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Painlevé Analysis for (2 + 1) Dimensional Non-Linear Schrödinger Equation

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Abstract

This paper investigates a real version of a (2 + 1) dimensional nonlinear Schrödinger equation through adoption of Painlevé test by means of which the (2 + 1) dimensional nonlinear Schrödinger equation is studied according to the Weiss *et al.* method and Kruskal's simplification algorithms. According to Painlevé test, it is found that the number of arbitrary functions required for explaining the Cauchy-Kovalevskaya theorem exist. Finally, the associated Bäcklund transformation and bilinear form is directly obtained from the Painlevé test.

Keywords

(2 + 1) Dimensional Nonlinear Schrödinger Equation, Painlevé Analysis, Bäcklund Transformation, Bilinear Form

1. Introduction

One of the most powerful nonlinear equations that has been discussed and used in different field of physics is the Non-linear Schrödinger equation (NLSE) in (2 + 1) dimension. The structure and integrability of (2 + 1) dimensional equations have received considerable attention in the last few years [1] [2]. Non-linear Schrödinger type equations are a particular case of interest. These equations were discovered by Calogero [3] and then discussed by Zakharov [4]. The geometrical properties of these equations have been studied [5].

Historically, more and more problems involved nonlinearity; much attention had been paid to the integrability of nonlinear models. Close relationships were observed between Painlevé test and the integrability of non-linear Partial Differential Equations (PDEs). Therefore it carries significant importance to investigate whether the non-linear PDEs have the Painlevé characteristics. In order to verify this proposition various methods have been historically used by researchers like Ablowitz-Ramani-Segur (ARS) method [6], the Weiss-Tabor-Carnevale (WTC) method [7], the Kruskal simplification method [8], Conte invariant method [9] and Pickering approach method [10].

Various methods have verified Painlevé property but the Kruskal's simplification and the Weiss-Tabor-Carnevale (WTC) methods are frequently employed for this verification. WTC method is significantly helpful in Painlevé test where non-linear PDEs are involved as tests of symmetry, Hirota's method of bilinear forms and special solutions. However, this is less significant when applied to the coupled equation system or equation where high resonance is observable. This study proposes another prospective approach *i.e.* to apply WTC method with Kruskal's simplification for verification of Painlevé test.

By observing chaotic behavior of certain systems and their non-integrability, significant advancement had been observed with respect to research on the integrability of non-linear PDEs thus leading to analysis of manifold classes of equations employing Painlevé test as a base [11] [12] [13]. Painlevé test had being used in various situations, although was neither observed as a necessity nor a condition for integrability of PDEs. The current study proposes a Painleve test analysis using (2 + 1) dimensions, for testing the integrability of Non-linear Schrödinger equation (NLSE). It was observed that such approach could satisfy requirements of Cauchy-Kovalevskaya theorem while providing satisfactory results.

The (2 + 1) dimensional NLSE under taken in this study is:

$$\begin{aligned} &p_z - \mu^2 p_{xx} - p_{yy} - pS - \delta p^2 q = 0, \\ &q_z + \mu^2 q_{xx} + q_{yy} + qS + \delta q^2 p = 0, \\ &S_{xx} - \mu^2 S_{yy} + 2\delta (pq)_{xx} = 0. \end{aligned} \tag{1}$$

In Equation (1), S is called a forcing term which shows the real quantity and $\mu^2 = 1$ or $\mu^2 = -1$ prescribes the elliptic or hyperbolic nature respectively.

2. Painlevé Analysis

Painlevé test is frequently used to determine integrability of non-linear equations. A PDE which lacks any movable dimension (Algebraic or Logarithmic) is called Painlevé-type (P-type) [14]. An ordinary differential equation (ODE) carries movable singularities without having movable branch points which place the same ODE into P-type equations. P-test to analyze the singularities structure for ODE was explained by Ablowitz *et al.* [15], further extended by Weiss-Tabor-Carnevale (WTC) to PDE [16] the method provides a great basis for investigation of integrability of various non-linear equations. A brief procedure for WTC is explained below.

Firstly, perform the leading order analysis, secondly, identify resonance position of arbitrary functions as per Laurent series and finally, existence of a significant number of arbitrary functions are verified without the introduction of movable critical manifold.

3. (2 + 1) Dimensional Non-Linear Schrodinger Equation

In this paper the Painlevé test is performed for (2 + 1) dimensional non-linear Schrödinger equation Equation (1) as described in three steps constituting the Kruskal's simplification algorithms such as determination of leading order, determination of resonance position and verification of the resonance conditions.

3.1. Leading Order Analysis

For estimation of leading singularity let's suppose that:

$$p \sim p_0\xi^{\alpha_1},\ q \sim q_0\xi^{\alpha_2},\ s \sim s_0\xi^{\alpha_3} \tag{2}$$

whereas $\alpha_j\left(j=1,2,3\right)$ will be all negative integers, $\xi = x - g\left(y,z\right)$ and $\left(p_0, q_0, s_0\right)$ all are functions of $\left(y,z\right)$. The same assumption were used by Jimbo *et al.* [17] and Goldstein & Infeld [18] the same were previously proposed in private communication by M. Kruskal [19]. As per original explanation of Weiss *et al.*, [17] ξ is a function of $\left(x,y,z\right)$. Substituting Equation (2) in Equation (1) followed by balancing the leading term of ξ, and considering that these exponents will be all negative integers, get the only possible values *i.e.* $\alpha_1 = \alpha_2 = -1$ and $\alpha_3 = -2$.

According to the condition

$$\delta p_0 q_0 = 2\left(\mu^2\xi_x^2 - \xi_y^2\right),\ s_0 = -4\mu^2\xi_x^2. \tag{3}$$

3.2. Determine the Resonance Position

As per WTC method, the system of Equation (1) which could satisfy the Painlevé test with the general solution is presented as:

$$P = \sum\nolimits_{i=0}^{\infty} p_i\xi^{i+\alpha_1},\ q = \sum\nolimits_{i=0}^{\infty} q_i\xi^{i+\alpha_2},\ S = \sum\nolimits_{i=0}^{\infty} s_i\xi^{i+\alpha_3}. \tag{4}$$

This equation contains six arbitrary functions with p_i, q_i, s_i in addition to the increasing function ξ, where the final solution is a single value regarding the arbitrary movable non-characteristic singular manifold as shown in the Laurent series.

Now substituting Equation (2), Equation (3) and Equation (4) into Equation (1) and equating the coefficients of $\left(\xi^{i-3}, \xi^{i-3}, \xi^{i-4}\right)$ we get

$$\begin{bmatrix} \delta q_0^2 & G - 4\xi_y^2 & q_0 \\ -G + 4\xi_y^2 & -\delta p_0^2 & -p_0 \\ 2\delta q_0 I\xi_x^2 & 2\delta p_0 I\xi_x^2 & I\left(\xi_x^2 - \mu^2\xi_y^2\right) \end{bmatrix} \begin{bmatrix} p_i \\ q_i \\ s_i \end{bmatrix} = 0, \tag{5}$$

where

$$G = \left(i-1\right)\left(i-2\right)\left(\mu^2\xi_x^2 + \xi_y^2\right)$$

And

$$I = \left(i-2\right)\left(i-3\right)$$

That value at which the determinant of the system matrix vanishes is called resonance position. It is establish that

$$\det[.] = i(i+1)(i-2)(i-3)^2(i-4)$$

Therefore we get the resonances at

$$i = -1, 0, 2, 3, 3, 4. \tag{6}$$

3.3. Coefficient of Expansion at the Resonance Position

Resonance at first position $i = -1$ is observed to relate the arbitrary singularity manifold ξ. For the Equation (1) to pass the Painlevé test, it is required that second resonance position at $i = 0$, third at $i = 2$, fourth & fifth at $i = 3$, 3 and sixth at $i = 4$ should be identically satisfied. We utilize the Kruskal's simplification for verification of resonance conditions, such as

$$\xi = x + g(y,z),\ \ p_i = p_i(y,z),\ \ q_i = q_i(y,z),\ \ s_i = s_i(y,z). \tag{7}$$

With the resolution of Equation (7), Equation (3) is decreased to

$$\delta p_0 q_0 = 2\left(\mu^2 - g_y^2\right),\ s_0 = -4\mu^2 \tag{8}$$

<u>**Case a:**</u> For $i = -1$.

This point to the arbitrariness of

$$\xi = x + g(y,z) \tag{9}$$

<u>**Case b:**</u> For $i = 0$.

This point to the arbitrariness of p_0 or q_0 (See Equation (3))

<u>**Case c:**</u> For $i = 1$,

Equating the coefficients of $\left(\xi^{-2}, \xi^{-2}, \xi^{-3}\right)$ we get,

$$p_1 = \frac{\left(\mu^2\left(3g_{yy} - g_z\right)g_y^2 + g_z + g_{yy}\right)q_0 + 2q_{0y}g_y\left(1 - \mu^2 g_y^2\right)}{\delta q_0^2\left(1 + \mu^2 g_y^2\right)} \tag{10}$$

$$q_1 = \frac{\mu^2\left(q_0 g_{yy} - q_0 g_z - 2q_{0y}g_y\right)}{\left(1 + \mu^2 g_y^2\right)} \tag{11}$$

$$s_1 = 0 \tag{12}$$

<u>**Case d**</u>: For $i = 2$,

Eliminating the coefficients of $\left(\xi^{-1}, \xi^{-1}, \xi^{-2}\right)$ we get,

$$p_{0z} + 4g_y^2 p_2 - \delta\left(q_2 p_0^2 + q_0 p_1^2\right) - p_0 s_2 - p_{0yy} - \left(s_1 + 2\delta q_1 p_0\right)p_1 = 0 \tag{13}$$

$$q_{0z} - 4g_y^2 q_2 + \delta\left(p_2 q_0^2 + p_0 q_1^2\right) + q_0 s_2 + q_{0yy} + \left(s_1 + 2\delta p_1 q_0\right)q_1 = 0 \tag{14}$$

$$s_{0yy} - 2s_{1y}g_y - s_1 g_{yy} = 0 \tag{15}$$

From Equation (8) it is observed that Equation (15) is identically satisfied viva Equation (8) and (12).

And hence there are only two Equations (13) and (14) for three unknown variables p_2, q_2 and s_2 which clarifies that one of them is arbitrary. Therefore the resonance condition at $i = 2$ is satisfied.

<u>**Case e:**</u> For $i = 3$ (double resonance).

By collecting the coefficients of $\left(\xi^0, \xi^0, \xi^{-1}\right)$ and according to Equation (8) we get,

$$\begin{aligned}&2\left(\mu^2-g_y^2\right)p_3+\left(g_{yy}+2\delta q_1 p_0+2\delta q_0 p_1-g_z+s_1\right)p_2+2p_{2y}g_y+p_{1yy}\\&+\left(\delta q_3 p_0+s_3\right)p_0+\left(\delta q_1 p_1+s_2+2\delta q_2 p_0\right)p_1-p_{1z}=0\end{aligned} \tag{16}$$

$$\begin{aligned}&2\left(\mu^2-g_y^2\right)q_3+\left(g_{yy}+2\delta q_0 p_1+2\delta q_1 p_0+g_z+s_1\right)q_2+2q_{2y}g_y+q_{1yy}\\&+\left(\delta p_3 q_0+s_3\right)q_0+\left(\delta p_1 q_1+s_2+2\delta p_2 q_0\right)q_1-q_{1z}=0\end{aligned} \tag{17}$$

$$s_{1yy}=0 \tag{18}$$

Obviously due to Equation (12), Equation (18) is identically satisfied. Now we solve s_3 from Equation (17) we get,

$$\begin{aligned}s_3=p_0^{-1}\Big[&p_{1z}-2\left(\mu^2-g_y^2\right)p_3-\left(g_{yy}+2\delta q_1 p_0+2\delta q_0 p_1-g_t+s_1\right)p_2\\&-\left(\delta q_1 p_1+s_2+2\delta q_2 p_0\right)p_1-p_{1yy}-2p_{2y}g_y-\delta q_3 p_0^2\Big]\end{aligned} \tag{19}$$

Now putting Equation (19) into Equation (17) and using Equations (10)-(14), we find that Equation (17) is satisfied identically, which shows that p_3 and q_3 must be arbitrary if s_3 is determined by Equation (19). Hence the resonance condition at $i = 3$ is satisfied.

Case f: for $i = 4$.

Vanishing the coefficients of $\left(\xi, \xi, \xi^0\right)$ and according to Equation (8) and Equation (11) we get,

$$\begin{aligned}&\left(3\mu^2+g_y^2\right)p_4+\delta p_0^2 q_4+p_0 s_4+\left(2\delta q_1 p_0 2g_z+2g_{yy}\right)p_3+4p_{3y}g_y-p_{2z}+p_{2yy}\\&+\left(2\delta q_3 p_0+s_3+q_2 p_1\right)p_1+\left(\delta q_0 p_2+s_2+2\delta p_1 q_1+2\delta q_2 p_0\right)p_2=0\end{aligned} \tag{20}$$

$$\begin{aligned}&\left(3\mu^2+g_y^2\right)q_4+\delta q_0^2 p_4+q_0 s_4+\left(2\delta q_0 p_1-2g_z+2g_{yy}\right)q_3+4q_{3y}g_y+q_{2t}+q_{2yy}\\&+\left(2\delta p_3 q_0+s_3+q_1 p_2\right)q_1+\left(\delta p_0 q_2+s_2+2\delta p_1 q_1+2\delta q_0 p_2\right)q_2=0\end{aligned} \tag{21}$$

$$4\delta\mu^2\left(q_0 p_4+p_0 q_4+p_3 q_1+p_2 q_2+p_1 q_3\right)+2\left(\mu^2-f_y^2\right)s_4-2s_{3y}f_y-s_3 f_{yy}-s_{2yy}=0 \tag{22}$$

Firstly we simplify Equations (20) and (21) for p_4 and q_4 and then putting into Equation (22) according to Equations (8) (10) (11) (13) (14), and (19) we can find that Equation (22) is identically satisfied.

So any one of p_4, q_4 and s_4 is arbitrary. Therefore the resonance condition at $i = 4$ is also satisfied.

Hence we proved that the Equation (1) satisfies the Painlevé test.

4. Associated Bäcklund Transformation

To construct the Bäcklund transformation of Equation (1), let us truncate the Laurent series at the constant level term to give

$$p=\frac{p_0}{\xi}+p_1,\ q=\frac{q_0}{\xi}+q_1,\ S=\frac{s_0}{\xi^2}+\frac{s_1}{\xi}+s_2. \tag{23}$$

Here $p_1=p_1\left(x,y,z\right)$, $q_1=q_1\left(x,y,z\right)$ and $s_1=s_1\left(x,y,z\right)$, $s_2=s_2\left(x,y,z\right)$. where the pair of functions $\left(p,p_1\right)$; $\left(q,q_1\right)$ and $\left(s,s_2\right)$ satisfies Equation (1), and hence Equation (23) may be treated as the associated Bäcklund transformation of Equation (1).

5. Bilinear Form

In order to derive the Hirota's bilinear form we consider the vacuum solutions $p_1 = q_1 = s_2 = 0$ in Equation (23) then we have

$$p = \frac{p_0}{\xi},\ q = \frac{q_0}{\xi},\ S = \frac{s_0}{\xi^2} + \frac{s_1}{\xi}. \tag{24}$$

This suggest that we take the Hirota's bilinear transformation in the form

$$P = \frac{V}{U},\ q = \frac{W}{U},\ S = 4\mu^2 \left(\ln U\right)_{xx}. \tag{25}$$

In Equation (25) U, V and W are functions of variables x, y and z where U is a real function and V, W are complex functions. Using Equation (25) and the Hirota's bilinear operator Equation (24), Equation (1) can be transforming into bilinear forms as,

$$\left(D_z - \mu^2 D_x^2 - D_y^2\right) W \cdot U = 0, \tag{26}$$

$$\left(D_z + \mu^2 D_x^2 + D_y^2\right) V \cdot U = 0, \tag{27}$$

$$\left(\mu^2 D_x^2 - D_y^2\right) U \cdot U + \delta V \cdot W = 0. \tag{28}$$

Here D is called the bilinear operator defined by [20].

The power-series can be further used with the bilinear form obtained in this paper to construct a soliton solution by expanding the dependent variables.

6. Conclusions

A demonstration on the system which passes the Painlevé test is presented in this paper and further explained that the system has multi-linear variable solutions having arbitrary function, also the associated Bäcklund transformation and bilinear form are obtained directly from the Painlevé test.

The analysis points that our equation having six positions of resonance at $i = -1, 0, 2, 3, 3, 4$, there are six arbitrary functions at these resonance positions. Therefore our equation conform the Painlevé criterion of integrability as explained in Cauchy-Kovalevskaya theorem. However, there is one further proposition of this paper that Painlevé analysis cannot be used to derive the Lax pair because in the case of coupled system there is still no specialized method for derivation of the Lax pair for coupled non-linear equations.

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