&\left(-k_1+\frac{\alpha_1}{w^2}+\frac{\beta_1}{w^4}\right)ch^4(\theta)+\left[-\frac{2\alpha_1}{w^2}-\frac{20\beta_1}{w^4}+\gamma_1\left(A_1^2+\sigma_1 A_2^2\right)\right]ch^2(\theta)\\&+\frac{24\beta_1}{w^4}-\delta_1\left(A_1^2+\sigma_2 A_2^2\right)^2=0,\end{aligned} \tag{19}$$

where we have defined $\theta = (t-az)/w$ For this equation to be satisfied, the coefficient of ch^4, ch^2 and the independent term (independent of *ch*) must be equal to zero. In this way we obtain the following three equations:

$$k_1 = \frac{\alpha_1}{w^2}+\frac{\beta_1}{w^4}\,, \tag{20}$$

$$-\frac{2\alpha_1}{w^2}-\frac{20\beta_1}{w^4}+\gamma_1\left(A_1^2+\sigma_1 A_2^2\right)=0\,, \tag{21}$$

$$\frac{24\beta_1}{w^4}-\delta_1\left(A_1^2+\sigma_2 A_2^2\right)^2=0\,. \tag{22}$$

Proceeding in a similar way, if we substitute (15) and (16) in Equation (10), we can obtain the following equations:

$$\omega_2 = 0\,, \tag{23}$$

$$k_2 = \frac{\alpha_2}{w^2}+\frac{\beta_2}{w^4}\,, \tag{24}$$

$$-\frac{2\alpha_2}{w^2}-\frac{20\beta_2}{w^4}+\gamma_2\left(A_2^2+\sigma_1 A_1^2\right)=0\,, \tag{25}$$

$$\frac{24\beta_2}{w^4}-\delta_2\left(A_2^2+\sigma_2 A_1^2\right)^2=0\,. \tag{26}$$

Now, if we solve Equation (22) for w^4, and substitute the resulting expression in (21), we obtain the following equation:

$$-2\alpha_1\left(\frac{\delta_1}{24\beta_1}\right)^{1/2}\left(A_1^2+\sigma_2A_2^2\right)-\frac{5}{6}\delta_1\left(A_1^2+\sigma_2A_2^2\right)^2+\gamma_1\left(A_1^2+\sigma_1A_2^2\right)=0\,. \tag{27}$$

In a similar way, from Equations (25) and (26) we can obtain:

$$-2\alpha_2\left(\frac{\delta_2}{24\beta_2}\right)^{1/2}\left(A_2^2+\sigma_2A_1^2\right)-\frac{5}{6}\delta_2\left(A_2^2+\sigma_2A_1^2\right)^2+\gamma_2\left(A_2^2+\sigma_1A_1^2\right)=0\,. \tag{28}$$

Moreover, from Equations (22) and (26) we can obtain two different expressions for w^{-4}. By equating these two expressions we obtain:

$$\frac{\delta_1}{\beta_1}\left(A_1^2+\sigma_2A_2^2\right)^2=\frac{\delta_2}{\beta_2}\left(A_2^2+\sigma_2A_1^2\right)^2, \tag{29}$$

and if we consider that the coefficients β_1, β_2, δ_1, δ_2 and σ_2 are all positive, Equation (29) implies that:

$$\left(\frac{\delta_1}{\beta_1}\right)^{1/2}\left(A_1^2+\sigma_2A_2^2\right)=\left(\frac{\delta_2}{\beta_2}\right)^{1/2}\left(A_2^2+\sigma_2A_1^2\right). \tag{30}$$

If we considered that the values of the 10 coefficients $\{\alpha_n,\beta_n,\gamma_n,\delta_n,\sigma_n \mid n=1,2\}$ had been chosen arbitrarily, then it would be impossible to find values of A_1^2 and A_2^2 which satisfy the three equations (27), (28) and (30), as we would have an overdetermined system of three equations with only two unknowns. However, if we consider that one of the coefficients is unknown (for example, γ_2), then the Equations (27), (28) and (30) would be a system of three equations with three unknowns (A_1^2, A_2^2 and γ_2), and it would be possible to obtain the solution of this system. However, if α_1 and α_2 are arbitrary real numbers, and $\sigma_1\neq\sigma_2$, it will not be possible to obtain the solution of this system in analytical form. Therefore, let us consider the simpler case when:

$$\sigma_1=\sigma_2\equiv\sigma\,. \tag{31}$$

In this case the Equations (27), (28) and (30) reduce to:

$$-2\alpha_1\left(\frac{\delta_1}{24\beta_1}\right)^{1/2}-\frac{5}{6}\delta_1\left(A_1^2+\sigma A_2^2\right)+\gamma_1=0\,, \tag{32}$$

$$-2\alpha_2\left(\frac{\delta_2}{24\beta_2}\right)^{1/2}-\frac{5}{6}\delta_2\left(A_2^2+\sigma A_1^2\right)+\gamma_2=0\,, \tag{33}$$

$$\left(\frac{\delta_1}{\beta_1}\right)^{1/2}\left(A_1^2+\sigma A_2^2\right)=\left(\frac{\delta_2}{\beta_2}\right)^{1/2}\left(A_2^2+\sigma A_1^2\right), \tag{34}$$

and if we obtain $\left(A_1^2+\sigma A_2^2\right)$ and $\left(A_2^2+\sigma A_1^2\right)$ from Equations (32) and (33), and substitute the resulting expressions into Equation (34), this equation transforms into:

$$\left(\frac{\delta_1\beta_2}{\delta_2\beta_1}\right)^{1/2}\frac{\delta_2}{\delta_1}\left[\gamma_1-2\alpha_1\left(\frac{\delta_1}{24\beta_1}\right)^{1/2}\right]=\gamma_2-2\alpha_2\left(\frac{\delta_2}{24\beta_2}\right)^{1/2}. \tag{35}$$

In this way, instead of the system (27), (28) and (30), now we just have a system of two Equations, (32) and (33), subject to the condition that the coefficients α_n, β_n, γ_n and δ_n must satisfy the condition (35). And this condition can be used to obtain γ_2 as a function of the remaining coefficients $\{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\delta_1,\delta_2\}$. If this function is now introduced in Equation (33), we can obtain A_1^2 and A_2^2 by solving the system (32)-(33), and the solution is:

$$A_1^2 = -\left[\frac{1}{5\delta_1\left(\sigma^2-1\right)}\right]\left[\sqrt{6}\alpha_1\sqrt{\delta_1/\beta_1}-6\gamma_1\right]\left[-1+\sigma\left(\frac{\beta_2\delta_1}{\beta_1\delta_2}\right)^{1/2}\right], \tag{36}$$

$$A_2^2 = \left[\frac{1}{5\delta_1\left(\sigma^2-1\right)}\right]\left[\sqrt{6}\alpha_1\sqrt{\delta_1/\beta_1}-6\gamma_1\right]\left[-\sigma+\left(\frac{\beta_2\delta_1}{\beta_1\delta_2}\right)^{1/2}\right]. \tag{37}$$

Therefore, if $\sigma^2 \neq 1$ and we have coefficients $\{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2\}$ which produce *positive* values of A_1^2 and A_2^2 when they are substituted in the right-hand-sides of Equations (36) and (37), then the system (9)-(10) will have a vector soliton solution of the form (15)-(16), with $\omega_1=\omega_2=0$. Once with the values of A_1^2 and A_2^2, we can calculate the value of w (the width of the solitons) with Equation (22) or (26), if we introduce in these equations the value of the parameter σ. And once with the value of w, we will be able to calculate the values of the wavenumbers k_1 and k_2 with the Equations (20) and (24). For example, let us consider the coefficients:

$$\alpha_1 = 0.5 \quad \text{and} \quad \alpha_2 = 0.25, \tag{38}$$

$$\beta_1 = 5 \quad \text{and} \quad \beta_2 = 4, \tag{39}$$

$$\gamma_1 = 5 \quad \text{and} \quad \gamma_2 = 4.44152, \tag{40}$$

$$\delta_1 = 1 \quad \text{and} \quad \delta_2 = 1, \tag{41}$$

$$\sigma = 2. \tag{42}$$

where the value of γ_2 was calculated with Equation (35). Substituting these coefficients in Equations (36)-(37) we obtain:

$$A_1^2 = 1.5489, \tag{43}$$

$$A_2^2 = 2.17078. \tag{44}$$

Then, substituting these values in (22) or (26), we obtain the value of w:

$$w = 1.36370, \tag{45}$$

and introducing this value in Equations (20) and (24) we obtain the values of the wavenumbers:

$$k_1 = 1.71459 \quad \text{and} \quad k_2 = 1.29101. \tag{46}$$

As these wavenumbers are positive, they are contained in the range of the linear dispersion relations of Equations (9) and (10), as these two equations have dispersion relations of the form:

$$k = \beta_n\omega^4 - \alpha_n\omega^2. \tag{47}$$

Therefore, the solitons (15) and (16), with the amplitudes, width and wavenumbers defined in Equations (43)-(46), are *embedded solitons*. This result proves that the NLS system (9)-(10) does indeed possess *embedded vector solitons* (EVSs).

In this way we have demonstrated that *embedded vector solitons* do indeed exist.

3.2. VK Stability of the EVSs of System (9)-(10)

In this sub-section we will use the VK criterion to estimate the stability of the EVSs of the NLS system (9)-(10). Solitons which are stable according to the VK criterion will be referred to as *VK-stable* solitons.

To determine the VK stability of the soliton solutions (15)-(16) of the system (9)-(10) we will need the integrals:

$$N_1 = \int_{-\infty}^{+\infty} |P|^2 \,\mathrm{d}t \quad \text{and} \quad N_2 = \int_{-\infty}^{+\infty} |Q|^2 \,\mathrm{d}t \,. \tag{48}$$

Substituting the functions (15) and (16) in these integrals we obtain:

$$N_1 = 2A_1^2 w \quad \text{and} \quad N_2 = 2A_2^2 w \,, \tag{49}$$

and to determine the stability of the solitons (15) and (16) we have to calculate the signs of the derivatives:

$$\frac{\mathrm{d}N_1}{\mathrm{d}k_1} \quad \text{and} \quad \frac{\mathrm{d}N_2}{\mathrm{d}k_2}\,. \tag{50}$$

Therefore, in order to calculate these derivatives, we need to express A_1^2, A_2^2 and w as functions of the wavenumbers k_1 and k_2. In the following we'll see how to obtain these expressions.

To begin with, we should observe that if we consider the simpler case when (31) holds, and we consider that $\sigma > 0$, then Equations (22) and (26) permit us to express w^{-2} in the following two forms:

$$\frac{1}{w^2} = \left(\frac{\delta_1}{24\beta_1}\right)^{1/2} \left(A_1^2 + \sigma A_2^2\right), \tag{51}$$

$$\frac{1}{w^2} = \left(\frac{\delta_2}{24\beta_2}\right)^{1/2} \left(A_2^2 + \sigma A_1^2\right). \tag{52}$$

On the other hand, from (20) and (24) we can obtain an expression for w^2 as a function of k_1 and k_2:

$$w^2 = \frac{\alpha_1 - \dfrac{\beta_1}{\beta_2}\alpha_2}{k_1 - \dfrac{\beta_1}{\beta_2}k_2}. \tag{53}$$

Then, the substitution of (53) in (51) leads to the following equation:

$$\left(A_1^2 + \sigma A_2^2\right)\left(\frac{\delta_1}{24\beta_1}\right)^{1/2}\left(\alpha_1 - \frac{\beta_1}{\beta_2}\alpha_2\right) = k_1 - \frac{\beta_1}{\beta_2}k_2. \tag{54}$$

In a similar way, from (52) and (53) we can obtain:

$$\left(A_2^2+\sigma A_1^2\right)\left(\frac{\delta_2}{24\beta_2}\right)^{1/2}\left(\alpha_2-\frac{\beta_2}{\beta_2}\alpha_2\right)=k_2-\frac{\beta_2}{\beta_1}k_1. \tag{55}$$

And from Equations (54) and (55) we can obtain the following equations:

$$\left(\sigma^2-1\right)A_1^2=\left(\frac{\sigma}{F_2}+\frac{\beta_1}{F_1\beta_2}\right)k_2-\left(\frac{\sigma\beta_2}{F_2\beta_1}+\frac{1}{F_1}\right)k_1, \tag{56}$$

$$\left(\sigma-\frac{1}{\sigma}\right)A_2^2=\left(\frac{1}{F_1}+\frac{\beta_2}{\sigma F_2\beta_1}\right)k_1-\left(\frac{1}{\sigma F_2}+\frac{\beta_1}{F_1\beta_2}\right)k_2, \tag{57}$$

where we have defined:

$$F_1=\left(\frac{\delta_1}{24\beta_1}\right)^{1/2}\left(\alpha_1-\frac{\beta_1}{\beta_2}\alpha_2\right), \tag{58}$$

$$F_2=\left(\frac{\delta_2}{24\beta_2}\right)^{1/2}\left(\alpha_2-\frac{\beta_2}{\beta_1}\alpha_1\right). \tag{59}$$

The Equations (53), (56) and (57) are particularly important. They reveal how the solitons' parameters w, A_1^2 and A_2^2 depend on the wavenumbers k_1 and k_2. It is worth observing that the Equations (36)-(37), obtained in Sec. 3.1, permitted us to calculate the values of the solitons' amplitudes A_1 and A_2, but those equations do not tell us how these amplitudes depend on the wavenumbers. Now, with the Equations (53), (56) and (57), we know how the integrals N_1 and N_2 [given in Equation (49)] depend on k_1 and k_2, and the knowledge of the functions $N_1\left(k_1,k_2\right)$ and $N_2\left(k_1,k_2\right)$ permits us to know the signs of the derivatives mentioned in (50).

Let us use the functions $N_1\left(k_1,k_2\right)$ and $N_2\left(k_1,k_2\right)$, obtained by substituting the Equations (53), (56) and (57) in (49), to determine the VK-stability of the solitons of the system (9)-(10) which were determined in Sec. 3.1. These solitons correspond to the coefficients shown in Equations (38)-(42). Therefore, introducing these coefficients in Equations (53), (56) and (57) we can obtain the form of the functions $N_1\left(k_1,k_2\right)$ and $N_2\left(k_1,k_2\right)$.

The VK-stability of the first soliton [the soliton P defined in Equation (15)] is defined by the slope of the curve $N_1\left(k_1,k_2=1.29101\right)$, where $k_2=1.29101$ is the wavenumber of the second soliton [the soliton Q defined in Equation (16)]. This curve is shown in **Figure 1**.

This figure shows that $\mathrm{d}N_1/\mathrm{d}k_1>0$, thus implying that the P-soliton defined in Equation (15) is VK-stable.

On the other hand, **Figure 2** shows the shape of the curve $N_2\left(k_1=1.71459,k_2\right)$, where $k_1=1.71459$ is the wavenumber of the first soliton [the soliton P shown in Equation (15)]. We can see that in this case $\mathrm{d}N_2/\mathrm{d}k_2<0$, thus implying that the Q-soliton, defined in Equation (16), is VK-unstable.

We have thus seen that with the coefficients shown in Equations (38)-(42) the P-soliton is VK-stable and the Q-soliton is VK-unstable. However, this situation may change if we use other coefficients. For example, let us consider the following coefficients:

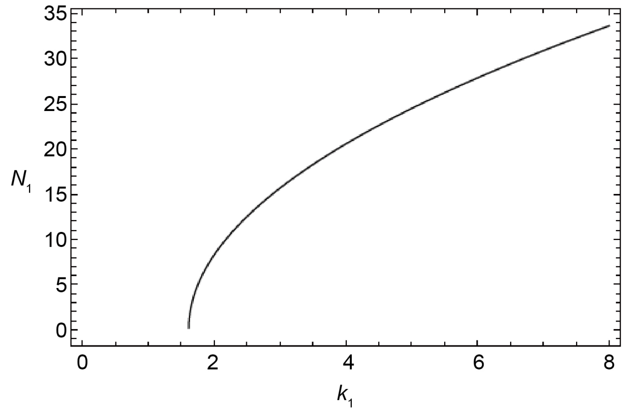

Figure 1. Integral N_1 as a function of the wavenumber k_1, corresponding to the coefficients shown in Equations (38)-(42).

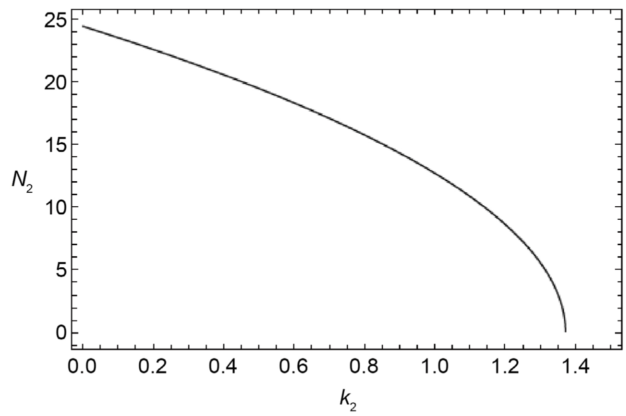

Figure 2. Integral N_2 as a function of the wavenumber k_2, corresponding to the coefficients shown in Equations (38)-(42).

$$\alpha_1 = 0.35 \quad \text{and} \quad \alpha_2 = 0.65, \tag{60}$$

$$\beta_1 = 5 \quad \text{and} \quad \beta_2 = 4, \tag{61}$$

$$\gamma_1 = 5 \quad \text{and} \quad \gamma_2 = 6.40008, \tag{62}$$

$$\delta_1 = 0.5 \quad \text{and} \quad \delta_2 = 1, \tag{63}$$

$$\sigma = 2, \tag{64}$$

where the value of γ_2 was calculated with Equation (35). With these coefficients the Equations (36)-(37) imply that:

$$A_1^2 = 1.05007, \tag{65}$$

$$A_2^2 = 5.42074. \tag{66}$$

Substituting these values in (22) we obtain the value of w:

$$w = 1.14139, \tag{67}$$

and introducing this value in Equations (20) and (24) we obtain the wavenumbers of the solitons P and Q defined in Equations (15)-(16):

$$k_1 = 3.21468 \quad \text{and} \quad k_2 = 2.85575\,. \tag{68}$$

Introducing the coefficients (60)-(64) in in Equations (53), (56) and (57) we can obtain w^2, A_1^2 and A_2^2 in terms of k_1 and k_2, and with these expressions we can obtain the forms of the functions $N_1(k_1, k_2 = 2.85575)$ and $N_2(k_1 = 3.21468, k_2)$. In **Figure 3** and **Figure 4** we can see the forms of these functions.

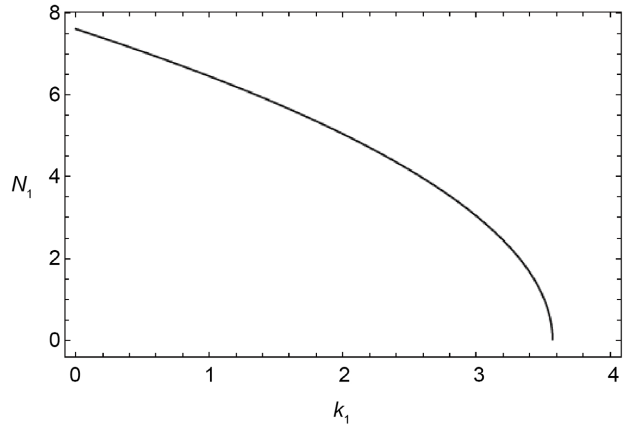

Figure 3. Integral N_1 as a function of the wavenumber k_1, corresponding to the coefficients shown in Equations (60)-(64).

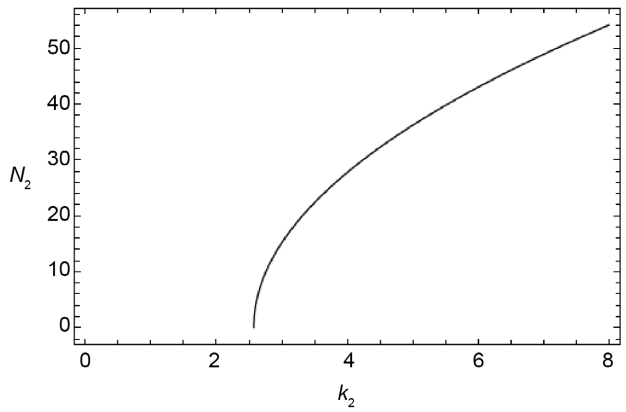

Figure 4. Integral N_2 as a function of the wavenumber k_2, corresponding to the coefficients shown in Equations (60)-(64).

We can see that in the two examples shown above [*i.e.*, with the coefficients (38)-(42), and also with the coefficients (60)-(64)], the VK criterion predicts that one of the solitons shown in (15)-(16) is VK-stable, while the other is VK-unstable. This result might be a bit unexpected at first sight. However, the VK prediction that the vector solitons of the NLS system (9)-(10) are neither fully stable, nor completely unstable, might be related to the fact that the embedded solitons of the cubic-quintic NLS equation with fourth-order dispersion [*i.e.*, Equation (9) with $\sigma_1 = \sigma_2 = 0$] are *semistable* objects, *i.e.*, they are stable in a linear approximation, but are subject to a one-sided nonlinear instability.

4. Embedded Vector Solitons of a cmKdV System

It is well-known that the standard NLS equation, with arbitrary *positive* coefficients in front of the dispersive and nonlinear terms, has bright solitons of *arbitrary* heights. On the other hand, we have seen that if we choose an adequate set of coefficients $\{\alpha_n, \beta_n, \gamma_n, \delta_n, \sigma_n\}$, for which we can obtain an analytical solution of the Equations (24)-(25), and this solution satisfies the condition (31), then the system (9)-(10) has *one* (and only one) *embedded vector soliton*. The uniqueness of this solution is not a surprise, since we know that the cubic-quintic NLS equation, with second and fourth-order dispersion [1, 11], has *one* (and only one) embedded soliton. However, it would be interesting to find out if there are systems of NLPDEs that accept families of different EVSs. In this section, we investigate this issue.

4.1. Existence of Solitons in a cmKdV System

Let us consider the following system of complex modified Korteweg-de Vries (cmKdV) equations:

$$u_z - \varepsilon_1 u_{ttt} - \gamma_1 \left(|u|^2 + \sigma |v|^2 \right) u_t = 0\,, \tag{69}$$

$$v_z - \varepsilon_2 v_{ttt} - \gamma_2 \left(|v|^2 + \sigma |u|^2 \right) v_t = 0\,. \tag{70}$$

And now let us investigate if this system accepts solutions of the form:

$$u = A_1 \operatorname{sech}\left(\frac{t - az}{w}\right) \mathrm{e}^{i(q_1 z + r_1 t)}, \tag{71}$$

$$v = A_2 \operatorname{sech}\left(\frac{t - az}{w}\right) \mathrm{e}^{i(q_2 z + r_2 t)}. \tag{72}$$

Substituting (71)-(72) in the system (69)-(70) we find that the functions (71)-(72) are indeed solutions of (69)-(70) if the following conditions are satisfied (for $n = 1$ and $n = 2$):

$$a = 3\varepsilon_n r_n^2 - \frac{\varepsilon_n}{w^2}, \tag{73}$$

$$q_n = -\varepsilon_n r_n^3 + \frac{3\varepsilon_n r_n}{w^2}, \tag{74}$$

$$\gamma_n \left(A_n^2 + \sigma A_{3-n}^2 \right) = \frac{6\varepsilon_n}{w^2}. \tag{75}$$

If we consider the two possible values, $n = 1$ and $n = 2$, in Equations (73)-(75), we obtain 6 equations. And these 6 equations involve the 8 parameters which appear in the soliton solutions (71)-(72), namely: a, w, A_1, A_2, q_1, q_2, r_1 and r_2. Therefore, we may choose the values of two of these 8 parameters, and the Equations (73)-(75) will permit us to obtain the values of the remaining 6 parameters. In the following we will see that it is convenient to choose the values of the parameters a and w [which are the parameters which appear in the two Equations (71)-(72)].

Once the values of the coefficients $\{\sigma, \varepsilon_n, \gamma_n\}$ have been specified, and the values of a and w have been chosen, Equation (73) will permit us to obtain the values of r_1 and r_2. And once the values of r_1 and r_2 are defined, the Equation (74) will permit us to calculate the values of q_1 and q_2. Finally, we only need to calculate the values of A_1 and A_2 in order to have the exact shape of the solitons (71)-(72).

We can see that Equation (75) defines a system of 2 linear equations for the variables A_1^2 and A_2^2. Solving this system we find that:

$$A_n^2 = \frac{6}{w^2\left(1-\sigma^2\right)}\left(\frac{\varepsilon_n}{\gamma_n} - \sigma\frac{\varepsilon_{3-n}}{\gamma_{3-n}}\right), \tag{76}$$

and these equations will give us the values of the solitons' amplitudes A_1 and A_2. As A_1^2 and A_2^2 must be positive, Equation (76) implies that the coefficients $\{\sigma, \varepsilon_n, \gamma_n\}$ must satisfy the condition:

$$\frac{1}{1-\sigma^2}\left(\frac{\varepsilon_n}{\gamma_n} - \sigma\frac{\varepsilon_{3-n}}{\gamma_{3-n}}\right) > 0. \tag{77}$$

Therefore, only of this condition is satisfied, the cmKdV system (69)-(70) will have soliton solutions of the form (71)-(72).

It should be emphasized that the soliton solutions (71)-(72) of the system (69)-(70) are indeed *embedded solitons*. This reason of this embedding can be easily understood if we observe that the linear dispersion relations of the Equations (69)-(70) have the following form:

$$k = \varepsilon_n \omega^3, \tag{78}$$

and the range of wavenumbers permitted by this dispersion relation is the entire real axis. Therefore, the solitons' wavenumbers q_1 and q_2 (irrespective of their particular values) will be necessarily contained in this range of wavenumbers, thus implying that the solitons (71)-(72) are indeed *embedded*, and consequently the Equations (71)-(72) constitute an *embedded vector soliton* of the system (69)-(70).

To close this sub-section, it is worth emphasizing that the cmKdV system (69)-(70) has *an infinity of EVSs*, since there exist EVSs of the form (71)-(72) for arbitrary values of the width, w, and the (inverse) velocity a.

4.2. VK Stability of the EVSs of System (69)-(70)

Headings, or heads, are organizational devices that guide the reader through your paper. There are two types: component heads and text heads.

In order to determine the VK stability of the solitons (71)-(72) we need to calculate the signs of the derivatives:

$$\frac{\mathrm{d}N_n}{\mathrm{d}q_n}, \tag{79}$$

where we have defined:

$$N_1 = \int_{-\infty}^{+\infty} |u|^2 \,\mathrm{d}t \quad \text{and} \quad N_2 = \int_{-\infty}^{+\infty} |v|^2 \,\mathrm{d}t . \tag{80}$$

Introducing the forms of the solitons (71)-(72) in these integrals we find that:

$$N_n = 2A_n^2 w = \frac{2A_n^2}{M^{1/2}}, \tag{81}$$

where we have defined $M = 1/w^2$, and $n = 1$ or $n = 2$.

Now, taking into account that Equation (76) implies that A_1^2 and A_2^2 depend on $M = 1/w^2$, and Equations (73)-(74) imply that q_1 and q_2 also depend on M, we can calculate the derivative shown in (79) in the following form:

$$\frac{\mathrm{d}N_n}{\mathrm{d}q_n} = \frac{\mathrm{d}}{\mathrm{d}M}\left(\frac{2A_n^2}{M^{1/2}}\right)\frac{1}{\mathrm{d}q_n/\mathrm{d}M}. \tag{82}$$

The first derivative which appears in the rhs (right-hand side) of (82) can be evaluated using (76), and in this way we find:

$$\frac{\mathrm{d}N_n}{\mathrm{d}q_n} = \frac{6}{\left(\sigma^2 - 1\right)M^{1/2}}\left(\sigma\frac{\varepsilon_{3-n}}{\gamma_{3-n}} - \frac{\varepsilon_n}{\gamma_n}\right)\frac{1}{\mathrm{d}q_n/\mathrm{d}M}. \tag{83}$$

Therefore, the sign of $\mathrm{d}N_n/\mathrm{d}q_n$ will be determined by the values of the coefficients σ, ε_n and γ_n, and the sign of the derivative $\mathrm{d}q_n/\mathrm{d}M$. And to calculate this last derivative we need an expression which gives us q_n as a function of M. To obtain this function we get r_n from (73), and introduce this expression of r_n in (74). In this way we obtain:

$$q_n^2 = \sum\nolimits_{k=1}^{5} h_k^{(n)} M^k, \tag{84}$$

where the coefficients $h_k^{(n)}$ have been defined as follows:

$$h_1^{(n)} = \frac{a^5}{243\varepsilon_n}, \tag{85}$$

$$h_2^{(n)} = -\frac{32a^4}{243}, \tag{86}$$

$$h_3^{(n)} = \frac{128}{81}a^3\varepsilon_n, \tag{87}$$

$$h_4^{(n)} = -\frac{2048}{243}a^2\varepsilon_n^2, \tag{88}$$

$$h_5^{(n)} = \frac{4096}{243}a\varepsilon_n^3. \tag{89}$$

With Equations (84)-(89) we can calculate the derivative $\mathrm{d}q_n/\mathrm{d}M$, and replacing $M = 1/w^2$ in the resulting expression, we obtain:

$$\frac{\mathrm{d}q_n}{\mathrm{d}M} = \frac{Q_1^{(n)}\left(a, w, \varepsilon_n\right)}{Q_2^{(n)}\left(a, w, \varepsilon_n\right)}, \tag{90}$$

where we have defined:

$$Q_1^{(n)}\left(a, w, \varepsilon_n\right) = \frac{a^5}{243\varepsilon_n} + \frac{20480a\varepsilon_n^3}{243w^8} - \frac{8192a^2\varepsilon_n^2}{243w^6} + \frac{128a^3\varepsilon_n}{27w^4} - \frac{64a^4}{243w^2}, \tag{91}$$

$$Q_2^{(n)}(a,w,\varepsilon_n) = -2\varepsilon_n \left(\frac{a}{3\varepsilon_n}+\frac{1}{3w^2}\right)^{3/2} + \frac{6\varepsilon_n}{w^2}\left(\frac{a}{3\varepsilon_n}+\frac{1}{3w^2}\right)^{1/2}. \quad (92)$$

Therefore, the Equations (83) and (90)-(92) permit us to obtain the values of the derivatives $\mathrm{d}N_n/\mathrm{d}q_n$, and the signs of these derivatives will define the VK stability of the solitons of the cmKdV system (69)-(70). In **Table 1** we show the signs of the derivatives $D1 \equiv \mathrm{d}N_1/\mathrm{d}q_1$ and $D2 \equiv \mathrm{d}N_2/\mathrm{d}q_2$, corresponding to four different choices of the set of parameters $\{\sigma, a, w, \varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2\}$.

Table 1. Signs of D1 and D2 for different values of the parameters σ, a, w, ε_1, ε_2, γ_1 and γ_2.

σ	a	w	ε_1	ε_2	γ_1	γ_2	Sign D1	Sign D2
2	2	$\sqrt{8/5}$	2	0.5	6	2	+	+
2	3	$\sqrt{8/5}$	1.25	0.5	6	2	+	+
2	6	$\sqrt{8/5}$	1.25	0.25	6	2	+	+
2	8	$\sqrt{8/5}$	2	0.5	6	2	+	+

Table 1 shows that in the four cases considered in the table, the two bright solitons (71)-(72) of the cmKdV system (69)-(70) turned out to be VK-stable. This is the result that we might have guessed, as the solitons of the standard cmKdV equation are always stable [4].

5. Dark Vector Solitons of the NLS system (9)-(10)

We have seen that the NLS system (9)-(10) has bright vector solitons (VSs) of the form (15)-(16). Now, in this section, we will see that this system also has a dark VS of the form:

$$P = A_1 \tanh\left(\frac{t}{w}\right)\mathrm{e}^{iq_1 z}, \quad (93)$$

$$Q = A_2 \tanh\left(\frac{t}{w}\right)\mathrm{e}^{iq_2 z}. \quad (94)$$

Substituting these functions in the system (9)-(10) we find that the following equation must be satisfied (for $n=1$ and $n=2$):

$$\begin{aligned}&\left[\gamma_n\left(A_n^2+\sigma_1 A_{3-n}^2\right)-\delta_n\left(A_n^2+\sigma_2 A_{3-n}^2\right)^2-q_n\right] sh\left(\frac{t}{w}\right) ch^4\left(\frac{t}{w}\right)\\ &+\left[2\delta_n\left(A_n^2+\sigma_2 A_{3-n}^2\right)^2-\gamma_n\left(A_n^2+\sigma_1 A_{3-n}^2\right)-\frac{8\beta_n}{w^4}-\frac{2\alpha_n}{w^2}\right] sh\left(\frac{t}{w}\right) ch^2\left(\frac{t}{w}\right)\\ &+\left[\frac{24\beta_n}{w^4}-\delta_n\left(A_n^2+\sigma_2 A_{3-n}^2\right)^2\right] sh\left(\frac{t}{w}\right)=0.\end{aligned} \quad (95)$$

And for this equation to be satisfied it is necessary that the coefficients of *sh ch²*, *sh* and *sh ch⁴* must be equal to zero. In this way the following conditions are obtained:

$$2\delta_n\left(A_n^2+\sigma_2 A_{3-n}^2\right)^2-\gamma_n\left(A_n^2+\sigma_1 A_{3-n}^2\right)-\frac{8\beta_n}{w^4}-\frac{2\alpha_n}{w^2}=0, \tag{96}$$

$$\frac{24\beta_n}{w^4}-\delta_n\left(A_n^2+\sigma_2 A_{3-n}^2\right)^2=0, \tag{97}$$

$$\gamma_n\left(A_n^2+\sigma_1 A_{3-n}^2\right)-\delta_n\left(A_n^2+\sigma_2 A_{3-n}^2\right)^2-q_n=0. \tag{98}$$

Now let's see how to obtain the values of the amplitudes (A_n), width (w) and wavenumbers (q_n) from these equations.

Solving (97) for $1/w^4$, and substituting the resulting expression in (96), the following equation is obtained:

$$\frac{5}{3}\delta_n\left(A_n^2+\sigma_2 A_{3-n}^2\right)^2-\gamma_n\left(A_n^2+\sigma_1 A_{3-n}^2\right)-2\alpha_n\left(\frac{\delta_n}{24\beta_n}\right)^{1/2}\left(A_n^2+\sigma_2 A_{3-n}^2\right)=0. \tag{99}$$

In order to simplify this equation, we will consider the particular case when $\sigma_1=\sigma_2\equiv\sigma$. In this case Equation (99) reduces to:

$$\left(A_n^2+\sigma A_{3-n}^2\right)-\frac{3}{5\delta_n}\left[\gamma_n+2\alpha_n\left(\frac{\delta_n}{24\beta_n}\right)^{1/2}\right]=0. \tag{100}$$

This equation represents a system of two equations for A_1^2 and A_2^2 which can be readily solved, and the solution is the following:

$$A_n^2=\frac{\sigma f_{3-n}-f_n}{\sigma^2-1}, \tag{101}$$

where we have defined:

$$f_n=\frac{3}{5\delta_n}\left[\gamma_n+2\alpha_n\left(\frac{\delta_n}{24\beta_n}\right)^{1/2}\right]. \tag{102}$$

Once with the values of A_1^2 and A_2^2, we can obtain the value of w by means of Equation (97), and the values of q_1 and q_2 with Equation (98). In this way the dark vector soliton solution (93)-(94) of the system (9)-(10) is completely defined.

There is, however, an important issue that we must observe. The solution obtained by means of Equations (101), (97) and (98) only exists if the coefficients $\{\alpha_n,\beta_n,\gamma_n,\delta_n,\sigma\}$, which enter in the system (9)-(10), satisfy two conditions. To find these conditions we must observe that Equation (97) gives us two different expressions for w^4. If we require these two expressions to be equal, the following equation is obtained:

$$\frac{A_1^2}{A_2^2}=\frac{g-\sigma}{1-g\rho}, \tag{103}$$

where we have defined:

$$g=\left(\frac{\beta_1\delta_2}{\beta_2\delta_1}\right)^{1/2}. \tag{104}$$

On the other hand, from Equation (101) we can obtain the following equation:

$$\frac{A_1^2}{A_2^2} = \frac{\sigma f_2 - f_1}{\sigma f_1 - f_2}. \tag{105}$$

Notice that A_1^2 / A_2^2 is a positive quantity, and therefore Equation (105) implies that the coefficients $\{\alpha_n, \beta_n, \gamma_n, \delta_n, \sigma\}$ must satisfy the following inequality:

$$\frac{\sigma f_2 - f_1}{\sigma f_1 - f_2} > 0. \tag{106}$$

Moreover, as the right-hand sides of Equations (103) and (105) must be equal, it follows that the coefficients $\{\alpha_n, \beta_n, \gamma_n, \delta_n, \sigma\}$ must also satisfy the following condition:

$$\frac{\sigma f_2 - f_1}{\sigma f_1 - f_2} = \frac{g - \sigma}{1 - g\rho}. \tag{107}$$

If we substitute in this equation the definitions of the functions f_1, f_2 and g [given by Equations (102) and (104)], we can see the condition (107) can be written in the form:

$$\begin{aligned} &\delta_1^{-1}\left(\rho M_1 + M_2\right)\left[\gamma_1 + 2\alpha_1\left(\frac{\delta_1}{24\beta_1}\right)^{1/2}\right] \\ &= \delta_2^{-1}\left(\rho M_2 + M_2\right)\left[\gamma_2 + 2\alpha_2\left(\frac{\delta_2}{24\beta_2}\right)^{1/2}\right], \end{aligned} \tag{108}$$

where M_1 and M_2 have been defined in the following form:

$$M_1 = g - \sigma, \tag{109}$$

$$M_2 = 1 - g\sigma. \tag{110}$$

There are infinite sets of coefficients $\{\alpha_n, \beta_n, \gamma_n, \delta_n, \sigma\}$ which satisfy the condition (108). And an easy way to generate a set of coefficients which satisfy (108), is to choose the values of all the coefficients, except γ_2, and then use (108) to determine the adequate value of γ_2. Then, if the coefficients so defined satisfy the inequality (106), the system (9)-(10) will have a dark vector soliton of the form (93)-(94), where the values of A_1^2, A_2^2, w, q_1 and q_2 can be found with the Equations (101), (97) and (98).

Let us exemplify the procedure just described with one example. Let us choose the coefficients:

$$\sigma = 2, \tag{111}$$

$$\alpha_1 = 10/9 \quad \text{and} \quad \alpha_2 = 1, \tag{112}$$

$$\beta_1 = 2 \quad \text{and} \quad \beta_2 = 1.5, \tag{113}$$

$$\gamma_1 = 0.75, \tag{114}$$

$$\delta_1 = 0.5 \quad \text{and} \quad \delta_2 = 1. \tag{115}$$

These coefficients satisfy the condition (106). Now, let us substitute these values in Equation (108). In this way we obtain an equation for γ_2. And solving this equation we find that:

$$\gamma_2 = 0.863003. \quad (116)$$

Therefore, the coefficients (111)-(116) satisfy the conditions (106) and (108). And now, substituting these coefficients in the equations defined in (101), we can obtain the following values of A_1 and A_2:

$$A_1 = 0.296332 \quad \text{and} \quad A_2 = 0.736326. \quad (117)$$

Now, substituting these values in the two equations that we obtain by taking $n = 1$ and $n = 2$ in Equation (98), we can obtain the values of the wavenumbers q_1 and q_2:

$$q_1 = 0.192138 \quad \text{and} \quad q_2 = 0.104226. \quad (118)$$

And finally, substituting the values of A_1 and A_2 in Equation (97) [and using either $n = 1$ or $n = 2$], we obtain the value of the width w:

$$w = 2.89117. \quad (119)$$

In this way we have obtained the values of the amplitudes (A_1 and A_2), wavenumbers (q_1 and q_2) and width (w), of the dark vector soliton of the NLS system (9)-(10), given by the Equations (93)-(94), corresponding to the coefficients (111)-(116). Following the same procedure we can obtain the shapes of the dark solitons of the system (9)-(10) corresponding to other coefficients.

It is worth observing that if we have an acceptable set of coefficients $\{\sigma, \alpha_n, \beta_n, \gamma_n, \delta_n\}$ satisfying the conditions (106) and (108), the NLS system (9)-(10) has a unique dark vector soliton (DVS), not a continuous family of DVSs. The same occurs in the case of the single cubic-quintic NLS equation with higher-order dispersive terms [11]: for a given set of coefficients, it has only one bright soliton and only one dark soliton.

6. Dark Vector Solitons of the cmKdV System (69)-(70)

In this section we will see that the cmKdV system (69)-(70) also has dark vector solitons (DVSs).

Let us consider a tentative solution of the form:

$$u = A_1 \tanh\left(\frac{t - az}{w}\right) e^{i(q_1 z + r_1 t)}, \quad (120)$$

$$v = A_2 \tanh\left(\frac{t - az}{w}\right) e^{i(q_2 z + r_2 t)}. \quad (121)$$

Substituting these functions in Equations (69) and Equation (70), we arrive at two equations of the form:

$$\begin{aligned} &\left[\frac{2\varepsilon_n}{w^3} + \frac{3\varepsilon_n r_n^2}{w} - \frac{a}{w} - \frac{6\varepsilon_n}{w^3} - \frac{\gamma_n}{w}\left(A_n^2 + \sigma A_{3-n}^2\right)\right]\frac{1}{ch^2(\theta)} \\ &+ i\left[q_n + \varepsilon_n r_n^3 - \gamma_n r_n \left(A_n^2 + \sigma A_{3-n}^2\right)\right]\frac{sh(\theta)}{ch(\theta)} \\ &+ \frac{1}{w}\left[\frac{6\varepsilon_n}{w^2} + \gamma_n\left(A_n^2 + \sigma A_{3-n}^2\right)\right]\frac{1}{ch^4(\theta)} \\ &+ i\left[\frac{6\varepsilon_n r_n}{w^2} + \gamma_n r_n\left(A_n^2 + \sigma A_{3-n}^2\right)\right]\frac{sh(\theta)}{ch^3(\theta)} = 0, \end{aligned} \quad (122)$$

where we defined $\theta = (t - az)/w$. For this equation to be satisfied it is necessary that the four coefficients which appear in front of the hyperbolic functions, are equal to zero. Setting equal to zero these coefficients we can obtain the following equations:

$$3\varepsilon_n r_n^2 = a - \frac{2\varepsilon_n}{w^2}, \tag{123}$$

$$q_n = -\frac{6\varepsilon_n r_n}{w^2} - \varepsilon_n r_n^3, \tag{124}$$

$$A_n^2 + \sigma A_{3-n}^2 = -\frac{6\varepsilon_n}{\gamma_n w^2}. \tag{125}$$

Taking $n=1$ and $n=2$ in (123)-(125) we obtain 6 equations which involve the 8 parameters which appear in Equations (120)-(121): A_1, A_2, q_1, q_2, r_1, r_2, a and w. Therefore, we can choose the values of 2 of these parameters, and then use the Equations (123)-(125) to obtain the values of the remaining 6 parameters. Suppose, for example, that we have chosen the values of a and w. Then (123) will give us the value of r_n, and having r_n we can obtain the value of q_n with (124). And, finally, the system of equations obtained from (125) will give us the values of A_1 and A_2.

It should be observed, however, that the procedure described in the previous paragraph will give us the values of the parameters that enter in the dark solitons (120)-(121), only if the coefficients which appear in Equations (69)-(70) satisfy certain conditions. Therefore, let us determine these conditions, considering that $\sigma > 0$, which is the physically realistic case.

To begin with, we can see that Equation (125) implies that so that the system (69)-(70) may have dark solitons, the coefficients ε_n and γ_n must satisfy the condition:

$$\frac{\varepsilon_n}{\gamma_n} < 0. \tag{126}$$

Moreover, from the equations shown in (125) we can obtain the following equation:

$$A_n^2 = \frac{6}{w^2(\sigma^2 - 1)}\left(\frac{\varepsilon_n}{\gamma_n} - \sigma\frac{\varepsilon_{3-n}}{\gamma_{3-n}}\right), \tag{127}$$

and this expression implies that the coefficients σ, ε_n and γ_n must also satisfy the condition:

$$\frac{1}{(\sigma^2 - 1)}\left(\frac{\varepsilon_n}{\gamma_n} - \sigma\frac{\varepsilon_{3-n}}{\gamma_{3-n}}\right) > 0. \tag{128}$$

Evidently, there are sets of coefficients $\{\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2, \sigma\}$ which satisfy (128), and coefficients which do not satisfy this condition. For example, the coefficients $\{\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2, \sigma\} = \{1.25, 0.5, -6, -2, 2\}$ satisfy (128), for $n=1$ and $n=2$. On the other hand, the coefficients $\{\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2, \sigma\} = \{1.25, 0.25, -6, -6, 2\}$ do not satisfy (128) when $n=1$.

Finally, there are two additional conditions that must be satisfied if the dark solitons (120)-(121) are going to be solutions of the cmKdV system (69)-(70). These conditions are consequences of the Equation (123). As r_n^2 must be a positive quantity, Equation (123) implies that a, w and ε_n must satisfy the following inequality:

$$\frac{a}{\varepsilon_n} > \frac{2}{w^2}, \tag{129}$$

and this inequality implies that ε_1 and ε_2 must be of the same sign, otherwise it would be impossible to find a value of a capable of satisfying (129) for $n=1$ and $n=2$. Therefore, the following condition must also be satisfied:

$$\varepsilon_1 \varepsilon_2 > 0 . \tag{130}$$

Therefore, when we have a set of coefficients $\{\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2, \sigma\}$ which satisfy the conditions (126), (128), and (130), the cmKdV system (69)-(70) has an infinite family of dark vector solitons of the form (120)-(121), where the parameters a and w can have any values which satisfy the inequality (129). Then, for these values of $\{\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2, \sigma, a, w\}$, the values of r_n, q_n and A_n can be obtained with the Equations (123)-(125).

7. Mixed Vector Solitons of the NLS System (9)-(10)

In this section, we will investigate if the NLS system (9)-(10) accepts *mixed vector solitons* (MVSs) *i.e.*, vector solitons composed by a *bright* and a *dark* soliton. In the first part of this section, we will prove that the system (9)-(10) does indeed possess MVSs. And in the second part we will use the VK criterion to estimate the stability of the bright solitons which enter in these MVSs.

7.1. Existence of Mixed Vector Solitons in the System (9)-(10)

Let us prove that the system (9)-(10) accepts mixed vector solitons composed by a bright and a dark soliton of the form:

$$P = A_1 \operatorname{sech}\left(\frac{t}{w}\right) e^{iq_1 z}, \tag{131}$$

$$Q = A_2 \tanh\left(\frac{t}{w}\right) e^{iq_2 z}. \tag{132}$$

As we did in Secs. 3.1 and 5, in order to simplify the calculations and being able to obtain analytical solutions, we will consider the particular case when $\sigma_1 = \sigma_2 \equiv \sigma$. In this way, substituting (131) and (132) in Equations (9) and (10) we obtain the following two equations:

$$\begin{aligned}
&\left(-q_1 + \frac{\alpha_1}{w^2} + \frac{\beta_1}{w^4} + \gamma_1 \sigma A_2^2 - \delta_1 \sigma^2 A_2^4\right) \frac{1}{ch(\theta)} \\
&+\left[-\frac{2\alpha_1}{w^2} - \frac{20\beta_1}{w^4} + \gamma_1\left(A_1^2 - \sigma A_2^2\right) - 2\delta_1 \sigma A_2^2\left(A_1^2 - \sigma A_2^2\right)\right] \frac{1}{ch^3(\theta)} \\
&+\left[\frac{24\beta_1}{w^4} - \delta_1\left(A_1^2 - \sigma A_2^2\right)^2\right] \frac{1}{ch^5(\theta)} = 0,
\end{aligned} \tag{133}$$

$$-q_2+\gamma_2A_2^2-\delta_2A_2^4 +\left[-\frac{2\alpha_2}{w^2}-\frac{8\beta_2}{w^4}-\gamma_2\left(A_2^2-\sigma A_1^2\right)+2\delta_2A_2^2\left(A_2^2-\sigma A_1^2\right)\right]\frac{1}{ch^2(\theta)} +\left[\frac{24\beta_2}{w^4}-\delta_2\left(A_2^2-\sigma A_1^2\right)^2\right]\frac{1}{ch^4(\theta)}=0, \tag{134}$$

where we defined $\theta=t/w$. In order to satisfy (133) and (134) all the coefficients of $ch(\theta)^{-n}$ (with $n=1,\cdots,5$) must be equal to zero. Therefore, setting equal to zero the coefficients of $ch(\theta)^{-5}$ and $ch(\theta)^{-4}$, we obtain the following two expressions for w^{-4}:

$$\frac{1}{w^4}=\frac{\delta_1}{24\beta_1}\left(A_1^2-\sigma A_2^2\right)^2, \tag{135}$$

$$\frac{1}{w^4}=\frac{\delta_2}{24\beta_2}\left(A_2^2-\sigma A_1^2\right)^2. \tag{136}$$

From these two equations it follows that the values of A_1^2 and A_2^2 must satisfy the following equation:

$$\frac{\delta_1}{\beta_1}\left(A_1^2-\sigma A_2^2\right)^2=\frac{\delta_2}{\beta_2}\left(A_2^2-\sigma A_1^2\right)^2, \tag{137}$$

which can also be written in the form:

$$\left(\frac{\delta_1}{\beta_1}-\frac{\delta_2\sigma^2}{\beta_2}\right)x^2+\left(\frac{\delta_1\sigma^2}{\beta_1}-\frac{\delta_2}{\beta_2}\right)y^2+\left(\frac{2\sigma\delta_2}{\beta_2}-\frac{2\sigma\delta_1}{\beta_1}\right)xy=0, \tag{138}$$

where we have defined $x=A_1^2$ and $y=A_2^2$. It should be noticed that Eq, (137) implies that the coefficients β_n and δ_n must satisfy the condition:

$$\frac{\beta_1\delta_2}{\beta_2\delta_1}>0, \tag{139}$$

and therefore this inequality is a necessary condition if the system (9)-(10) is to have vector soliton solutions of the form (131)-(132).

If we know the values of the coefficients which appear in Equations (9) and (10), then the only unknown parameters in Equation (138) are $x=A_1^2$ and $y=A_2^2$. Now we will obtain a second equation whose only unknown parameters are also x and y.

Substituting the value of w^{-4} given by Equation (135) in the equation obtained by setting equal to zero the coefficient of $ch^{-3}(\theta)$ which appears in Equation (133), we obtain the following equation:

$$-\frac{2}{w^2}=\frac{5\delta_1}{6\alpha_1}\left(A_1^2-\sigma A_2^2\right)^2-\frac{\gamma_1}{\alpha_1}\left(A_1^2-\sigma A_2^2\right)+\frac{2\delta_1\sigma}{\alpha_1}A_2^2\left(A_1^2-\sigma A_2^2\right). \tag{140}$$

In a similar way, substituting the value of w^{-4} given by Equation (136) in the equation obtained by setting equal to zero the coefficient of $ch^{-2}(\theta)$ in Equation (134), we obtain:

$$-\frac{2}{w^2}=\frac{\delta_2}{3\alpha_2}\left(A_2^2-\sigma A_1^2\right)^2+\frac{\gamma_2}{\alpha_2}\left(A_2^2-\sigma A_1^2\right)-\frac{2\delta_2}{\alpha_2}A_2^2\left(A_2^2-\sigma A_1^2\right). \tag{141}$$

And now, as the right-hand-sides of Equations (140) and (141) must be equal, we can obtain the following equation:

$$\left(c_1 - c_3\sigma^2\right)x^2 + \left(c_1\sigma^2 - c_2\sigma - c_3 + c_4\right)y^2 + \left(-2c_1\sigma + c_2 + 2c_3\sigma - c_4\sigma\right)xy + \left(c_5 - \frac{\gamma_1}{\alpha_1}\right)x + \left(c_6 - \frac{\gamma_2}{\alpha_2}\right)y = 0, \tag{142}$$

where we have defined:

$$c_1 = \frac{5\delta_1}{6\alpha_1}, \tag{143}$$

$$c_2 = \frac{2\delta_1\sigma}{\alpha_1}, \tag{144}$$

$$c_3 = \frac{\delta_2}{3\alpha_2}, \tag{145}$$

$$c_4 = \frac{2\delta_2}{\alpha_2}, \tag{146}$$

$$c_5 = \frac{\gamma_2\sigma}{\alpha_2}, \tag{147}$$

$$c_6 = \frac{\gamma_1\sigma}{\alpha_1}. \tag{148}$$

In this way we have obtained two equations involving *x* and *y*: Equations (138) and (142). Solving these two equations we can find the values of $x = A_1^2$ and $y = A_2^2$. And once having these values we can obtain the value of *w* with Equation (135) or (136). Then, the value of q_1 can be determined by substituting the values of *w*, A_1^2 and A_2^2, in the equation obtained by setting equal to zero the coefficient of $ch^{-1}(\theta)$ which appears in Equation (133), namely the equation:

$$q_1 = \frac{\alpha_1}{w^2} + \frac{\beta_1}{w^4} + \gamma_1\sigma A_2^2 - \delta_1\sigma^2 A_2^4. \tag{149}$$

And finally, the value of q_2 can be found by substituting the value of A_2^2 in the equation obtained by setting equal to zero the independent term (*i.e.*, independent of hyperbolic functions) in Equation (134), *i.e.*, the equation:

$$q_2 = \gamma_2 A_2^2 - \delta_2 A_2^4. \tag{150}$$

Let's exemplify this procedure using the same coefficients that we used in Sec. 3.1, *i.e.*, those shown in Equations (38)-(42). If we solve the Equations (138) and (142) using these coefficients, the solution of the system is the following:

$$x = A_1^2 = 1.71645, \quad y = A_2^2 = 1.78143. \tag{151}$$

If we now substitute these values in Equation (135) we obtain the value of *w*:

$$w = 2.43574. \tag{152}$$

And finally, substituting these values of A_1^2, A_2^2 and w in Equations (149) and (150), we obtain the following values of the wavenumbers:

$$q_1 = 5.34666, \quad q_2 = 4.73876. \tag{153}$$

It is important to observe that the positivity of q_1 implies that the bright soliton defined by Equation (131) is an *embedded soliton*. Therefore, the *mixed vector soliton* defined by Equations (131)-(132) contains an *embedded bright soliton*.

7.2. VK Stability of the Bright Soliton of the MVS of System (9)-(10)

Now let's apply the VK criterion to estimate the stability of the bright soliton (131). We should remember that the VK criterion cannot be used to study the stability of dark solitons.

As we have already mentioned in Secs. 3.2 and 4.2, the VK-stability of a bright soliton of the form (131) is defined by the sign of the derivative:

$$\frac{\mathrm{d}N_1}{\mathrm{d}q_1}, \tag{154}$$

where N_1 is defined as:

$$N_1 = \int_{-\infty}^{+\infty} |P|^2 \,\mathrm{d}t, \tag{155}$$

and consequently, substituting (131) in (155), we obtain:

$$N_1 = 2A_1^2 w. \tag{156}$$

Therefore, in order to be able to calculate the derivative (154), we will need expressions for A_1^2, A_2^2 and w in terms of the wavenumbers q_1 and q_2.

The Equation (150) already tells us which is the relation between A_2^2 and q_2. Therefore, we need a second equation that tells us which is the relation between A_1^2 and q_1.

If we consider the six equations that are obtained by setting equal to zero, the coefficients of $ch^n(\theta)$ which appear in Equations (133) and (134), we will note that q_1 only appears in the coefficient of $ch^{-1}(\theta)$ in Equation (133). And when we set equal to zero this coefficient we obtain Equation (149). Therefore, we have to use this equation to obtain the second equation that we need (*i.e.*, an equation relating A_1^2 and q_1). At first sight, it might seem that A_1^2 does not appear in Equation (149). However, A_1^2 is hidden in the w which appears in (149), since Equation (135) [or also Equation (136)], shows that w is a function of A_1^2 and A_2^2. Therefore, let us substitute the expression for w^{-4} that is given in Equation (135) in Equation (149). In this way we obtain the following equation:

$$\frac{\delta_1}{24}x^2 - \frac{23}{24}\delta_1\sigma^2 y^2 - \frac{\delta_1\sigma}{12}xy + \gamma_1\sigma y - q_1 + \frac{\alpha_1}{w^2} = 0\,, \tag{157}$$

where we have defined $x = A_1^2$ and $y = A_2^2$. This equation, in conjunction with Equation (150), will define a system of equations which permit us to obtain A_1^2 and A_2^2 as functions of q_1 and q_2. And substituting these functions $A_1^2(q_1, q_2)$ and $A_2^2(q_1, q_2)$ in Equation (135), we will also obtain an expression for w as a function of q_1 and q_2. Finally, substituting the functions $A_1^2(q_1, q_2)$ and $w(q_1, q_2)$ in Equation (156) we will obtain an expression for N_1 as a function of q_1 and q_2. And with this function we will able to calculate the sign of the derivative (154), which defines the VK-stability of the bright soliton given in Equation (131).

There is, however, a subtlety in this procedure. To obtain the Equation (157) we substituted the value of w^{-4} given by Equation (135) in the second term on the right-hand-side (rhs) of Equation (149), but we will use the exact value of w [given by Equation (152)] in the first term of this equation. If we introduced Equation (135) in the first term on the rhs of Equation (149) we would obtain a different equation instead of Equation (157). And the solution of this new equation, in combination with Equation (150), leads to eight different solutions for $A_1^2(q_1,q_2)$ and $A_2^2(q_1,q_2)$. These eight solutions are given by extraordinarily long expressions, *and none of them is the correct solution*. It is interesting to observe that even though the determination of the functions $A_1^2(q_1,q_2)$, $A_2^2(q_1,q_2)$ and $w(q_1,q_2)$ from Equations (135), (149) and (150) seems to be a simple, straightforward, algebraic problem, it is not so simple to understand why none of the eight solutions obtained by substituting Equation (135) in the two terms which contain w in Equation (149) gives us the correct expressions of A_1^2 and A_2^2 as functions of q_1 and q_2, while, on the other hand, the solutions $A_1^2(q_1,q_2)$ and $A_2^2(q_1,q_2)$, obtained from the Equations (150) and (157) [introducing the value of w given by Equation (152) in the last term of Equation (157)], *are the correct solutions*. The interested reader might find it worthwhile to examine this issue more closely.

Now let us calculate the VK-stability of the bright soliton (131) following the procedure described above, *i.e.*, calculating $A_1^2(q_1,q_2)$ and $A_2^2(q_1,q_2)$ with Equations (150) and (157), and then calculating $w(q_1,q_2)$ with Equation (135). We will consider the coefficients (38)-(42), which were the coefficients used to obtain the solutions shown in Equations (151)-(153). Using these coefficients, and following the procedure just described, we can obtain the function $N_1(q_1,q_2)$. And if we give to q_2 the exact value given in Equation (153), we find that the function $N_1(q_1,q_2=4.73876)$ has the form shown in **Figure 5**.

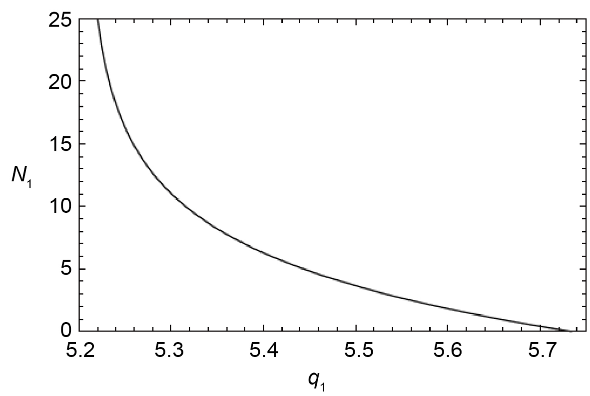

Figure 5. Integral N_1 corresponding to the bright soliton (131) [which enters in the mixed vector soliton (131)-(132)], as a function of the wavenumber q_1, using the coefficients shown in Equations (38)-(42).

As the slope of the function $N_1(q_1,q_2=4.73876)$ at $q_1=5.34666$ (the wavenumber of the exact bright soliton) is *negative*, it follows that the bright soliton (131) is VK-unstable [if we use the coefficients (38)-(42)].

Now let's investigate if the VK-stability of the bright soliton (131) changes if we use a different set of coefficients. In Sec. 3.2 we found that the VK-stability of the bright solitons changed when we used the coefficients (60)-(64), instead of the coefficients (38)-(42). Therefore, let us consider the coefficients (60)-(64). With these coefficients the solution of the Equations (138) and (142) is the following:

$$x = A_1^2 = 2.6997\,, \quad y = A_2^2 = 3.13781\,, \tag{158}$$

and substituting these values in Equation (135) the following value of w is obtained:

$$w = 2.08142\,. \tag{159}$$

Finally, substituting these values of A_1^2, A_2^2 and w in Equations (149) and (150), we obtain the following values of the wavenumbers:

$$q_1 = 12.0336\,, \quad q_2 = 10.2364\,. \tag{160}$$

As in the case of the bright soliton corresponding to the coefficients (38)-(42), in this case, with the coefficients (60)-(64), the positivity of q_1 implies that also in this case the bright soliton is *embedded*.

Now, let us find the form of the function $N_1(q_1, q_2 = 10.2364)$. To find this function we start by solving the Equations (150) and (157). In this way we obtain the functions $A_1^2(q_1, q_2)$ and $A_2^2(q_1, q_2)$, Then, substituting these functions in Equation (135) we obtain $w(q_1, q_2)$, and with these functions we can calculate $N_1(q_1, q_2) = 2A_1^2(q_1, q_2)\,w(q_1, q_2)$. Following this procedure, we find that the function $N_1(q_1, q_2 = 10.2364)$ has the form shown in **Figure 6**.

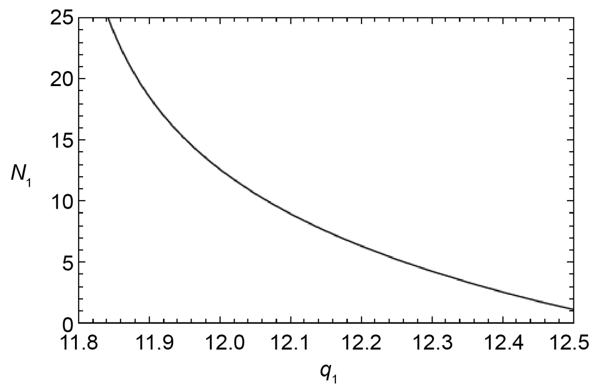

Figure 6. Integral N_1 corresponding to the bright soliton (131) [which enters in the mixed vector soliton (131)-(132)], as a function of the wavenumber q_1, using the coefficients shown in Equations (60)-(64).

The shape of this curve shows that also in this case we have $\mathrm{d}N_1/\mathrm{d}q_1 < 0$ at $q_1 = 12.0336$, thus implying that also with the coefficients given in Equations (60)-(64) the bright soliton given by (131) is VK-unstable. Therefore, in this case, when we have mixed vector solitons of the form (131)-(132), the change the coefficients

(38)-(42) by the new coefficients (60)-(64) does not change the VK-stability of the bright soliton given by Equation (131).

We have, thus, proved that *mixed vector solitons* (MVSs) do indeed exist. We have seen the NLS system (9)-(10) has MVSs of the form (131)-(132), when the coefficients have the values shown in (38)-(42) or (60)-(64).

8. Mixed Vector Solitons of the cmKdV System (69)-(70)

In this section we investigate if the cmKdV system (69)-(70) has mixed vector solitons (MVSs).

8.1. Existence of Mixed Vector Solitons in the System (69)-(70)

Let's show that the cmKdV system (69)-(70) has a continuous family of MVSs of the form:

$$u = A_1 \operatorname{sech}\left(\frac{t-az}{w}\right) \mathrm{e}^{i(q_1 z + r_1 t)}, \tag{161}$$

$$v = A_2 \tanh\left(\frac{t-az}{w}\right) \mathrm{e}^{i(q_2 z + r_2 t)}. \tag{162}$$

Substituting these two expressions into Equation (69) we obtain:

$$\begin{aligned} &\left(\frac{a}{w}+\frac{\varepsilon_1}{w^3}-\frac{3\varepsilon_1 r_1^2}{w}+\frac{\gamma_1\sigma A_2^2}{w}\right)\frac{sh(\phi)}{ch^2(\phi)} \\ &+i\left(q_1+\frac{3\varepsilon_1 r_1}{w^2}+\varepsilon_1 r_1^3-\frac{6\varepsilon_1 r_1}{w^2}-\gamma_1\sigma A_2^2 r_1\right)\frac{1}{ch(\phi)} \\ &+\left(-\frac{6\varepsilon_1}{w^3}+\frac{\gamma_1 A_1^2}{w}-\frac{\gamma_1\sigma A_2^2}{w}\right)\frac{sh(\phi)}{ch^4(\phi)} \\ &+i\left(\frac{6\varepsilon_1 r_1}{w^2}-\gamma_1 A_1^2 r_1+\gamma_1\sigma A_2^2 r_1\right)\frac{1}{ch^3(\phi)}=0, \end{aligned} \tag{163}$$

where $\phi = (t-az)/w$. For this equation to be satisfied it is necessary that each of the coefficients of the four terms with different hyperbolic functions be equal to zero. Therefore, setting equal to zero the coefficient of sh/ch^2, we find:

$$3\varepsilon_1 r_1^2 = a + \frac{\varepsilon_1}{w^2} + \gamma_1\sigma A_2^2. \tag{164}$$

In a similar way, setting equal to zero the coefficient of ch^{-1}, we find:

$$q_1 = -\varepsilon_1 r_1^3 + \frac{3\varepsilon_1 r_1}{w^2} + \gamma_1\sigma A_2^2 r_1. \tag{165}$$

And finally, equating to zero the coefficient of ch^{-3} (or that of sh/ch^4), we obtain:

$$\frac{1}{w^2} = \frac{1}{6\varepsilon_1}\left(\gamma_1 A_1^2 - \gamma_1\sigma A_2^2\right). \tag{166}$$

In a similar way, substituting (161)-(162) in Equation (70), the following equation is obtained:

$$\begin{aligned}&\left(-\frac{a}{w}+\frac{2\varepsilon_2}{w^3}+\frac{3\varepsilon_2 r_2^2}{w}-\frac{6\varepsilon_2}{w^3}-\frac{\gamma_2 A_2^2}{w}\right)\frac{1}{ch^2(\phi)}\\&+i\left(q_2+\varepsilon_2 r_2^3-\gamma_2 A_2^2 r_2\right)\frac{sh(\phi)}{ch(\phi)}\\&+\left(\frac{6\varepsilon_2}{w^3}+\frac{\gamma_2 A_2^2}{w}-\frac{\gamma_2\sigma A_1^2}{w}\right)\frac{1}{ch^4(\phi)}\\&+i\left(\frac{6\varepsilon_2 r_2}{w^2}+\gamma_2 A_2^2 r_2-\gamma_2\sigma A_1^2 r_2\right)\frac{sh(\phi)}{ch^3(\phi)}=0,\end{aligned}\tag{167}$$

and setting equal to zero the coefficients of the terms with different hyperbolic functions, we obtain the following three equations:

$$3\varepsilon_2 r_2^2=a+\frac{4\varepsilon_2}{w^2}+\gamma_2 A_2^2\tag{168}$$

$$q_2=\gamma_2 A_2^2 r_2-\varepsilon_2 r_2^3,\tag{169}$$

$$\frac{1}{w^2}=\frac{1}{6\varepsilon_2}\left(\gamma_2\sigma A_1^2-\gamma_2 A_2^2\right).\tag{170}$$

We can see that the Equations (166) and (170) constitute a system of two algebraic equations for A_1^2 and A_2^2. Solving this system we can obtain A_1^2 and A_2^2 in terms of ε_n, γ_n and w, as follows:

$$A_1^2=\frac{6}{\left(\sigma^2-1\right)w^2}\left(\sigma\frac{\varepsilon_2}{\gamma_2}-\frac{\varepsilon_1}{\gamma_1}\right),\tag{171}$$

$$A_2^2=\frac{6}{\left(\sigma^2-1\right)w^2}\left(\frac{\varepsilon_2}{\gamma_2}-\sigma\frac{\varepsilon_1}{\gamma_1}\right).\tag{172}$$

Now let us consider the particular case when $\sigma=2$. In this case we have:

$$A_1^2=\frac{2}{w^2}\left(2\frac{\varepsilon_2}{\gamma_2}-\frac{\varepsilon_1}{\gamma_1}\right),\tag{173}$$

$$A_2^2=\frac{2}{w^2}\left(\frac{\varepsilon_2}{\gamma_2}-2\frac{\varepsilon_1}{\gamma_1}\right),\tag{174}$$

and these expressions have an interesting implication. As A_1^2 and A_2^2 must be positive, (173) and (174) show that the cmKdV system (69)-(70) will only have MVSs if the coefficients ε_n and γ_n satisfy the condition:

$$\frac{\varepsilon_2}{\gamma_2}>2\frac{\varepsilon_1}{\gamma_1}.\tag{175}$$

If this condition is satisfied, then the cmKdV system (69)-(70) will have a continuous family of MVSs of the form (161)-(162), where the amplitudes are defined by Equations (173)-(174). Once with the values of A_1^2 and A_2^2, we can obtain r_1 and r_2 with Equations (164) and (168). And once with A_n^2 and r_n, the Equations (165) and (169) will give us the values of q_1 and q_2. This process defines a continuous family of MVSs because the values of a and w can be chosen arbitrarily (as in Sec. 4.1).

8.2. Embedding and VK Stability

The bright solitons that enter in the MVSs of the cmKdV system (69)-(70) have two characteristics which deserve to be emphasized: (*i*) they are *embedded solitons*, and (*ii*) all of them are VK-stable.

The *embedding* of the bright solitons of the MVSs of system (69)-(70) is evident, as the range of wavenumbers of the linear dispersion relation (LDR) of Equation (69) is the entire real axis [see Equation (78)]. Therefore, the wavenumber q_1 of a bright soliton (161) will *always* be contained in the range of wavenumbers permitted by the LDR of Equation (69), irrespective of the particular value that q_1 may have. Therefore, all the bright solitons that enter in the MVSs of system (69)-(70) are *embedded solitons*.

Now let us determine the VK stability of the bright soliton (161). As we have already seen in Sec. 4.2, if we define:

$$N = \int_{-\infty}^{+\infty} |u|^2 \, \mathrm{d}t, \tag{176}$$

and we substitute (161) in this equation, we obtain:

$$N = 2wA_1^2, \tag{177}$$

and then, the VK stability of the soliton (161) will be defined by the sign of the derivative:

$$\frac{\mathrm{d}N}{\mathrm{d}q_1} = \frac{\mathrm{d}}{\mathrm{d}q_1}\left(2wA_1^2\right). \tag{178}$$

To calculate this derivative we must remember that we have seen that the six equations (164)-(166) and (168)-(170) permit us to determine the values of six of the eight parameters that enter in the definition of the solitons (161)-(162). Mor,e precisely, these equations permit us to determine the values of A_1, A_2, q_1, q_2, r_1 and r_2, and we are free to choose the values of the parameters a and w, which are the only two parameters that appear in both solitons (161) and (162). Therefore, the value assigned to w is completely independent of the values of A_n, q_n and r_n. Due to this independence, we can calculate the derivative (178) in the following way:

$$\frac{\mathrm{d}N}{\mathrm{d}q_1} = 2w\frac{\mathrm{d}A_1^2}{\mathrm{d}q_1} = 2w\left(\frac{\mathrm{d}q_1}{\mathrm{d}A_1^2}\right)^{-1}. \tag{179}$$

Now, in order to calculate the derivative $\mathrm{d}q_1/\mathrm{d}A_1^2$, we need an equation which gives us q_1 *exclusively* as a function of A_1^2 [*i.e.*, not involving any of the other soliton parameters $\left(A_2^2, q_2, r_1, r_2\right)$, which are also functions of q_1]. Such an equation can be obtained from Equations (164)-(166), and the result is:

$$q_1 = \left(\frac{a}{3\varepsilon_1} - \frac{5}{3w^2} + \frac{\gamma_1}{3\varepsilon_1}A_1^2\right)^{1/2}\left(-\frac{a}{3} - \frac{4\varepsilon_1}{3w^2} + \frac{2}{3}\gamma_1 A_1^2\right). \tag{180}$$

From this function we can obtain the desired derivative:

$$\frac{\mathrm{d}q_1}{\mathrm{d}A_1^2} = \frac{\gamma_1}{6\varepsilon_1}\left(\frac{a}{3\varepsilon_1} - \frac{5}{3w^2} + \frac{\gamma_1}{3\varepsilon_1}A_1^2\right)^{-1/2}\left(-\frac{a}{3} - \frac{4\varepsilon_1}{3w^2} + \frac{2}{3}\gamma_1 A_1^2\right) + \frac{2\gamma_1}{3}\left(\frac{a}{3\varepsilon_1} - \frac{5}{3w^2} + \frac{\gamma_1}{3\varepsilon_1}A_1^2\right)^{1/2}. \tag{181}$$

This function defines the VK stability of the bright soliton (161). For example, if we choose $a = 2$, $w = \sqrt{8/5}$, and the coefficients $\sigma = 2$, $\varepsilon_1 = 0.5$, $\varepsilon_2 = 2$, $\gamma_1 = 6$ and $\gamma_2 = 8$ [which satisfy the condition (175)], then from Equation (171) we obtain $A_1^2 = 0.5208$, and Equation (181) gives us the value:

$$\frac{\mathrm{d}q_1}{\mathrm{d}A_1^2} = 7.462 > 0, \tag{182}$$

thus implying that in this case the soliton (161) is VK-stable. However, the Equation (181) permits us to obtain a far more general result, as we will see in the following.

Let us investigate when the bright soliton (161) will be VK-unstable. If we demand that the right-hand-side of Equation (181) gives a negative value, and we consider the case when $\varepsilon_1 > 0$, we arrive at the condition:

$$a + 2\gamma_1 A_1^2 < 8\frac{\varepsilon_1}{w^2}. \tag{183}$$

If we now substitute in this inequality the value of A_1^2 given in Equation (173) [valid when $\sigma = 2$], and we consider the case when $\gamma_1 > 0$, the condition (183) will be transformed into:

$$\frac{\varepsilon_2}{\gamma_2} < \frac{3}{2}\frac{\varepsilon_1}{\gamma_1} - \frac{aw^2}{8\gamma_1}, \tag{184}$$

and this condition *will never be satisfied* if the coefficients ε_n and γ_n satisfy the condition (175), and we have chosen positive values for γ_1 and the parameter *a*. Therefore, if $\sigma = 2$, $a > 0$, $\varepsilon_1 > 0$, $\gamma_1 > 0$, and we have coefficients ε_n and γ_n satisfying the condition (175), then the bright soliton (161) *will always be VK-stable*.

9. Summary and Conclusions

The principal purpose of this article has been to find the answer to the following question:

- Are there systems of nonlinear partial differential equations (NLPDEs) which have, among their solutions, vector solitons where each of the solitons is an *embedded soliton*?

This question can be phrased concisely as follows:

- Do embedded vector solitons (EVSs) exist?

And in this article we show that the answer to this question is:

- YES, EVSs do indeed exist, and we present *two different systems of NLPDEs* which have EVSs:
- the first one is a system of two cubic-quintic nonlinear Schrödinger (NLS) equations with higher-order dispersion [the system (9)-(10)],

- and the second one is a system of two complex modified Korteweg-de Vries (cmKdV) equations [the system (69)-(70)].

Then we show that these two systems of NLPDEs are indeed interesting, because, in addition of having EVSs, they also have *dark vector solitons* (DVSs), and *mixed vector solitons* (MVSs), where one of the solitons is a bright one, and the second one is a dark soliton. Therefore, we have found six types of new vector solitons: EVSs, DVSs and MVSs for the NLS system (9)-(10), and EVSs, DVSs and MVSs for the cmKdV system (69)-(70).

It is worth observing that the EVSs, the DVSs and the MVSs of the NLS system are always *isolated* solutions. On the contrary, in the case of the cmKdV system, there exist continuous families of EVSs, DVSs and MVSs.

It should be noted that the existence of each of the six types of vector solitons thus found, requires that the coefficients of the equations being considered satisfy certain conditions. These conditions are summarized in **Appendix 1**.

Moreover, we applied the Vakhitov-Kolokolov (VK) criterion of stability in order to determine the VK-stability of the bright solitons which appear in these vector solitons. The results concerning the VK criterion are summarized in **Appendix 2**.

To close this paper, we would like to emphasize that the NLS and cmKdV systems studied in this article [*i.e.*, systems (9)-(10) and (69)-(70)], are the only systems known to date which have *embedded vector solitons* (EVSs) among their solutions. And the precise analytical forms of these EVSs are presented in this paper. In future communications, it would be interesting to calculate numerical solutions of the systems (9)-(10) and (69)-(70) corresponding to initial conditions that are slightly different from exact EVSs. Such numerical solutions will show how perturbed EVSs behave. In particular, such results will show if the bright solitons that conform the EVSs of the NLS and the cmKdV systems, as well as the bright solitons that enter in the MVSs of these systems, emit monochromatic radiation when they are perturbed, as occurs with the embedded solitons (ESs) of the cubic-quintic NLS equation with fourth-order dispersion, and the ESs of the standard cmKdV equation. To obtain these numerical solutions we could replace the time derivatives that appear in these systems with finite difference approximations, and we may calculate the evolution along the z direction using a fourth-order Runge-Kutta algorithm.

As a final comment, we would like to mention that other types of nonlinear systems [different from the systems (9)-(10) and (69)-(70)] might also have EVSs. We are presently studying this issue.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix 1: Necessary Conditions

In this Appendix we summarize the conditions that the coefficients of the systems (9)-(10) and (69)-(70) must satisfy in order that the six types of EVSs found in this communication may indeed exist.

- The EVSs of system (9)-(10) that we found in Sec. 3 exist if the following two conditions are *satisfied*:

  (i) the coefficients of the system are positive,

  (*ii*) the coefficients satisfy Equation (35).

  If not all the coefficients are positive, EVSs might exist, but this case has not been investigated in this communication.
- The EVSs of system (69)-(70) only exist if the condition (77) is satisfied.
- The DVSs of system (9)-(10) only exist if the conditions (106) and (108) are satisfied.
- The DVSs of system (69)-(70) only exist if the conditions (126), (128), (129) and (130) are satisfied.
- The MVSs of system (9)-(10) only exist if the condition (139) is satisfied.
- The MVSs of system (69)-(70) only exist if the condition (175) is satisfied.

Appendix 2: Vakhitov-Kolokolov Criterion

The VK criterion of stability was used to estimate the stability of the bright embedded solitons that were found in the sections 3.2, 4.2, 7.2 and 8.2 of this communication. And the results found with the VK criterion were the following:

- In the case of the NLS system, the examples analyzed in Sec. 3.2 show that one of the solitons is VK-stable, and the other is VK-unstable.
- In the cmKdV system we found that the two bright solitons (71) and (72) turned out to be VK-stable in all the examples analyzed. This result is consistent with the fact that the bright solitons of the cmKdV equation are always stable.
- In the case of the MVSs of the NLS system, the bright solitons turned out to be VK-unstable in the examples analyzed.
- Concerning the bright solitons which enter in the MVSs of the cmKdV system, we found that if we choose $\varepsilon_1 > 0$, $\gamma_1 > 0$ and $a > 0$, *all the bright solitons* in these MVSs are *always* VK-stable (when $\sigma = 2$). No VK-unstable bright solitons exist in this case.