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Introduction to Interplanetary Travel
—Pocket Edition

Jan Vrbik

Mathematics Department, Brock University, St. Catharines, Canada
Email: jvrbik@brocku.ca

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Abstract

The main part of space travel within our Solar system consists of following an orbit determined, to a good approximation, by Sun's gravity only. In this article, we introduce a new technique for dealing with the resulting Keplerian motion, both elliptical and hyperbolic. This new method provides a simple way to establish parameters of the ensuing trajectory and compute the location and velocity of a spacecraft at any future time. Another essential feature of such travel, when visiting several planets in a single mission, is the utilization of each planet's gravity to increase and re-direct a spacecraft's velocity to enable it to reach the next, more distant planet. We derive and demonstrate all the relevant formulas required to understand the basic principles of designing such an interplanetary journey.

Keywords

Keplerian Orbits, KS Regularization, Transfer Orbits, Flyby Maneuver

1. Introduction

In this article, we concentrate on a single major component of space travel, namely utilizing Sun's gravitational field to effortlessly coast from one planet to another. Nevertheless, for readers' orientation, we first mention the remaining essentials (and some less important ingredients) of planetary missions.

Propulsion. Travelling on Earth, we are used to fuel usage being proportional to distance traveled. Things work very differently in space, where fuel is burnt and the ensuing gases are ejected only when a change in velocity is required. Each time this happens, the mass of the rocket decreases according to the following formula [1]:

$$|\Delta \boldsymbol{v}| = v_g \ln\left(\frac{m_a}{m_b}\right)$$

where $\Delta \boldsymbol{v}$ is the velocity change, v_g is the escaping gases' nozzle speed, and m_b and m_a denote the mass of the rocket before and after the completion of such a maneuver (the mass loss is due to the corresponding fuel expenditure). This implies that, for example, if a rocket goes through three velocity changes which reduce its mass by 15, 25, and 20 percent respectively, its final mass will be $0.85 \times 0.75 \times 0.80 = 0.51$ of the original one. Note that a typical value of v_g is 4.5 km/sec, while Earth's **escape velocity** [2] is 11.2 km/sec; this means that $\frac{m_a}{m_b} = \exp\left(\frac{11.2}{4.5}\right) \simeq 12$, *i.e.*, a rocket consumes (optimistically) over 91% of its original mass just to leave Earth.

There are several ways of minimizing fuel consumption: 1) by using the highest possible value of v_g with maximum thrust, 2) discarding empty fuel tanks, usually in stages, 3) imparting velocity changes in the direction of current velocity (whether to increase or decrease its magnitude), 4) designing and following an optimum trajectory to reach the mission's target. The first point results in minimizing the duration of each engine activation, allowing us to consider it (and the corresponding velocity change) to be practically *instantaneous.*

Navigation. It is essential to monitor the rocket's position and velocity throughout its travel. This is done by measuring distances to selected objects by establishing the time it takes for a signal (travelling at the speed of light) to be reflected back. Similarly, velocities in the line-of-sight are established by the Doppler effect observed in the same signal; velocities perpendicular to it may require triangulation (measuring distances and their change in time, to several objects). The rocket's attitude is established relative to the celestial sphere, changed by engine-induced rotation, and maintained with the help of a set of gyroscopes. A change in velocity is gauged by an accelerometer and imparted by firing the rocket's engines, in a specific direction and for a given time interval. All such maneuvers are executed by mission control and onboard computers.

Miscellaneous. There is a multitude of other factors affecting a rocket's motion beyond the Sun's gravity and occasional engine-induced velocity changes. Their list includes planets' gravitational forces, atmospheric drag, Earth's rotation and oblateness, Solar radiation pressure, etc. Worth mentioning (but not utilized by us) are **parking orbits** [3], which allow a rocket to orbit a planet before it is propelled back into interplanetary space.

We now proceed to study a rocket's motion under the influence of the Sun's gravity, while discarding all the other (usually minute) forces acting on it. Throughout the article, we use a technique of **patched conics** [4] which assumes that, to a good approximation, a trajectory of a rocket can be built by considering only the predominant gravitational field (be it the Sun's or of a nearby planet) acting on it at the moment. This breaks it into individual stretches (or patches) of (different) conic shapes; these are then stitched together to create the resulting complete path.

2. Linearized Kepler Problem

Our first objective is to formulate and solve the equation of motion of a rocket orbiting a massive **primary** by utilizing the Kustaanheimo-Stiefel transformation [5]. Since the resulting motion is planar, to describe it we introduce an inertial plane whose points are identified by their x and y coordinates combined into a single *complex* number $x+\mathrm{i}y$; we also assume that the primary is placed at its origin. Note that locations and velocities are then also represented by complex numbers, and their lengths (*i.e.*, distances and speeds) by the respective magnitudes. Also note that multiplying a complex number by $\exp(\mathrm{i}\psi)$ rotates (counter-clockwise) the corresponding vector by angle ψ around the origin.

We now need to solve the following second-order differential equation [5] for a two-coordinate vector (represented by a complex number) of the rocket's location (denoted by $\boldsymbol{r}$ and implicitly considered a function of time t)

$$\ddot{\boldsymbol{r}}+\mu\frac{\boldsymbol{r}}{r^3}=\boldsymbol{0} \tag{1}$$

where the two dots imply double differentiation with respect to t, r is the magnitude of $\boldsymbol{r}$, and μ is the gravitational constant multiplied by the primary's mass (its *gravitational mass*).

To find a solution to (1), we introduce a new (complex) *dependent* variable $\mathbb{U}$ and a new *independent* variable s called **modified time** (a dimensionless scalar) [6], related to the old variables in the following manner

$$\boldsymbol{r}=\mathbb{U}^2 \tag{2}$$

$$\frac{\mathrm{d}t}{\mathrm{d}s}=2r\sqrt{\frac{a}{\mu}} \tag{3}$$

where $a>0$ is a real and at this point arbitrary *constant*, having the dimension of length.

It is easy to see that

$$r:=|\boldsymbol{r}|=\mathbb{U}\bar{\mathbb{U}} \tag{4}$$

$$\boldsymbol{v}:=\dot{\boldsymbol{r}}=\frac{2\mathbb{U}\mathbb{U}'}{2r\sqrt{\frac{a}{\mu}}}=\frac{\mathbb{U}\mathbb{U}'}{r}\sqrt{\frac{\mu}{a}}=\frac{\mathbb{U}'}{\bar{\mathbb{U}}}\sqrt{\frac{\mu}{a}}$$

$$\begin{aligned}\ddot{\boldsymbol{r}}&=\left(\frac{(\mathbb{U}')^2+\mathbb{U}\mathbb{U}''}{r}-\frac{\mathbb{U}\mathbb{U}'\left(\mathbb{U}\bar{\mathbb{U}}'+\mathbb{U}'\bar{\mathbb{U}}\right)}{r^2}\right)\cdot\frac{\sqrt{\frac{\mu}{a}}}{2r\sqrt{\frac{a}{\mu}}}\\&=\left(\frac{(\mathbb{U}')^2+\mathbb{U}\mathbb{U}''}{r}-\frac{r(\mathbb{U}')^2+\mathbb{U}^2\mathbb{U}'\bar{\mathbb{U}}'}{r^2}\right)\cdot\frac{\mu}{2ar}\\&=\left(\frac{\mathbb{U}\mathbb{U}''}{r^2}-\frac{\mathbb{U}^2\mathbb{U}'\bar{\mathbb{U}}'}{r^3}\right)\cdot\frac{\mu}{2a}\end{aligned}$$

where $'$ indicates differentiation with respect to s, and a bar denotes complex *conjugation* (these two operations commute). The second line further implies that

$$v^2=\frac{\mu}{a}\frac{\mathbb{U}'\bar{\mathbb{U}}'}{r} \tag{5}$$

based on $v^2 = \boldsymbol{v}\overline{\boldsymbol{v}}$, where $v := |\boldsymbol{v}|$ (our notation for magnitude).

Adding $\mu \frac{\mathbb{U}^2}{r^3}$ to $\ddot{\boldsymbol{r}}$, multiplying each term by $\frac{2ar^2}{\mu \mathbb{U}}$ and making the resulting sum equal to zero is then equivalent to (1); the new equation thus reads

$$\mathbb{U}'' - \frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r} \cdot \mathbb{U} = 0 \tag{6}$$

It is easy to show that $\frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r}$ is a **constant of motion** [7], since its (modified-time) derivative is

$$\begin{aligned}&\frac{\mathbb{U}''\overline{\mathbb{U}}' + \mathbb{U}'\overline{\mathbb{U}}''}{r} - \frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r^2}\left(\mathbb{U}\overline{\mathbb{U}}' + \mathbb{U}'\overline{\mathbb{U}}\right)\\ &= \frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r} \cdot \frac{\mathbb{U}\overline{\mathbb{U}}' + \mathbb{U}'\overline{\mathbb{U}}}{r} - \frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r^2}\left(\mathbb{U}\overline{\mathbb{U}}' + \mathbb{U}'\overline{\mathbb{U}}\right) = 0\end{aligned}$$

after eliminating $\mathbb{U}''$ and $\overline{\mathbb{U}}''$ based on (6). Note that, due to (5),

$$\frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r} = \frac{2a}{\mu}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) \tag{7}$$

where the expression in parentheses is the total (kinetic and gravitational) energy of the rocket, denoted E (thus, also a constant of motion). Incidentally, solving $E = 0$ yields the following formula for the primary's *escape velocity* (to leave its gravitational field, starting at its surface):

$$V_e = \sqrt{\frac{2\mu}{R}} \tag{8}$$

where R is the primary's radius. *Proof*: E becomes zero in the $r \to \infty$ and $v \to 0$ limit (definition of "escape"); to have the *same* zero value at the primary's surface, v must equal to V_e.

To find a solution to (6), we need to consider three distinct possibilities, depending on whether the rocket's initial value of E is negative (resulting in an elliptical orbit), equals zero (parabolic orbit), or is positive (hyperbolic orbit).

2.1. Elliptical Case

Assuming that (7) is negative, we can simplify the solution to (6) by choosing a so that $\frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r}$ is equal to -1, *i.e.*,

$$a = \left(\frac{2}{r_0} - \frac{v_0^2}{\mu}\right)^{-1} \tag{9}$$

where v_0 and r_0 are the initial values of v and r, respectively.

The general solution to $\mathbb{U}'' + \mathbb{U} = 0$ is then given by

$$\mathbb{U} = \mathbb{A}\exp(\mathrm{i}s) - \mathbb{B}\exp(-\mathrm{i}s) \tag{10}$$

where $\mathbb{A}$ and $\mathbb{B}$ are *complex* numbers. To find their values, we need the rocket's initial position $\boldsymbol{r}_0$ and velocity $\boldsymbol{v}_0$. Using zero as the initial value of s, solving for these coefficients is then quite routine (note that $\boldsymbol{v}_0$ needs to be multiplied by the initial value of the RHS of (3) to convert it from ordinary velocity $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$ to $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = 2\mathbb{U}\mathbb{U}'$). The resulting equations then read

$$\mathbb{A} - \mathbb{B} = \sqrt{\boldsymbol{r}_0} \tag{11}$$

$$\mathbb{A} + \mathbb{B} = \frac{\boldsymbol{v}_0\sqrt{\boldsymbol{r}_0}}{\mathrm{i}}\sqrt{\frac{a}{\mu}} = -\mathrm{i}\frac{\boldsymbol{v}_0}{v_0}\sqrt{\boldsymbol{r}_0}\sqrt{\frac{2a}{r_0}-1}$$

where we have replaced $\sqrt{\frac{a}{\mu}}$ in (3) by $\frac{1}{v_0}\sqrt{\frac{2a}{r_0}-1}$, utilizing (7) being equal to -1.

The last two equations are easily solved for $\mathbb{A}$ and $\mathbb{B}$; note that we can choose either sign of $\sqrt{\boldsymbol{r}_0}$. Using (2) and the first two lines of (4), we can then compute the location and velocity of the rocket at any future value of s; to convert s into *actual* time t, we use

$$t = 2\sqrt{\frac{a}{\mu}}\left(as + \mathrm{Im}\left(\mathbb{A}\overline{\mathbb{B}}\left(1-\exp\left(2\mathrm{i}s\right)\right)\right)\right) \tag{12}$$

assuming $t_0 = 0$.

The technique is sufficient for the purpose of this article.

Orbital elements (optional) Nevertheless, to prove that the resulting trajectory is an **ellipse** [8], and to make contact with the usual description of this kind of motion, we express $\mathbb{A}$ in its *polar* form of $A\exp(-\mathrm{i}\alpha)$, and $\mathbb{B}$ similarly as $B\exp(\mathrm{i}\beta)$, so that (10) now reads

$$\begin{aligned}\mathbb{U} &= A\exp\left(\mathrm{i}\left(s-\alpha\right)\right) - B\exp\left(-\mathrm{i}\left(s-\beta\right)\right)\\ &= \exp\left(\mathrm{i}\frac{\psi}{2}\right)\left(A\exp\left(\mathrm{i}\left(s-s_p\right)\right) - B\exp\left(-\mathrm{i}\left(s-s_p\right)\right)\right)\end{aligned}$$

where $s_p := \frac{\alpha+\beta}{2}$ is the value of s at the trajectory's **pericenter** [1], and $\psi := \beta - \alpha$.

Note that $\frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r} = -1$ then yields

$$\begin{aligned}2a &= A^2 + B^2 + 2AB\cos\left(2\left(s-s_p\right)\right)\\ &\quad + A^2 + B^2 - 2AB\cos\left(2\left(s-s_p\right)\right)\\ &= 2\left(A^2+B^2\right)\end{aligned}$$

implying that $a = A^2 + B^2$.

For the rocket's location at any future value of s, we get

$$\begin{aligned}\boldsymbol{r} &= \exp\left(\mathrm{i}\psi\right)\left(A^2\exp\left(2\mathrm{i}\left(s-s_p\right)\right) + B^2\exp\left(-2\mathrm{i}\left(s-s_p\right)\right) - 2AB\right)\\ &= \exp\left(\mathrm{i}\psi\right)\left(A^2+B^2\right)\left(\cos 2\left(s-s_p\right) + \mathrm{i}\frac{A^2-B^2}{A^2+B^2}\sin 2\left(s-s_p\right) - \frac{2AB}{A^2+B^2}\right)\end{aligned} \tag{13}$$

This confirms that the actual path is an ellipse with the length of its **semi-major**

axis [1] equal to $a = A^2 + B^2$, its **eccentricity** equal to $\varepsilon := \frac{2AB}{A^2 + B^2}$ and one (the rightmost) of its foci at the origin (where the primary's center is) further rotated (around the origin) by angle ψ; these are the usual *orbital elements*.

Proof: ignoring the $\exp(\mathrm{i}\psi)$ rotation, the magnitude of the remaining expression (*i.e.*, distance to the origin) is $(A^2 + B^2)(1 - AB\cos 2(s - s_p))$, while the distance to the second focus (located at $-4AB$) is $(A^2 + B^2)(1 + AB\cos 2(s - s_p))$; the sum of the two distances is $2(A^2 + B^2)$, which is the usual way of describing an ellipse.

We can also compute the velocity of the rocket at the *modified* time s by

$$\boldsymbol{v} = \mathrm{i}\sqrt{\frac{\mu}{a}}\exp(i\psi)\frac{A^2\exp(2\mathrm{i}(s - s_p)) - B^2\exp(-2\mathrm{i}(s - s_p))}{A^2 + B^2 - 2AB\cos 2(s - s_p)}$$

To convert s to actual time t, we integrate (3), getting

$$\begin{aligned} t &= 2\sqrt{\frac{a}{\mu}}\int\left(A^2 + B^2 - 2AB\cos 2(s - s_o)\right)\mathrm{d}s \\ &= \sqrt{\frac{a^3}{\mu}}\left(2s - \varepsilon\sin 2(s - s_p) - \varepsilon\sin 2s_p\right) \end{aligned}$$

assuming the initial value of t is also set to zero.

Note that when s advances by π, the rocket returns to its original location; this implies that $T = 2\pi\sqrt{\frac{a^3}{\mu}}$ is the time it takes the rocket to complete a full orbit.

Also note that, when $B = 0$, the motion is circular, when $A = B$, it becomes *rectilinear* (running directly towards or away from the Sun), and when $B > A$, the rocket then orbits the primary in the *retrograde* (clockwise) direction.

2.2. Parabolic Case

When (7) is equal to zero, the value of a no longer affects the equation; we make it equal to 1/2 to simplify the solution. Solving $\mathbb{U}'' = 0$ results in

$$\mathbb{U} = \mathbb{A} + \mathbb{B}s$$

Starting at a specific initial location $\boldsymbol{r}_0$ requires that $\mathbb{A} =: A\exp(-\mathrm{i}\alpha) = \sqrt{\boldsymbol{r}_0}$; to make the initial velocity equal to $\boldsymbol{v}_0$, we also need

$$\frac{\mathbb{U}\mathbb{U}'}{r}\sqrt{\frac{\mu}{a}} = \frac{\mathbb{B}}{A\exp(\mathrm{i}\alpha)}\sqrt{2\mu} = \boldsymbol{v}_0$$

Since (7) is equal to 0, the magnitude of $\boldsymbol{v}_0$ must equal to $A\sqrt{2\mu}$, implying that $|\mathbb{B}| = 1$. By choosing $\mathbb{B} := \mathrm{i}\exp(\mathrm{i}\beta)$, the previous equation can be easily solved for β.

Orbital elements (optional) The solution can be expressed in the following form

$$\mathbb{U} = \exp(\mathrm{i}\beta)\left(A\cos(\alpha + \beta) + \mathrm{i}(s - s_v)\right)$$

where $s_v = A\sin(\alpha + \beta)$ is the value of s at the closest (to primary) approach (*i.e.*, the path's **vertex**).

The resulting trajectory is then

$$\boldsymbol{r} = \exp(2\mathrm{i}\beta)\left(A^2\cos^2(\alpha+\beta) + 2A\mathrm{i}(s-s_v)\cos(\alpha+\beta) - (s-s_v)^2\right)$$

which is a **parabola** [9] with its focus at the origin and its vertex at $A^2\sin^2(\alpha+\beta)$, further rotated by 2β. To prove this, we show that the distance to the origin, namely

$$r = (s-s_v)^2 + A^2\cos^2(\alpha+\beta)$$

is the same as the pre-rotation distance from $\boldsymbol{r}$ to the **directrix**, namely a vertical line with its x coordinate equal to $2A^2\cos^2(\alpha+\beta)$, as can be easily verified.

This time

$$t = \sqrt{\frac{2}{\mu}}\left(\frac{s^3}{3} - s_v s^2 + \left(A^2\cos^2(\alpha+\beta) + s_v^2\right)s\right)$$

When $\cos(\alpha+\beta) < 0$, the motion is retrograde.

2.3. Hyperbolic Case

Finally, when (7) is positive, we choose the value of a so that $\dfrac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r} = 1$ and (6) thus reads $\mathbb{U}'' - \mathbb{U} = 0$; its general solution is

$$\mathbb{U} = \mathbb{A}\exp(s) + \mathbb{B}\exp(-s)$$

where

$$\mathbb{A} + \mathbb{B} = \sqrt{\boldsymbol{r}_0}$$

$$\mathbb{A} - \mathbb{B} = \frac{\boldsymbol{v}_0 r_0}{\sqrt{\boldsymbol{r}_0}}\sqrt{\frac{a}{\mu}} = \frac{\boldsymbol{v}_0}{v_0}\overline{\sqrt{\boldsymbol{r}_0}}\sqrt{\frac{2a}{r_0}+1}$$

with

$$a = \frac{\mu}{2E_0}$$

based on the initial value of (7).

Using (2) and (4), we can then easily compute the location and velocity of the rocket at any future value of s; the following expression converts this value into the corresponding real time t (assuming that its initial value is also 0)

$$t = \sqrt{\frac{a}{\mu}}\left(|\mathbb{A}|^2(\exp(2s)-1) - |\mathbb{B}|^2(\exp(-2s)-1) + 2\left(\overline{\mathbb{A}}\mathbb{B} + \mathbb{A}\overline{\mathbb{B}}\right)s\right)$$

Orbital elements (optional) To show that the resulting trajectory is one branch of a **hyperbola** [10], we introduce $\mathbb{A} := A\exp(\mathrm{i}\alpha)$ and $\mathbb{B} := B\exp(-\mathrm{i}\beta)$; this enables us to write

$$\mathbb{U} = \sqrt{AB}\exp\left(\mathrm{i}\frac{\psi}{2}\right)\left(\exp\left(s - s_v + \mathrm{i}\frac{\gamma}{2}\right) + \exp\left(-s + s_v - \mathrm{i}\frac{\gamma}{2}\right)\right) \tag{14}$$

where $\gamma = \alpha+\beta$, $\psi = \alpha-\beta$ and $s_v = \ln\sqrt{\frac{B}{A}}$ is the value of s at the hyperbola's **vertex**.

Based on $\frac{\mathbb{U}'\overline{\mathbb{U}}'-2a}{r}=1$, we get

$$\begin{aligned}2a &= AB\left(\exp\left(2\left(s-s_v\right)\right)+\exp\left(-2\left(s-s_v\right)\right)-2\cos\gamma\right)\\ &\quad -AB\left(\exp\left(2\left(s-s_v\right)\right)+\exp\left(-2\left(s-s_v\right)\right)+2\cos\gamma\right)\\ &= -4AB\cos\gamma\end{aligned}$$

implying that $a=-AB\cos\gamma$.

(14) further implies that

$$\boldsymbol{r}=\exp\left(\mathrm{i}\psi\right)AB\left(2+\exp\left(\mathrm{i}\gamma\right)\mathrm{e}^{2\left(s-s_v\right)}+\exp\left(-\mathrm{i}\gamma\right)\mathrm{e}^{-2\left(s-s_v\right)}\right)$$

which represents the left branch of a hyperbola with its focus at the origin, its vertex at $2AB\left(1+\cos\gamma\right)$, its center at $2AB$, and two asymptotes going through the center at angles $\pm\gamma$, further rotated by ψ. This can be shown by comparing $r=AB\left(\exp\left(2\left(s-s_v\right)\right)+\exp\left(-2\left(s-s_v\right)\right)+2\cos\gamma\right)$ with the pre-rotational distance of $\boldsymbol{r}$ to the *second* focus at $4AB$, which turns out to be $AB\left(\exp\left(2\left(s-s_v\right)\right)+\exp\left(-2\left(s-s_v\right)\right)-2\cos\gamma\right)$; the constant difference, namely $-4AB\cos\gamma$, then yields $2a$ (where a now represents the apex-to-center distance). Note that $\cos\gamma<0$ as γ must be within the $\pi/2$ and π range; the lower limit results in only a small deviation from the rocket's original direction, while, at the upper limit, its direction is nearly reversed. When $\pi<\gamma<3\pi/2$, the motion becomes retrograde (note that γ is measured from the $+x$ axis).

3. Transfer Orbits

Let us consider two planets, each moving in a *circular* orbit around a central Sun of gravitational mass μ, both of their orbits being in the *same* plane (this simplification is sufficient to demonstrate the main features of space travel). We want to study the motion of a rocket sent from the mother planet (Earth) to land on the target planet (we will use Mars as its generic name). Leaving Earth requires attaining the corresponding escape velocity (a major part of fuel consumption) to enable the rocket to break free from (and further ignore) Earth's gravity, but not gain any additional speed; after that, the rocket needs an extra (practically instantaneous) velocity boost to put it in a **transfer orbit** [1], capable of reaching the **target** orbit of Mars. Once done, the rocket keeps on *coasting* under the influence of the Sun's gravity until it crosses Mars' orbit. It then needs to modify its velocity again to match Mars' velocity at the point of this encounter (assuming that leaving Earth has been carefully *synchronized* with the motion of Mars to achieve such a rendezvous). Once the velocities match, the rocket can execute a landing maneuver, with a fuel cost given by Mars' escape velocity. Knowing that total fuel consumption is determined by the sum of all required velocity changes enables us to compare different transfer orbits in terms of their fuel efficiency; this is done in a subsequent example.

3.1. Elliptical Case

It can be shown that, to maximize an increase in total energy and thus enable it to reach higher planetary orbits, the *extra* velocity imparted to the rocket upon leaving Earth must be in the direction of the Earth's own motion; the resulting orbit is called **tangential**. Denoting the total (Earth-inherited plus the extra boost) speed by V, the radius of Earth's orbit by a_0 and the radius of the target orbit by a_1 (do not forget: both are assumed to be circular), our first task is to establish when and at what point will the rocket reach the target orbit.

Placing the Sun at the origin of our coordinates and orienting the x axis so that the rocket's initial location is a_0 (real and positive), and its initial velocity $\boldsymbol{v}_0$ equals $\mathrm{i}V$ (thus *tangential* to Earth's orbit), we can routinely compute the *transfer* orbit's a value, based on (9), by

$$a = \left(\frac{2}{a_0} - \frac{V^2}{\mu}\right)^{-1}$$

and construct the corresponding solution $\mathbb{U}$. In this section, we assume that $V < \sqrt{\frac{2\mu}{a_0}}$, making the orbit *elliptical*; for larger velocities, the transfer orbit becomes *hyperbolic* (to be dealt with next). We also need to verify that $a \geq \frac{a_0 + a_2}{2}$, otherwise, the resulting trajectory never reaches the target orbit.

Deriving the next set of formulas is quite routine; we skip the details.

First, we get

$$\mathbb{U} = \sqrt{a_0}\cos s + \mathrm{i}\sqrt{2a - a_0}\sin s$$

implying that the rocket's distance from the Sun is

$$r = a - (a - a_0)\cos 2s$$

The target orbit is reached as soon as r becomes equal to a_1, yielding the following solution for the corresponding value of s (denoted s_1)

$$\cos 2s_1 = \frac{a - a_1}{a - a_0}$$

We already know that this solution exists only when $a \geq \frac{a_0 + a_1}{2}$; in the case of exact equality, the solution yields $2s_1 = \pi$ and the target orbit is also reached *tangentially* (the corresponding transfer orbit is then called **cotangential**). For this orbit, the initial value V needs to be equal to

$$\sqrt{\frac{2a_1\mu}{a_0(a_0 + a_1)}} \tag{15}$$

while the target orbit is approached with the final speed of

$$\sqrt{\frac{2a_0\mu}{a_1(a_0 + a_1)}} \tag{16}$$

We now return to a *general* solution: the *location* of where the transfer and target orbits intersect is clearly at distance a_1 from the Sun; we thus need only the corresponding polar-coordinate *angle* (denoted θ). It is found based on the *real* component of $\boldsymbol{r} = \mathbb{U}^2$ evaluated at $s = s_1$, which turns out to be

$$a_0 - \frac{a_1 - a_0}{a - a_0} a$$

The last expression, divided by a_1 provides the value of $\cos\theta$, and consequently of θ.

To establish the rocket's velocity at this point, we evaluate

$$\boldsymbol{v} = \sqrt{\frac{\mu}{a}} \frac{\mathbb{U}\mathbb{U}'}{r}$$

at modified time s_1, thus getting

$$\boldsymbol{v}_1 := \sqrt{\frac{\mu}{a}} \left(-\frac{a}{a_1} \sin 2s_1 + \mathrm{i} \frac{\sqrt{(2a - a_0) a_0}}{a_1} \cos 2s_1 \right)$$

And finally, the *real* time of the same event (setting the *initial* value of time to 0) is given by

$$t = \sqrt{\frac{a}{\mu}} \left(2as_1 - (a - a_0) \sin 2s_1 \right)$$

Note that the velocity of the target planet (to be encountered at that location) is

$$\boldsymbol{v}_2 := \mathrm{i} \sqrt{\frac{\mu}{a_1}} \exp(\mathrm{i}\theta)$$

As mentioned already, the rocket now needs to use its engines to reduce the difference between the last two velocities to zero before it can land safely on the target planet.

Example The value of Sun's gravitational mass μ is, to a sufficient approximation, equal to 1.33 × 10^{20} meter^3/second^2; to simplify our notation, we make 10^{10} meters our unit of distance, and 10^6 seconds (about 11.6 days) our unit of time, implying that 10 km/sec is our velocity unit.

Let us now explore transfer orbits taking a rocket from Earth to Mars; the Sun's μ then equals 133 units, and the distance of Earth (Mars) from the Sun is 15 (22.5) units (the final reminder: both orbits are assumed to be circular and in the same plane). The two escape velocities (to leave Earth and to land on Mars; both are major contributors to fuel consumption) are 1.12 and 0.50 units, respectively (same for all transfer orbits). For simplicity (and maximum efficiency), we keep the initial velocity $\boldsymbol{v}_0$ (after adding an extra boost $\Delta\boldsymbol{v}_0$ to its Earth-inherited value) *tangential* to the Earth's orbit.

In the following table, we vary the a value of the (always elliptic, by our choice) transfer orbit, from the smallest (cotangential), up to its $a \to \infty$ limit (effectively parabolic), while quoting the corresponding magnitude of the change in velocity needed at each end of the journey (*after* escaping Earth's gravity and *before* landing on Mars), the total travel time t, and the location of the actual arrival (in terms of θ).

$\frac{1}{a}$	$\frac{2}{a_0+a_1}$	$\frac{1.5}{a_0+a_1}$	$\frac{1}{a_0+a_1}$	$\frac{0.5}{a_0+a_1}$	$\frac{0}{a_0+a_1}$
$\lvert\Delta v_0\rvert$	0.284	0.546	0.789	1.02	1.23
$\lvert v_1-v_2\rvert$	0.257	0.996	1.41	1.73	2.02
t	22.12	10.09	7.84	6.65	5.88
θ	180°	100°	84°	76°	71°

Due to a substantial increase in fuel consumption (an exponential function of velocity changes), only solutions close to cotangential (the first column) are practical.

3.2. Hyperbolic Case

Going (more quickly) over the same steps when $V > \sqrt{\frac{2\mu}{a_0}}$, we get

$$a = \left(\frac{V^2}{\mu} - \frac{2}{a_0}\right)^{-1}$$

where a is the orbit's center-to-apex distance, and

$$\mathbb{U} = \sqrt{a_0}\cosh s + \mathrm{i}\sqrt{2a+a_0}\sinh s$$

implying

$$r = -a + (a+a_0)\cosh 2s$$

Setting $r = a_1$ and solving for $2s_1$, we get

$$\cosh 2s_1 = \frac{a+a_1}{a+a_0}$$

resulting in an angle θ of the point of encounter, established from

$$\cos\theta = \frac{a_0}{a_1} - \frac{a}{a_1}\left(\frac{a_1-a_0}{a+a_0}\right)$$

and the corresponding rocket's velocity of

$$\boldsymbol{v}_1 = \sqrt{\frac{\mu}{a}}\left(-\frac{a}{a_1}\sinh 2s_1 + \mathrm{i}\frac{\sqrt{(2a+a_0)a_0}}{a_1}\cosh 2s_1\right)$$

The *time* of the encounter is

$$t = \sqrt{\frac{a}{\mu}}\left(-2as_1 + (a+a_0)\sinh 2s_1\right)$$

while the velocity of the target planet (being in that location at that time) is given by

$$\boldsymbol{v}_2 = \mathrm{i}\sqrt{\frac{\mu}{a_1}}\exp(\mathrm{i}\theta)$$

4. Flyby Orbits

These are used by a rocket to closely approach a planet (or several planets in succession), utilizing each such visit to increase the rocket's speed and (equally importantly) to re-direct its trajectory, so that the rocket can not only reach the next, more distant orbit, but also find the corresponding planet in the right location.

To explore details of each such near encounter, we first realize that, for the duration of the **flyby** maneuver (as these are called), the planet's gravitational pull becomes so predominant that even the Sun's gravity may be ignored. What cannot be ignored is the planet's own motion around the Sun, which, during the same, relatively *brief* interval, can be considered *uniform* [11]. We already know how to describe the rocket's (now always *hyperbolic*) trajectory in an inertial coordinate frame *moving* with the planet (located at the origin of this new frame): the rocket's direction is deflected (left or right) by an angle that can vary from zero to its largest possible value, while its final speed is the same as the initial one. The *maximum* deflection is established based on the planet's gravitational mass μ_p, the rocket's *relative* (*i.e.* in the new frame) speed of approach (denoted V_r) and the planet's radius R, based on the following argument: clearly, the distance of the hyperbola's apex from the planet's center, already shown to be $\frac{-a}{\cos\gamma}(1+\cos\gamma)$, must be bigger than R, where

$$a=\frac{\mu_p}{V_r^2} \tag{17}$$

which results from making (7) equal to 1 and r equal to ∞ (visualize the rocket approaching the planet from *far away* at speed V_r). This implies that, to get the upper limit (denoted γ_m) for the γ value, we must solve

$$-\frac{\mu_p}{V_r^2}\left(\frac{1}{\cos\gamma_m}+1\right)=R$$

Since the planet's escape velocity is $V_e=\sqrt{\frac{2\mu_p}{R}}$ due to (8), the solution can be simplified to

$$\cos\gamma_m=-\left(1+\frac{2V_r^2}{V_e^2}\right)^{-1} \tag{18}$$

while the resulting maximum *deflection* becomes $\phi_m=2\gamma_m-\pi$. Within the permissible range of ϕ values, specified by $\phi<|2\gamma_m-\pi|$, we must then find a ϕ that will facilitate a synchronized encounter with the next planet (see the next example).

Let us recall that, to achieve a *specific* value of ϕ, the rocket must align its direction (before the planet's gravity takes its effect) with the corresponding asymptote, *i.e.*, a straight line whose closest approach to the planet's center is given by the following **impact parameter**.

$$-\frac{\mu_p}{2V_r^2}\cdot\frac{\sin 2\gamma}{1+\cos\gamma}$$

where $\gamma=\frac{\phi+\pi}{2}$ (this follows from (17) and basic geometry of a hyperbola). Having done that, the planet's gravity then automatically (without *any* fuel expenditure) starts bending the rocket's path to eventually complete the desired hyperbolic maneuver.

Up to this point, all quantities have been measured relative to the new, moving frame (something quite natural for the rocket to do, as the moving planet becomes its major point of reference, which *appears* to be stationary). Furthermore, seen as a part of a whole interplanetary journey, the flyby maneuver may be considered to be just an instantaneous change of the rocket's velocity (both in speed and direction), while ignoring the inner details just described [11]. The only formula needed is to find the rocket's new velocity $\boldsymbol{v}_0^{\mathrm{n}}$ in terms of its old velocity $\boldsymbol{v}_1$, the deflection angle ϕ, and the planet's velocity $\boldsymbol{v}_2$ (all expressed in the original heliocentric frame of reference). This is achieved by summarizing what has just been shown to happen in the moving reference frame, namely

$$\boldsymbol{v}_0^{\mathrm{n}}-\boldsymbol{v}_2=\left(\boldsymbol{v}_1-\boldsymbol{v}_2\right)\exp\left(\mathrm{i}\phi\right) \tag{19}$$

where $\boldsymbol{v}_1-\boldsymbol{v}_2$ and $\boldsymbol{v}_0^{\mathrm{n}}-\boldsymbol{v}_2$ are the rocket's in and out velocities (both having the same magnitude); $\boldsymbol{v}_0^{\mathrm{n}}$ is then used as the initial velocity of the next leg of the journey.

Example. This time we consider a rocket launched, using a *cotangential* transfer orbit, to first visit Jupiter; we then show how to re-direct its motion (using a Jupiter's flyby) to commence a new transfer orbit, capable of reaching Saturn. We use a frame of reference whose $+x$ axis is in the direction of Jupiter at the rocket's flyby. We already have a complete solution for the Earth-Jupiter leg of the journey, specifically: we know that the final (upon reaching Jupiter) location of the rocket is 77.8 and the corresponding velocity, based on (16), is $\boldsymbol{v}_1=0.74\mathrm{i}$, while Jupiter's velocity at the point of this encounter is $\boldsymbol{v}_2=1.31\mathrm{i}$, based on Jupiter's orbital speed of $\sqrt{\frac{133}{77.8}}$. Substituting the magnitude of the difference, namely $V_r=0.57$, and Jupiter's escape velocity of 6 into (18) tells us that the maximum value of $|\phi|$ is 152 degrees. Using several admissible values of ϕ, we then use (19) to compute the initial velocity $\boldsymbol{v}_0^{\mathrm{n}}$ of the Jupiter-Saturn leg (the initial location stays equal to 77.8), find the corresponding transfer orbit and compute its duration t, the location θ of the (potential) encounter with Saturn, and the rocket's velocity $\boldsymbol{v}_1^{\mathrm{n}}$ at that point.

We know that, when $\left|\boldsymbol{v}_0^{\mathrm{n}}\right|<\sqrt{\frac{2\cdot 133}{77.8}}$ (which, incidentally, happens always to be the case in this example—hyperbolic transfer orbits are impossible), the transfer orbit is elliptical; failing to solve the $r=143$ (the last number being the Sun-to-Saturn distance) equation, which happens when $|\phi|<92^\circ$, indicates that $\left|\boldsymbol{v}_0^{\mathrm{n}}\right|$ is insufficient to reach Saturn.

$\phi =$	-150°	-140°	-120°	-100°	…	100°	120°	140°	150°
v_0^n	-0.29 $+1.8i$	-0.37 $+1.7i$	-0.49 $+1.6i$	-0.56 $+1.4i$	…	0.56 $+1.4i$	0.49 $+1.6i$	0.37 $+1.7i$	0.29 $+1.8i$
t	115	126	160	224	…	125	96	89	115
θ	106°	117°	150°	150°	…	68°	63°	66°	69°
v_1^n	-1.2 $+0.59i$	-1.2 $+0.32i$	-1.0 $-0.40i$	0.04 $-0.86i$	…	-0.56 $+0.65i$	-0.46 $+1.0i$	-0.53 $+1.2i$	-0.60 $+1.2i$

Note that when ϕ is negative, the rocket's counter-clockwise motion is deflected to the left (*i.e.*, towards the Sun) by this flyby; the opposite is true when ϕ is positive.

The remaining critical task is to determine *which* of these ϕ values allows the rocket to meet Saturn (whose motion is known) in the location θ at time t. This is achieved by utilizing the well-known solution to **Lambert problem** [12], or obtained directly from our table (some of its refinement and subsequent interpolation may be required for sufficient accuracy); we will not do it here.

But, since our table's range of θ values (and the corresponding travel times t) covers only *parts* of Saturn's orbit, finding such ϕ may not always be possible (unless willing to wait for the right moment to start the mission; this may take several years). Nevertheless, *not* insisting on a cotangential transfer orbit to get from Earth to Jupiter provides the extra flexibility (at moderate fuel cost) to substantially extend the range of achievable θ and t values and thus eliminate this problem. That arranging for such consecutive rendezvous of several planets is possible (e.g., after Jupiter and Saturn to continue, in the same flyby manner, to visit Uranus and Neptune) was demonstrated by the journey of Voyager 2 [13].

Note that when such a mission is properly planned and executed, fuel expenditure is needed only to leave Earth; nevertheless, due to various errors, occasional small corrections to the rocket's velocity (instantaneously correcting its location is impossible) are always required.

5. Conclusions

In this article, we have concentrated on the cruising part of interplanetary travel, ignoring less important forces and consequently, without much emphasis on the high accuracy of the results. This means that to plan an actual mission, the techniques we have used here can help only to create an initial road map, but the final trajectory has to be computed using more sophisticated numerical methods, capable of considering several forces acting on the rocket at the same time, and accommodating orbits' eccentricities and inclinations.

In this context, it also needs to be recognized that following an exact schedule is necessarily frustrated by small errors in propulsion and other causes. This implies that the rocket's location and velocity must be frequently monitored and corrected whenever a deviation from the originally planned trajectory is detected. This necessitates a quick computation and immediate implementation of the required (usually minute) velocity change, as well as the corresponding readjustment of the journey's schedule. The computation's accuracy must to be *only* on par with the expected errors in its execution.

We acknowledge that to fully understand all components of a successful space mission requires a lot more than what has been covered by this article. Nevertheless, it is hoped that we have been able to elucidate its most important feature, namely, the utilization of classical Keplerian orbits.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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