Applied Mathematics, 2023, 14, 324-335
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Adomian Decomposition Method for Solving Time Fractional Burgers Equation Using Maple

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How to cite this paper: Alwehebi, F., Hobiny, A. and Maturi, D. (2023) Adomian Decomposition Method for Solving Time Fractional Burgers Equation Using Maple. *Applied Mathematics*, **14**, 324-335.
https://doi.org/10.4236/am.2023.145020

Received: April 21, 2023
Accepted: May 28, 2023
Published: May 31, 2023

Abstract

In this paper, the Adomian decomposition method was used to solve the Time Fractional Burger equation using Mabel program. This method was applied to a number of examples of the Time Fractional Burger Equation. The obtained numerical results were presented in the form of tables and graphics. The difference between the exact solutions and the numerical solutions shows us the effectiveness of the solution using the Mabel program and that this method gave accurate results and was close to the exact solution, in addition to its ability to obtain the numerical solution quickly and efficiently using the Mabel program.

Keywords

Adomian Decomposition Method, Time Fractional Burgers Equation, Maple 18

1. Introduction

The branch of mathematics studies fractional calculus properties of derivatives and integrals of non-integer orders which are called fractional derivatives and integrals the concept and methods of using fractional derivatives to solve differential equations. Particularly fascinating are the applications of science in subjects like physics, chemistry, and engineering (fluid flow, viscoelasticity, electrical networks, optics and signal processing, etc). The development of Newton's and Leibniz's classical calculus coincided with the beginning of fractional calculus. Fractional calculus was first introduced in l'Hopital's (1695) letter to Leibniz. The possibility of a derivative of order 1/2 was questioned. Where Leibniz predicted the beginning of what is known as fractional calculus [1] [2] [3] [4]. Euler was the second person after Leibniz to identify the non-integer orders problem. The issue of non-integer orders was first identified by Euler (1738), who came after Leibniz [3]. As the first definition of a derivative of any positive order, Fourier (1822) suggested an integral representation for the definition of a derivative [3] [5]. An equal time problem involving the solution of an integral equation was the subject of Abel's fractional calculus application in (1826) [3] [5]. The definition Liouville proposed, known as Liouville's first definition, was based on the derivation of the exponential function (1832). Liouville's second definition, which is now known as the Liouville version for the integration of noninteger order, is expressed in terms of an integral. The most significant paper after several of Liouville's works was made by Riemann ten years after his passing [6]. We see that the complementary function is present in the Liouville and Riemann formulations. It is a problem that must be resolved using the Liouville and Riemann method that integration introduced.

It was independently developed by Grünwald [7] and Letnikov [8] to analyze the derivatives of noninteger orders in terms of a simple convergent series. Letnikov has demonstrated consistency between his definition and those put out by Riemann and Liouville for specific order values, under a useful explanation of the so-called difference of non-integer orders. According to a work by Hadamard (1892) [5] the derivative of non-integer orders of an analytical function must be stated in terms of a Taylor series.

Fractional calculus rapidly developed after (1900), and numerous definitions were created in an effort to formulate particular problems, some of which we list. Weyl [9] constructs a derivative in order to solve a problem regarding a particular class of functions, the periodic functions. The Fourier transform formula is established by Riesz [10] [11] who also establishes the mean value theorem for fractional integrals Marchaud (1927) gives a completely new definition of the order of non-integer derivatives that is compatible with Liouville's concept of (sufficiently good) functions [3] [5]. Non-integer orders were defined separately by Erde lyi-Kober (1940) [3] [5]. Compared to Liouville and Riemann, Caputo (1967) [12] proposes a definition that is more precise, but is better suited for discussions of issues involving fractional differential equations with initial conditions [13]-[21]. This approach will be contrasted with Liouville and Riemann's formula. Caputo's formulation reflects the order of the integral and derivative operators due to the significance of his version with the derivative of non-integer orders from Liouville and Riemann. We'll compare the two formulas. In Caputo, the derivative of non-integer orders is computed first, and then the integral of non-integer orders. The integral of non-integer orders is calculated in the Liouville and Riemann equation first, and then the derivative of integer orders. It is significant to highlight that issues when the function's initial conditions are satisfied and each of them has integer derivatives can be solved using the Caputo derivative. Fractional calculus has developed since the first conference at the University of New Haven in (1974), and as a result, numerous applications in numerous scientific fields have appeared. There are numerous approaches to dealing with derivatives-related problems.

Fractional differential equations are beginning to enjoy widespread application in many real-life modeling problems. Time Fractional Burger equation is kind of subdiffusion convection equation used in fluid mechanics. In the study of turbulent flow, they are used to represent a wide range of phenomena, such as the propagation of shallow water waves and nonlinear acoustic waves in gas pipelines [22] [23] shock propagation, electromagnetic waves, turbulence, porous medium flows, pollutant flow, and temperature and pressure waves, medical sciences, etc. Among others, these models aid in better explanation and understanding [24] [25] [26] [27]. Shock waves propagating through viscous material are another illustration. This equation is commonly employed by researchers as a test case for determining the efficacy of novel numerical methods. By using a fractional derivative in place of the first-order time derivative, this equation can be derived from the classical Burger equation.

The classical Burger equation or the Pittman-Burger equation is a fundamental partial differential equation. The equation was first introduced by Harry Pittman in (1915) and later studied by Johannes Martinus Berger in (1948) to solve nonlinear equation systems. Fractional differential equations can be challenging to solve precisely at times. Therefore, the purpose of this study is to use the Adomian decomposition method by Mabel 18 program to solve the fractional time Burger equation.

The fractional time Berger equation is solved, and the numerical examples and error estimates from the Mabel program are explained.

2. Comparing Adomian Decomposition Method with Common Numerical Methods for Solving Time Fractional Burger Equation

The numerical methods needed to solve the Time Fractional Burger problem make it a challenging partial differential equation. The Adomian decomposition method (ADM), a potent numerical methodology for resolving nonlinear partial differential equations, is one such approach. However, contrasting ADM with other widely used numerical techniques can offer a more thorough understanding of its benefits and drawbacks.

First off, partial differential equations are frequently solved using finite difference techniques (FDM). The discretization of the equation into a system of algebraic equations using FDM is simple and easy to do. FDM needs a lot of grid points to produce accurate results and has difficulties when solving nonlinear equations.

Secondly, finite element methods (FEM) are also commonly used for solving partial differential equations. FEM discretizes the domain into smaller elements and approximates the solution using a polynomial function. FEM is efficient for irregular geometries and has a high degree of accuracy. However, FEM requires a significant computational effort to solve large-scale problems.

Thirdly, spectral methods (SM) are another class of numerical methods used for solving partial differential equations. SM uses orthogonal basis functions to approximate the solution, which provides high accuracy and convergence rates.

However, SM has limitations in solving nonlinear equations and requires a large number of basis functions for accurate results.

In comparison, ADM is a powerful numerical method that has several advantages over other methods. ADM is a non-iterative method that does not require the discretization of the domain, making it computationally efficient. Moreover, ADM can handle nonlinear equations and provides accurate results with a small number of terms in the decomposition series.

In conclusion, comparing ADM with other commonly used numerical methods for solving the Time Fractional Burger equation enriches the discussion and provides a better understanding of its advantages and limitations. While FDM, FEM, and SM are effective methods, ADM offers a unique approach that is efficient, accurate, and suitable for nonlinear equations.

We'll start by providing some definitions for fractional calculus.

3. Definitions

3.1. Definition (1)

Let $\alpha \in R_+$ The operator J_a^α defined on the usual Lebesque space $L_1[a,b]$ by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t$$

$$J_a^0 f(x) = f(x)$$

for $a \le x \le b$, is called the Riemann-Liouville fractional integral operator of order α.

Properties of the operator J_a^α can be found in [28] we mention the following: For $f \in L_1[a,b]$, $\alpha, \beta \ge 0$ and $\gamma > -1$,

1) $J_a^\alpha f(x)$ exists for almost every $x \in [a,b]$,
2) $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$,
3) $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$,
4) $J_a^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$

We offer the Caputo-proposed modified fractional differential operator D^α for the theory of viscoelasticity [12] [29] [30] [31] [32].

3.2. Definition (2)

The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\mathrm{d}t,$$

For $m-1 < \alpha \le m$, $m \in \mathbb{N}$, $x > 0$

Two of its fundamental characteristics are also required here.

3.3. Lemma

If $m-1 < \alpha \le m$ and $f \in L_1[a,b]$, then

$$D_a^\alpha J_a^\alpha f(x) = f(x)$$

and,

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{(x-a)^k}{k!}, x > 0$$

3.4. Definition (3)

For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial t^m}\mathrm{d}\tau, & \text{for } m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N} \end{cases}$$

4. Adomian Decomposition Method (ADM)

The In this study, the Adomian decomposition method (ADM) is used to solve Time Fractional Burgers Equation in Maple. Adomian put forth a fresh approach to solving several sorts of nonlinear functional equations at the start of the 1980s. The method actually entails breaking down the nonlinear operators into a set of functions. These series terms are all Adomian's polynomials, which are a type of polynomial. Now, let's go through the decomposition method's fundamental tenets [33] [34].

Consider the one-dimensional Time Fractional Burgers equation for $0 < \alpha \le 1$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \varepsilon u(x,t)\frac{\partial u(x,t)}{\partial x} = v\frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0 \tag{1}$$

subject to the initial condition

$$u(x,0) = g(x) \tag{2}$$

Following [33] [34], we write (1) in an operator form

$$D_t^\alpha u(x,t) = vL_x u - \varepsilon u u_x$$

where $u = u(x,t)$, $L_x = \frac{\partial^2 u}{\partial x^2}$, L is the highest order derivative in the equation and the fractional differential operator $D_t^\alpha = \frac{\partial^\alpha u}{\partial t^\alpha}$

The inverse operator of L_x namely L_x^{-1} is a twofold integral operator given by

$$L_x^{-1}(.) = \int_0^x \int_0^x (.)\mathrm{d}x\mathrm{d}x$$

Operating with $J^\alpha = J_0^\alpha$ on both sides of Equation (1) and using the initial condition (2) yields

$$u(x,t) = u(x,0) + J^{\alpha}\left[vL_x u - \varepsilon\phi(u)\right] \tag{3}$$

where $\phi(u) = uu_x$.

The Adomian decomposition method [33] [34] assumes a series solution for $u(x,t)$ given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{4}$$

The Adomian polynomials can be calculated for all forms of nonlinearity $\phi(u)$ according to specific algorithms constructed by Adomian [33] [34] The general form of formula for A_n Adomian polynomials is given by

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\phi\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0} \tag{5}$$

We give here the first few terms of Adomian's polynomials A_n for the non-linear functions $\phi(u)$ as

$$\begin{aligned} A_0 &= u_0(u_0)_x \\ A_1 &= u_1(u_0)_x + u_0(u_1)_x \\ A_1 &= u_1(u_0)_x + u_0(u_1)_x \\ A_2 &= u_2(u_0)_x + u_1(u_1)_x + u_0(u_2)_x \\ A_3 &= u_3(u_0)_x + u_2(u_1)_x + u_1(u_2)_x + u_0(u_3)_x \end{aligned} \tag{6}$$

Following Adomian method analysis, Equation (3) is transformed into a set of recursive relations given by

$$u_0 = g(x),\ u_{n+1} = J^{\alpha}\left[vL_x u_n - \varepsilon A_n\right] \tag{7}$$

Where $\phi(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \cdots, u_n)$ is called the Adomian polynomials. Using the above recursive relationship and Mathematica, the first three terms of the decomposition series are given by

$$\begin{aligned} u_0 &= u(x,0) = g(x) \\ u_1 &= J^{\alpha}\left[vL_x u_0 - \varepsilon A_0\right] = \left[vg'' - \varepsilon gg'\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ u_2 &= J^{\alpha}\left[vL_x u_1 - \varepsilon A_1\right] \\ &= \left[2\varepsilon^2 gg'^2 + \varepsilon^2 g^2 g'' - 4\varepsilon vg'g'' - 2\varepsilon vgg''' + v^2 g^{(4)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\vdots \end{aligned} \tag{8}$$

and so on, in this manner, the rest of components of the decomposition series can be obtained. The solution in series form is given by

$$\begin{aligned} u(x,t) &= g(x) + \left[vg'' - \varepsilon gg'\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad + \left[2\varepsilon^2 gg'^2 + \varepsilon^2 g^2 g'' - 4\varepsilon vg'g'' - 2\varepsilon vgg''' + v^2 g^{(4)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \end{aligned} \tag{9}$$

5. Application to Obtain the Numerical Solution of Time Fractional Burgers' Equation

5.1. Example 1

We consider one-dimensional Time Fractional Burgers Equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^2 u}{\partial x^2}-2u\frac{\partial u}{\partial x}+\frac{\partial u^2}{\partial x}=0 \tag{10}$$

With initial conditions $u(x,0)=\sin x$

$$0<\alpha\le 1,\ t>0,\ x\in R$$

$$\alpha=1$$

5.2. Example 2

We consider one-dimensional Time Fractional Burgers Equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^2 u}{\partial x^2}-2u\frac{\partial u}{\partial x}+\frac{\partial u^2}{\partial x}=0 \tag{11}$$

With initial conditions $u(x,0)=\sin x$

$$0<\alpha\le 1,\ t>0,\ x\in R$$

$$\alpha=1.5=\frac{3}{2}$$

Figure 1 and Figure 2 show the exact and approximate solutions. This problem was solved by ADM and their results are shown in Table 1 and Table 2 using maple.

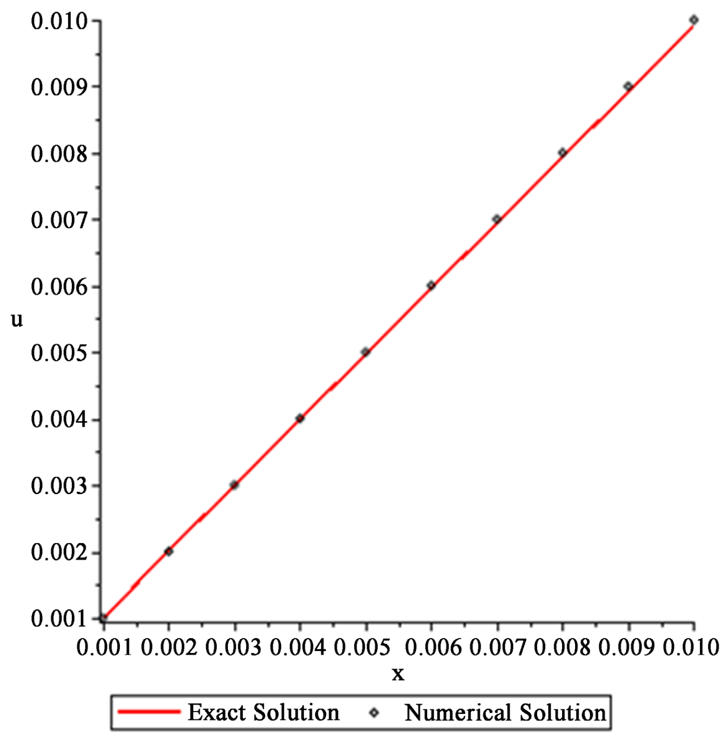

Figure 1. Graph showing the correspondence between exact and approximate solutions result of time-fractional Burgers equation in Example 1.

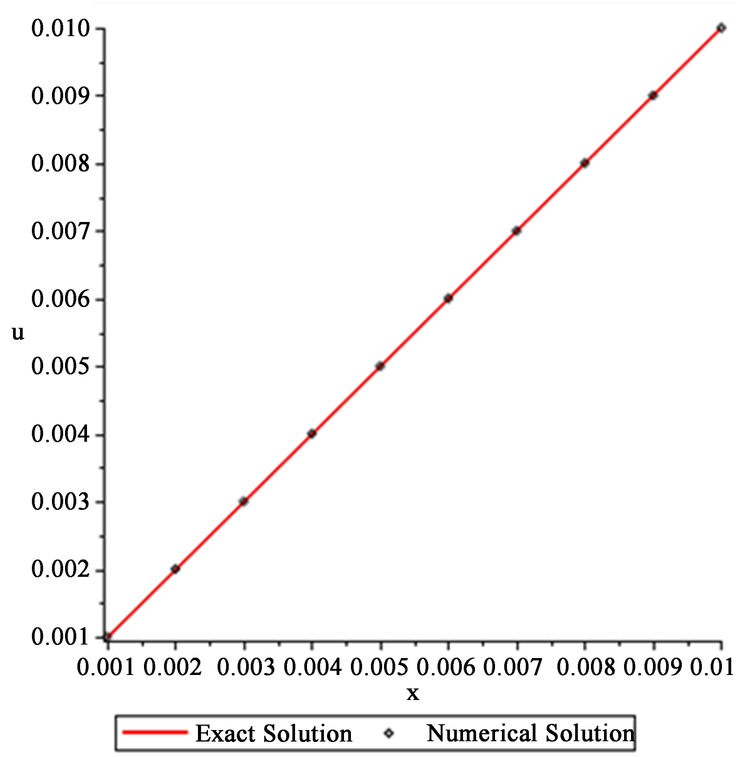

Figure 2. Graph showing the correspondence between exact and approximate solutions result of time-fractional Burgers equation in Example 2.

Table 1. Numerical results and Exact solution of one-dimensional Time Fractional Burgers equation for Example 1.

x	y Exac	z Numerical	Error
0.00100000	0.00099900	0.00100000	0.00000100
0.00200000	0.00199600	0.00200000	0.00000400
0.00300000	0.00299101	0.00300000	0.00000899
0.00400000	0.00398402	0.00399999	0.00001597
0.00500000	0.00497504	0.00499998	0.00002494
0.00600000	0.00596407	0.00599996	0.00003589
0.00700000	0.00695111	0.00699994	0.00004883
0.00800000	0.00793617	0.00799991	0.00006374
0.00900000	0.00891924	0.00899988	0.00008064
0.01000000	0.00990033	0.00999983	0.00009950

Table 2. Numerical results and Exact solution of one-dimensional Time Fractional Burgers equation for Example 2.

x	y Exac	z Numerical	Error
0.00100000	0.00099998	0.00100000.	0.00000002
0.00200000	0.00199988	0.00200000	0.00000012

Continued

0.00300000	0.00299967	0.00300000	0.00000033
0.00400000	0.00399931	0.00399999	0.00000067
0.00500000	0.00499880	0.00499998	0.00000118
0.00600000	0.00599811	0.00599996	0.00000186
0.00700000	0.00699721	0.00699994	0.00000273
0.00800000	0.00799610	0.00799991	0.00000382
0.00900000	0.00899476	0.00899988	0.00000512
0.01000000	0.00999317	0.00999983	0.00000666

6. Conclusion

The Adomian decomposition method is used in Maple 18 to solve the time-fractional Burgers equation. The results were compared with the exact solution corresponding to the Berger fractional time equation by comparing the numerical results. This demonstrated the effectiveness of the procedure and the capability of using Maple 18 software to quickly and effectively create a numerical solution that was related to the exact solution while noting the error value, making the accuracy of the solutions obtained very satisfactory. We can see that the numerical solution is frequently linked to the exact solution. Most engineering and mathematics topics can be numerically calculated with Maple 18. As Mabel 18 is an arithmetic system and a programming language at the same time. In addition, the solution has been graphically displayed. Using the package version of Mabel, these results are displayed in **Table 1**, **Table 2**, **Figure 1** and **Figure 2**. **Table 1**, **Table 2**, **Figure 1** and **Figure 2** show the difference between exact solutions and numerical solutions using the Adomian decomposition method by Mabel program; we were able to get close to the exact solutions of the equations. The findings demonstrate how effective the current approach is for locating numerical and exact solutions for fractional-time Burger equations. The primary goal of this study is to automate Adomian decomposition method calculation using the Maple software. By doing this, we might be able to obtain preliminary estimations of the solutions, which will make it simpler to employ Mabel in the future to address more challenging Issues.

7. Recommendations for Future Research

A strong numerical technique for resolving partial differential equations, such as the Time Fractional Burgers Equation, is the Adomian Decomposition Method (ADM). The equation is broken down into an endless series of functions using the non-iterative ADM approach, which is simple to solve using Maple.

The fact that ADM does not necessitate the discretization of the domain makes it computationally effective for dealing with complex issues, which is one of its key merits. ADM also works with nonlinear equations and delivers precise results with just a few terms in the decomposition series.

However, ADM has some limitations that should be taken into account. First, the convergence of the series may not be guaranteed for certain problems, which can lead to inaccurate results. Second, ADM may not be suitable for problems with irregular geometries or complex boundary conditions.

Therefore, future research should focus on improving the convergence properties of ADM and extending its applicability to more complex problems. This can be achieved by combining ADM with other numerical methods, such as finite element methods or spectral methods, which can improve the accuracy and efficiency of the solution.

In addition, it is important to investigate the stability and error analysis of ADM for solving the Time Fractional Burgers Equation. This can help to determine the optimal number of terms in the decomposition series and provide guidelines for choosing the appropriate parameters in the method.

Finally, the application of ADM to real-world problems, such as fluid dynamics or heat transfer, can provide valuable insights into the performance and limitations of the method. This can help to identify the potential applications of ADM in various fields and guide the development of more advanced numerical methods.

Acknowledgements

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University. The author, therefore, acknowledges with thanks DSR's technical and financial support.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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