

Weighted Estimates for Marcinkiewicz Integral Operator Associated with Schrödinger Operator and Its Commutator on Generalized Fractional Morrey Spaces

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Abstract

The aim of this paper is to investigate the boundedness of a Marcinkiewicz integral operator μ_j^c associated with Schrödinger operator and its commutator on weighted generalized fractional Morrey spaces $L^{p,\eta,\varphi}(\mathbb{R}^n, \omega)$. Under assumption that the function φ satisfies doubling conditions, the author proves that the μ_j^c is bounded on spaces $L^{p,\eta,\varphi}(\mathbb{R}^n, \omega)$, where $0 < \eta < \frac{n}{m}$, $\frac{n}{m\eta} < p < \frac{n}{\eta}$, and $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$. Furthermore, the boundedness of the commutator $[b, \mu_j^c]$ formed by $b \in \text{BMO}(\mathbb{R}^n)$ and the μ_j^c on spaces $L^{p,\eta,\varphi}(\mathbb{R}^n, \omega)$ is also obtained.

Keywords

Marcinkiewicz Integral, Schrödinger Operator, Generalize Fractional Morrey Space, Weighted Boundedness, Commutator

1. Introduction

In 2014, Chen and Zhou investigated the boundedness of $[b, \mu_j^c]$ on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces when $b \in \text{BMO}(\sigma)$ in [1]. In 2023, Zhang *et al.* established the boundedness of μ_j^c on $L^p(\omega)$ spaces and further derived corresponding results for the commutator $[b, \mu_j^c]$ in [2], where $b \in \text{BMO}(\sigma)$ and $\omega \in A_p^\rho$. In 2020, Xuan *et al.* introduced weighted generalized Morrey-type spaces associated with Schrödinger operators and established the boundedness of singular in-

tegrals in [3]. More researches about the Marcinkiewicz integral operators associated with Schrödinger operators on various function spaces can be seen in [4]-[7]. Building on these results, this paper primarily investigates the weighted boundedness of Marcinkiewicz integral and its commutators associated with Schrödinger operators on the generalized fractional Morrey spaces. Before stating the organization of this paper, we first recall some necessary notation.

The following definition of Laplacian operator is defined by

$$\mathcal{L} = -\Delta + V(x), \quad x \in \mathbb{R}^n, \quad n \geq 3.$$

V is a non-negative potential, and belongs to the inverse Hölder class RH_s . For $s \geq \frac{n}{2}$, there exists a constant C such that for an arbitrary sphere $B \subset \mathbb{R}^n$. We have

$$\left(\frac{1}{|B|} \int_B V(y)^s \, dy \right)^{\frac{1}{s}} \leq \frac{C}{|B|} \int_B V(y) \, dy. \tag{1}$$

Let $V \in RH_s$, for any $x \in \mathbb{R}^n$, the critical radius function $\rho(x)$ is dependent on V as follows:

$$\rho(x) := \sup_{r>0} \left\{ r : \frac{1}{n-2} \int_{B(x,r)} V(y) \, dy \leq 1 \right\}. \tag{2}$$

The Marcinkiewicz integral $\mu_j^{\mathcal{L}}$ associated with the Schrödinger operator \mathcal{L} is defined as

$$\mu_j^{\mathcal{L}}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq h} \mathcal{K}_j^{\mathcal{L}}(x, y) f(y) \, dy \right|^2 \frac{dh}{h^3} \right)^{\frac{1}{2}}, \tag{3}$$

where $\mathcal{K}_j^{\mathcal{L}}(x, y) = \widetilde{K}_j^{\mathcal{L}}(x, y)|x - y|$ is the kernel of $\mathcal{R}_j^{\mathcal{L}} = \frac{\partial}{\partial x_j} \mathcal{L}^{\frac{1}{2}}$ for $1 \leq j \leq n$.

Let $V = 0$ and $\widetilde{K}_j^{\mathcal{L}}(x, y)$ be the kernel of classical Riesz transform

$\mathcal{R}_j = \frac{\partial}{\partial x_j} \Delta^{-\frac{1}{2}}$. Then

$$\mathcal{K}_j^{\Delta}(x, y) = \widetilde{K}_j^{\Delta}(x, y)|x - y| = \frac{(x_j - y_j)|x - y|}{|x - y|^{n-1}}.$$

The following definition of spaces $BMO_{\theta, \psi}$ is from [8].

Definition 1.1 *In this article, we define*

$$\psi_\theta(B) := \left(1 + \frac{r}{\rho(x)} \right)^\theta, \quad \theta > 0,$$

where B represents a sphere with x as its center and r as its radius.

A new class of BMO spaces is introduced in the literature, that is given a local function b , and

$$\|b\|_{BMO_\theta(\rho)} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\psi_\theta(B)|B|} \int_B |b(y) - b_B| \, dy < \infty,$$

where

$$b_B = \frac{1}{|B|} \int_B f(y) dy.$$

The commutator generated by μ_j^C and given function $b \in \text{BMO}_\theta(\rho)$ is defined by

$$[b, \mu_j^C](f)(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq h} \mathcal{K}_j^C(x, y) (b(y) - b_B) f(y) dy \right|^2 \frac{dh}{h^3} \right)^{\frac{1}{2}}. \quad (4)$$

Let $1 < p < \infty$, weighted function $\omega(x) \in A_p(\mathbb{R}^n)$ refers to an arbitrary sphere B in \mathbb{R}^n . There exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{\frac{1}{p-1}} dx \right)^{p-1} \leq C.$$

Weighted function $\omega(x) \in RH_m(\mathbb{R}^n)$ is refer to the arbitrary sphere B in \mathbb{R}^n . There exists a constant $C > 0$ and $m > 1$ such that the following reverse Hölder inequation sets up

$$\left(\frac{1}{|B|} \int_B \omega(x)^m dx \right)^{\frac{1}{m}} \leq C \left(\frac{1}{|B|} \int_B \omega(x) dx \right).$$

Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n, \omega)$ and $b \in \text{BMO}(\mathbb{R}^n)$ respectively defined as

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} < \infty,$$

$$\|b\|_* = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx < \infty,$$

where $b_B = \frac{1}{|B|} \int_B b(y) dy$.

Definition 1.2 ([9]) Let φ be a non-negative increasing function on $(0, +\infty)$. For any $r \geq 0$ satisfy

$$\varphi(2r) \leq D_\varphi \varphi(r), \quad (5)$$

where $D(\varphi) \geq 1$ is independent of r , $0 \leq \eta < n$ and $1 \leq p < \frac{n}{\eta}$. The generalized fractional weighted Morry space is defined by

$$L^{p,\eta,\varphi}(\mathbb{R}^n, \omega) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\eta,\varphi}(\omega)} < \infty \right\},$$

where

$$\|f\|_{L^{p,\eta,\varphi}(\omega)} = \sup_{x \in \mathbb{R}^n, r > 0} \left(\frac{1}{\varphi(r)^{1-\frac{\eta}{n}}} \int_{B(x,r)} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}},$$

$$B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}.$$

Definition 1.3 ([10]) Let $1 < p < \infty$. If $B \subseteq \mathbb{R}^n$, there exists a constant C such

that

$$\left(\frac{1}{\psi_\theta(B)|B|} \int_B \omega(y) dy\right) \left(\frac{1}{\psi_\theta(B)|B|} \int_B \omega(y)^{\frac{1}{p-1}} dy\right)^{p-1} \leq C. \tag{6}$$

It is called a non-negative measurable function $\omega \in A_p^{\rho,\theta}$, so we have $A_p^{\rho,\infty} = U_{\theta>0} A_p^{\rho,\theta}$ and $A_\infty^{\rho,\infty} = U_{\theta>0} A_p^{\rho,\infty}$. For $\psi_\theta(B) \geq 1$, it is evident that $A_p \subset A_p^{\rho,\theta}$ where $1 \leq p < \infty$.

Remark (a): It is evident that when $\eta = 0$, the space $L^{p,0,\varphi}(\mathbb{R}^n, \omega)$ coincides with the generalized weighted Morrey space $L^{p,\varphi}(\mathbb{R}^n, \omega)$ introduced by Guliyev in the literature in [11], which extends the classical weighted Morrey space. For $\varphi(r) = r^\sigma$, it reduces to the classical Morrey space $L^{p,\sigma}(\mathbb{R}^n, \omega)$ proposed by Komori and Shirai.

(b): When $\omega = 1$, $L^{p,\sigma}(\mathbb{R}^n, \omega)$ corresponds to the classical Morrey space $L^{p,\sigma}(\mathbb{R}^n, \omega)$ in [12], originally introduced by Morrey in the literature to investigate the local behavior of solutions to second-order elliptic partial differential equations. Further advancements and properties of Morrey spaces can be referenced in existing studies in [13] [14], notably for $\varphi(r) = 1$, $L^{p,\varphi}(\mathbb{R}^n, \omega)$ simplifies to the weighted Lebesgue space $L^{p,\sigma}(\mathbb{R}^n, \omega)$ as established in prior works in [15]. Our main results are as follows:

Theorem 1.1 Let $1 \leq D(\varphi) \leq 2^n$, $m > 1$, $0 < \eta < \frac{n}{m}$, $\frac{n}{m\eta} < p < \frac{n}{\eta}$ and $\omega(x) \in A_p^{\rho,\theta}$. There exists a positive constant C that is independent of f such that

$$\|\mu_j^C(f)\|_{L^{p,\eta,\varphi}(\omega)} \leq C \|f\|_{L^{p,\eta,\varphi}(\omega)}.$$

Theorem 1.2 Let $1 \leq D(\varphi) \leq 2^n$, and $[b, \mu_j^C]$ is defined by (1.4), $b \in \text{BMO}_\rho$. If $m > 1$, $0 < \eta < \frac{n}{m}$, $\frac{n}{m\eta} < p < \frac{n}{\eta}$ and $\omega(x) \in A_p^{\rho,\theta}$, then there exists a positive constant C that is independent of f such that

$$\|[b, \mu_j^C](f)\|_{L^{p,\eta,\varphi}(\omega)} \leq C \|f\|_{L^{p,\eta,\varphi}(\omega)}.$$

2. Preliminary Knowledge

To prove the main theorems, in this section, we need to recall some necessary lemmas, respectively.

Lemma 2.1 ([2]) Let $1 < p < \infty$ and $\omega \in A_p^\rho$. There exists a positive constant C such that

$$\|\mu_j^C(f)\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}.$$

Lemma 2.2 ([2]) Let $1 < p < \infty$, $b \in \text{BMO}_\rho$ and $V \in RH_s$. If $\omega \in A_p^\rho$, then there exists a positive constant C such that

$$\|[b, \mu_j^C](f)\|_{L^p(\omega)} \leq C \|b\|_{\text{BMO}_\rho} \|f\|_{L^p(\omega)}.$$

Now we state the characterizations of spaces $BMO_{\theta,\varphi}(X)$ as follows.

Lemma 2.3 ([16]) Let $1 \leq p < \infty$. If $\omega \in A_p^\rho$ such that

$$\frac{1}{\psi(B)|B|} \int_B |f(y)| dy \leq C \left(\frac{1}{\omega(5B)} \int_B |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}},$$

where $\omega(E) = \int_E \omega(y) dy$.

Lemma 2.4 ([17]) Let $1 \leq p < \infty$. If $\omega \in A_p^\rho$, then there exists positive constant $0 < \delta < 1$, γ and C . For any measurable set E of sphere B , we have

$$\frac{\omega(E)}{\omega(B)} \leq C \left(1 + \frac{r}{\rho(x)} \right)^\gamma \left(\frac{|E|}{|B|} \right)^\delta.$$

Lemma 2.5 ([18]) Let $\omega \in RH_m$, $m \geq 1$. For any measurable subset E of an arbitrary sphere B , then there exists constant $C > 0$ such that

$$\frac{\omega(E)}{\omega(B)} \leq C \left(\frac{|E|}{|B|} \right)^{\frac{m-1}{m}}.$$

Lemma 2.6 ([19]) Let $V \in RH_s$. For any $N > 0$, we have (1). There exists a constant C such that

$$|\mathcal{K}_j^v(x, y)| \leq \frac{C}{(1 + |x - y| \rho(x)^{-1})^N} \frac{1}{|x - y|^{n-1}}.$$

(2) For $0 < \rho_0 < 1 - \frac{n}{q}$, there exists a constant C such that

$$|\mathcal{K}_j^v(x, y) - \mathcal{K}_j^v(y, z)| \leq \frac{C}{(1 + |x - y| \rho(x)^{-1})^N} \frac{|x - y|^{\rho_0}}{|x - z|^{n-1+\rho_0}},$$

where $|x - y| < \frac{2}{3}|x - z|$.

Lemma 2.7 ([20]) Let $1 \leq p < \infty$. If $f \in BMO_\rho$ and $\omega \in A_p^\rho$, then there exist two normal numbers C and β such that sphere $B \subset \mathbb{R}^n$. We have

$$\left(\frac{1}{\omega(B)} \int_B |f(y) - f_B| \omega(y) dy \right)^{\frac{1}{p}} \leq C \psi(B)^{\frac{\beta}{p}} \|f\|_{BMO_\rho}.$$

Lemma 2.8 ([21]) Let $1 \leq p < \infty$, where $b \in BMO(\mathbb{R}^n)$. We have

$$\sup \left(\frac{1}{|B|} \int_B |b(x) - b_B|^p dx \right)^{\frac{1}{p}} \leq C \|b\|_*$$

3. Proofs of Main Theorems

By the aid of some ideas and methods from the proofs, we mainly state the proof of Theorem 1.1 as follows.

Proof of Theorem 1.1. Let $f \in L^{p,\lambda,\varphi}(\omega)$ and a sphere $B = B(x_0, r_B)$ with x_0 as the center and r_B as the radius. Decompose $f = f_1 + f_2$, where

$f_1 = f \chi_{2B}$ and $f_2 = f \chi_{(2B)^c}$. We have

$$\begin{aligned} \|\mu_j^L(f)\|_{L^{p,\eta,\varphi}(\omega)} &= \frac{1}{\varphi(r)^{\frac{1-\eta}{p-n}}} \|\mu_j^L(f)\|_{L^p(\omega)} \\ &\leq \frac{1}{\varphi(r)^{\frac{1-\eta}{p-n}}} \|\mu_j^L(f_1)\|_{L^p(\omega)} + \frac{1}{\varphi(r)^{\frac{1-\eta}{p-n}}} \|\mu_j^L(f_2)\|_{L^p(\omega)} \\ &= E + F, \end{aligned}$$

which yields the desired result.

Firstly, we estimate E , from definition 1.1 and Lemma 2.1

$$\begin{aligned} E &\leq C \left(\frac{\varphi(2r_B)^{\frac{1-\eta}{p-n}}}{\varphi(r_B)^{\frac{1-\eta}{p-n}}} \right) \left(\frac{1}{\varphi(2r_B)^{\frac{1-\eta}{p-n}}} \right) \|f \chi_{2B}\|_{L^p(\omega)} \\ &\leq C \left(\frac{1}{\varphi(2r_B)^{\frac{1-\eta}{p-n}}} \right) \|f \chi_{2B}\|_{L^p(\omega)} = C \|f_1\|_{L^{p,\eta,\varphi}(\omega)} \\ &\leq C \left(\frac{1}{\varphi(r_B)^{\frac{1-\eta}{p-n}}} \right) \|f \chi_B\|_{L^p(\omega)} = C \|f\|_{L^{p,\eta,\varphi}(\omega)}. \end{aligned}$$

For the term F , from Minkowski's inequality and Lemma 2.6, we can get

$$\begin{aligned} \mu_j^L(f_2)(x) &= \left(\int_0^\infty \left| \int_{(2B)^c \cap \{y:|x-y|\leq h\}} \mathcal{K}_j^L(x,y) f(y) dy \right|^2 \frac{dh}{h^3} \right)^{\frac{1}{2}} \\ &\leq C \int_{(2B)^c} |\mathcal{K}_j^L(x,y)| |f(y)| \left(\int_{|x-y|\leq h} \frac{dh}{h^3} \right)^{\frac{1}{2}} dy \\ &\leq C \left(\int_{(2B)^c} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N} \frac{|f(y)|}{|x-y|^n} dy \right). \end{aligned}$$

Notice that where $x \in B$, $y \in 2^{j+1}B \setminus 2^jB$ and $j \in \mathbb{N}$, we have $|x-y| \sim 2^{j+1}r$. For F , from Lemma 2.3 and Lemma 2.4, we have

$$\mu_j^L(f_2)(x) \leq C \sum_{j=1}^\infty \left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |f(y)| dy.$$

Combining the inequality (1.6), we know that

$$\begin{aligned} &\left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |f(y)| dy \\ &\leq \frac{1}{|2^{j+1}B|} \left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \left(\int_{2^{j+1}B} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \left(\int_{2^{j+1}B} \omega(y)^{\frac{p'}{p}} dy \right)^{\frac{1}{p'}} \\ &\leq \frac{\varphi(2^{k+1}r_B)^{\frac{1-\eta}{p-n}}}{|2^{j+1}B|} \left(\frac{1}{\varphi(2^{j+1}r_B)^{\frac{1-p\eta}{p-n}}} \int_{2^{j+1}B} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned} & \times \left[\left(\frac{1}{\psi_N(2^{j+1}B)|2^{j+1}B|} \int_{2^{j+1}B} \omega(y)^{\frac{1}{p-1}} dy \right)^{p-1} \right. \\ & \left. \times \left(\frac{1}{\psi_N(2^{j+1}B)|2^{j+1}B|} \int_{2^{j+1}B} \omega(y) dy \right) \right]^{\frac{1}{p}} |2^{j+1}B| \omega(2^{j+1}B)^{\frac{1}{p}} \\ & \leq C \|f\|_{L^{p,\eta,\varphi}(\omega)} \varphi(2^{j+1}r_B)^{\frac{1}{p} - \frac{\eta}{n}} \omega(2^{j+1}B)^{\frac{1}{p}}. \end{aligned}$$

Thus, we can get

$$\mu_j^{\mathcal{L}}(f_2)(x) \leq C \|f\|_{L^{p,\eta,\varphi}(\omega)} \sum_{j=1}^{\infty} \varphi(2^{j+1}B)^{\frac{1}{p} - \frac{\eta}{n}} \omega(2^{j+1}B)^{\frac{1}{p}}. \tag{7}$$

Using (3.1), we can get

$$F \leq C \|f\|_{L^{p,\eta,\varphi}(\omega)} \sum_{j=1}^{\infty} \frac{\varphi(2^{j+1}r_B)^{\frac{1}{p} - \frac{\eta}{n}} \omega(B)^{\frac{1}{p}}}{\varphi(r_B)^{\frac{1}{p} - \frac{\eta}{n}} \omega(2^{j+1}B)^{\frac{1}{p}}}.$$

Because of $\omega \in A_p$, there exists $m > 1$ such that $\omega \in RH_m$. Therefore, use

(1.5) and Lemma 2.5 formula, and observe $0 < \eta < \frac{n}{m}$, $\frac{n}{m\eta} < p < \frac{n}{\eta}$. we have

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\varphi(2^{j+1}r_B)^{\frac{1}{p} - \frac{\eta}{n}} \omega(B)^{\frac{1}{p}}}{\varphi(r_B)^{\frac{1}{p} - \frac{\eta}{n}} \omega(2^{j+1}B)^{\frac{1}{p}}} & \leq C \sum_{j=1}^{\infty} \left(\frac{|B|}{|2^{j+1}B|} \right)^{\frac{m-1}{pm}} \left(\frac{D(\varphi)^{k+1} \varphi(r_B)}{\varphi(r_B)} \right)^{\frac{1}{p} - \frac{\eta}{n}} \\ & \leq C \sum_{j=1}^{\infty} \left(\frac{|B|}{|2^{j+1}B|} \right)^{\frac{1}{p} - \frac{\eta}{n}} \left(\frac{D(\varphi)^{k+1} \varphi(r_B)}{\varphi(r_B)} \right)^{\frac{1}{p} - \frac{\eta}{n}} \\ & \leq C \sum_{j=1}^{\infty} \left(\frac{D(\varphi)}{2^n} \right)^{(k+1)\left(\frac{1}{p} - \frac{\eta}{n}\right)}. \end{aligned}$$

Observing that $(k+1)\left(\frac{1}{p} - \frac{\eta}{n}\right) > 0$, $1 \leq D(\varphi) < 2^n$, we have

$$F \leq C \|f_2\|_{L^{p,\eta,\varphi}(\omega)}.$$

Combining with the estimates of E and F , we conclude that

$$\|\mu_j^{\mathcal{L}}(f)\|_{L^{p,\eta,\varphi}(\omega)} \leq C \|f\|_{L^{p,\eta,\varphi}(\omega)}.$$

Hence, the proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, decompose $f = f_1 + f_2$. Among which $f_1 = f \chi_{2B}$, we note that

$$\begin{aligned} & \frac{1}{\varphi(r_B)^{\frac{1}{p} - \frac{\eta}{n}}} \left(\int_B |[b, \mu_j^{\mathcal{L}}](f)(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \\ & \leq \frac{1}{\varphi(r_B)^{\frac{1}{p} - \frac{\eta}{n}}} \left(\int_B |[b, \mu_j^{\mathcal{L}}](f_1)(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\varphi(r_B)^{\frac{1}{p} - \frac{\eta}{n}}} \left(\int_B |[b, \mu_j^{\mathcal{L}}](f_2)(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \\
 & = I_1 + I_2.
 \end{aligned}$$

Similarly, then it follows that I_1 , from Lemma 2.2

$$\begin{aligned}
 I_1 & \leq C \left(\frac{\varphi(2r_B)^{\frac{1}{p} - \frac{\eta}{n}}}{\varphi(r_B)^{\frac{1}{p} - \frac{\eta}{n}}} \right) \left(\frac{1}{\varphi(2r_B)^{\frac{1}{p} - \frac{\eta}{n}}} \right) \|f_{\chi_{2B}}\|_{L^p(\omega)} \\
 & \leq C \|f\|_{L^{p,\eta,\varphi}(\omega)}.
 \end{aligned}$$

For the term I_2 , it is similar to the argument of F

$$\begin{aligned}
 [b, \mu_j^{\mathcal{L}}](f_2)(x) & = \left(\int_0^\infty \left| \int_{|x-y|\leq h} \mathcal{K}_j^{\mathcal{L}}(x, y) (b(y) - b_B) f_2(y) dy \right|^2 \frac{dh}{h^3} \right)^{\frac{1}{2}} \\
 & \leq C |b(x) - b_B| \left(\int_0^\infty \left| \int_{(2B)^c \cap \{|x-y|\leq h\}} \mathcal{K}_j^{\mathcal{L}}(x, y) f(y) dy \right|^2 \frac{dh}{h^3} \right)^{\frac{1}{2}} \\
 & \quad + C \left(\int_0^\infty \left| \int_{(2B)^c \cap \{|x-y|\leq h\}} \mathcal{K}_j^{\mathcal{L}}(x, y) (b(y) - b_B) f(y) dy \right|^2 \frac{dh}{h^3} \right)^{\frac{1}{2}} \\
 & := I_{21} + I_{22}.
 \end{aligned}$$

For I_{21} , use Hölder's inequality and the weight condition of $A_p^{\rho,\theta}$.

$$I_{21} \leq C |b(x) - b_B| \sum_{j=0}^\infty \left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |f(y)| dy. \tag{8}$$

Based on the above estimates (3.2), we have

$$\begin{aligned}
 & \frac{1}{\varphi(r_B)^{\frac{1}{p} - \frac{\eta}{n}}} \left(\int_B |I_{21}|^p \omega(x) dx \right)^{\frac{1}{p}} \\
 & \leq C \|f\|_{L^{p,\eta,\varphi}(\omega)} \sum_{j=1}^\infty \frac{\varphi(2^{j+1}r_B)^{\frac{1}{p} - \frac{\eta}{n}}}{\varphi(r_B)^{\frac{1}{p} - \frac{\eta}{n}}} \frac{\omega(B)^{\frac{1}{p}}}{\omega(2^{j+1}B)^{\frac{1}{p}}} \left(\frac{1}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x) dx \right)^{\frac{1}{p}} \\
 & \leq C \|f\|_{L^{p,\eta,\varphi}(\omega)} \left(\frac{1}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x) dx \right)^{\frac{1}{p}}.
 \end{aligned}$$

Because of $\omega \in A_p$, there exists $m > 1$ such that $\omega \in RH_m$. By the Hölder's inequality and Lemma 2.8, we can deduce that

$$\begin{aligned}
 & \left(\frac{1}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x) dx \right)^{\frac{1}{p}} \\
 & \leq \frac{1}{\omega(B)^{\frac{1}{p}}} \left(\int_B |b(x) - b_B|^{pm'} dx \right)^{\frac{1}{pm'}} \left(\frac{1}{|B|} \int_B \omega(x)^m dx \right)^{\frac{1}{pm}} |B|^{\frac{1}{pm}}
 \end{aligned}$$

$$\begin{aligned} &\leq C \frac{1}{\omega(B)^{\frac{1}{p}}} \left(\int_B |b(x) - b_B|^{pm'} dx \right)^{\frac{1}{pm'}} \left(\frac{1}{|B|} \int_B \omega(x) dx \right)^{\frac{1}{p}} |B|^{\frac{1}{pm}} \\ &\leq C \|b\|_* . \end{aligned}$$

Obviously, we can get

$$\frac{1}{\varphi(r_B)^{\frac{1}{p} \frac{\eta}{n}}} \left(\int_B |I_{21}|^p \omega(x) dx \right)^{\frac{1}{p}} \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)} .$$

Hence, we can divide I_{22} into two parts as follows:

$$\begin{aligned} I_{22} &\leq C \sum_{j=1}^{\infty} \left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}| |f(y)| dy \\ &\quad + C \sum_{j=1}^{\infty} \left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \frac{|b_{2^{j+1}B} - b_B|}{(2^{j+1}r)^n} \int_{2^{j+1}B} |f(y)| dy \\ &:= J_1 + J_2 . \end{aligned}$$

By the Hölder's inequality, from Lemma 2.7, we have

$$\begin{aligned} &\int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}| |f(y)| dy \\ &\leq \left(\int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} \omega(y)^{-\frac{p'}{p}} dy \right)^{\frac{1}{p'}} \left(\int_{2^{j+1}B} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \\ &\leq C \|f\|_{L^{p,\eta,\varphi}(\omega)} \varphi(2^{j+1}r_B)^{\frac{1}{p} \frac{\eta}{n}} \left(\int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} \omega(y)^{-\frac{p'}{p}} dy \right)^{\frac{1}{p'}} . \end{aligned}$$

The similar argument as what we have done in the proof of Theorem 1.1 yields that

$$\begin{aligned} &\frac{1}{|2^{j+1}B|} \left(\frac{1}{\omega(2^{j+1}B)} \right)^{\frac{1}{p}} \left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \left(\int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} \omega(y)^{-\frac{p'}{p}} dy \right)^{\frac{1}{p'}} \\ &\leq \frac{1}{|2^{j+1}B|} \left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \left(\frac{1}{\omega(2^{j+1}B)} \right)^{\frac{1}{p}} \left(\int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} \omega(y)^{-\frac{p'}{p}} dy \right)^{\frac{1}{p'}} \\ &\leq \frac{1}{|2^{j+1}B|} \left(\frac{1}{\omega(2^{j+1}B)} \right)^{\frac{1}{p}} \left(\int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'm'} dy \right)^{\frac{1}{p'm'}} \\ &\quad \times \left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \left(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} \omega(y)^{-\frac{p'}{p}} dy \right)^{\frac{1}{p'}} |2^{j+1}B|^{\frac{1}{p'm}} \\ &\leq C \|b\|_* . \end{aligned}$$

Combining the above inequality, following estimates are obtained

$$J_1 \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)} \sum_{j=1}^{\infty} \varphi(2^{j+1}B)^{\frac{1}{p} \frac{\eta}{n}} \omega(2^{j+1}B)^{\frac{1}{p}} . \tag{9}$$

Based on the above estimate (3.3), we can get

$$\begin{aligned} & \frac{1}{\varphi(r_B)^{\frac{1}{p}-\frac{\eta}{n}}} \left(\int_B J_1^p \omega(x) dx \right)^{\frac{1}{p}} \\ & \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)} \sum_{j=1}^{\infty} \frac{\varphi(2^{j+1}r_B)^{\frac{1}{p}-\frac{\eta}{n}}}{\varphi(r_B)^{\frac{1}{p}-\frac{\eta}{n}}} \frac{\omega(B)^{\frac{1}{p}}}{\omega(2^{j+1}B)^{\frac{1}{p}}} \\ & \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)} \sum_{j=1}^{\infty} \left(\frac{D(\varphi)}{2^n} \right)^{(k+1)\left(\frac{1}{p}-\frac{\eta}{n}\right)} \\ & \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)}. \end{aligned}$$

For J_2 , note that $|b_{2^{j+1}B} - b_B| \leq C(j+1)\|b\|_*$ [22]. We can get

$$\begin{aligned} J_2 & = C \sum_{j=0}^{\infty} \left(1 + \frac{2^{j+1}r}{\rho(x)} \right)^{-N} \frac{|b_{2^{j+1}B} - b_B|}{(2^{j+1}r)^n} \int_{2^{j+1}B} |f(y)| dy \\ & \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)} \sum_{j=1}^{\infty} (j+1) \varphi(2^{j+1}B)^{\frac{1}{p}-\frac{\eta}{n}} \omega(2^{j+1}B)^{\frac{1}{p}}. \end{aligned}$$

Similar to the argument of J_1 , we have

$$\begin{aligned} & \frac{1}{\varphi(r_B)^{\frac{1}{p}-\frac{\eta}{n}}} \left(\int_B J_2^p \omega(x) dx \right)^{\frac{1}{p}} \\ & \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)} \sum_{j=1}^{\infty} (j+1) \frac{\varphi(2^{j+1}r_B)^{\frac{1}{p}-\frac{\eta}{n}}}{\varphi(r_B)^{\frac{1}{p}-\frac{\eta}{n}}} \frac{\omega(B)^{\frac{1}{p}}}{\omega(2^{j+1}B)^{\frac{1}{p}}} \\ & \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)} \sum_{j=1}^{\infty} (j+1) \left(\frac{D(\varphi)}{2^n} \right)^{(j+1)\left(\frac{1}{p}-\frac{\eta}{n}\right)} \\ & \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)}. \end{aligned}$$

Synthesize the above results, we can get

$$\frac{1}{\varphi(r_B)^{\frac{1}{p}-\frac{\eta}{n}}} \left(\int_B |I_{22}|^p \omega(x) dx \right)^{\frac{1}{p}} \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)}. \tag{10}$$

From (3.4), we conclude that

$$I_2 \leq C \|b\|_* \|f\|_{L^{p,\eta,\varphi}(\omega)}.$$

Combined I_1 and I_2 estimates, the proof of Theorem 1.2 is completed.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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