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Hopf Modules in the Category of Yetter-Drinfeld Modules

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Abstract

We give the Fundamental Theorem for Hopf modules in the category of Yetter-Drinfeld modules ${}_L^L\mathcal{YD}$, where L is a quasitriangular weak Hopf algebra with a bijective antipode. We also show that H^* has a right H-Hopf module structure in the Yetter-Drinfeld category. As an application we deduce the existence and uniqueness of right integral from it.

Keywords

Weak Hopf Algebra, Hopf Module, Fundamental Theorem

1. Introduction

Weak Hopf algebras were introduced by G. Böhm and K. Szlachányi as a generalization of usual Hopf algebras and groupoid algebras [1] [2]. A weak Hopf algebra is a vector space that has both algebra and coalgebra structures related to each other in a certain self-dual fashion and possesses an analogue of the linearized inverse map [3]-[5]. The main difference between ordinary and weak Hopf algebras comes from the fact that the comultiplication of the latter is no longer required to preserve the unit (equivalently, the counit is not requires to be a homomorphism) and results in the existence of two canonical subalgebras playing the role of "non-commutative bases".

Paper [6] was shown what is a weak Hopf algebra in the braided category of modules over a weak Hopf algebra. In [7] we prove a Fundamental Theorem of Hopf modules for the categorical weak Hopf algebra motivation to study quasitriangular weak Hopf algebras is the so-called biproduct construction and interpreted in the terms of braided categories. More precisely, we are interested in a specific type of quaitriangular weak Hopf algebras.

we prove the Fundamental Theorem for Hopf modules in the category of Yetter-Drinfeld modules according to the fact that the matrix R gives rise to a natural braiding for ${}_L\mathcal{M}$ and ${}_L^L\mathcal{YD}$. Furthermore H^* is also a right H-Hopf module in the category Yetter-Drinfeld modules. Using this result we obtain the existence and uniqueness of integrals for a finite dimensional weak Hopf algebra in ${}_L^L\mathcal{YD}$.

2. Preliminaries

Throughout this paper we use Sweedler's notation for comultiplication, writing $\Delta(h) = h_1 \otimes h_2$. Let k be a fixed field and all weak Hopf algebras are finite dimensional.

Definition 1. A weak Hopf algebra is a vector space L with the structure of an associative unital algebra (L, m, μ) with multiplication $m: L \otimes L \to L$ and unit $1 \in L$ and a coassociative coalgebra (L, Δ, ε) with comultiplication $\Delta: L \to L \otimes L$ and counit $\varepsilon: L \to k$ such that

1) The comultiplication Δ is a (not necessarily unit-preserving) homomorphism of algebras such that

$$(\Delta \otimes id)\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1).$$

2) The counit satisfies the following identity

$$\varepsilon(kgl) = \varepsilon(kg_1)\varepsilon(g_2 l) = \varepsilon(kg_2)\varepsilon(g_1 l), \quad \forall k, g, l \in L.$$

3) There is a linear map $S_L: L \to L$ called an antipode, such that, for all $l \in L$

$$m(id \otimes S_L)\Delta(l) = (\varepsilon \otimes id)(\Delta(1)(l \otimes 1)),$$

$$m(S_L \otimes id)\Delta(l) = (id \otimes \varepsilon)((1 \otimes l)\Delta(1)),$$

$$S_L(l) = S_L(l_1) l_2 S_L(l_3).$$

The linear map defined in the above equations are called target and source counital maps and denoted by ε_t and ε_s respectively:

$$\varepsilon_t(l) = \varepsilon(1_{(1)} l) 1_{(2)} = \varepsilon(S_L(l) 1_{(1)}) 1_{(2)},$$

$$\varepsilon_s(l) = 1_{(1)} \varepsilon(l 1_{(2)}) = 1_{(1)} \varepsilon(1_{(2)} S_L(l)).$$

For all $l \in L$, we have

$$l_1 \otimes \varepsilon_t(l_2) = 1_{(1)} l \otimes 1_{(2)}, \quad \varepsilon_s(l_1) \otimes l_2 = 1_{(1)} \otimes l 1_{(2)},$$

$$l_1 \otimes \varepsilon_s(l_2) = l 1_{(1)} \otimes S_L(1_{(2)}), \quad \varepsilon_t(l_1) \otimes l_2 = S_L(1_{(1)}) \otimes 1_{(2)} l.$$

We will briefly recall the necessary definitions and notions on the weak Hopf algebras.

Definition 2. A quasitriangular weak Hopf algebra is a pair (L, R) where L is a weak Hopf algebra and $R \in \Delta^{op}(1)(L \otimes L)\Delta(1)$ (called the R-matrix) satisfying the following conditions:

$$\Delta^{op}(l) R = R\Delta(l)$$

for all $l \in L$, where Δ^{op} denotes the conditions apposite to Δ,

$$(id \otimes \Delta)(R) = R_{13} R_{12},$$

$$(\Delta \otimes id)(R) = R_{13} R_{23}.$$

where $R_{12} = R \otimes 1, R_{23} = 1 \otimes R$, etc. as usual, and such that there exits $\bar{R} \in \Delta(1)(L \otimes L)\Delta^{op}(1)$ with $R\bar{R} = \Delta^{op}(1), \bar{R}R = \Delta(1)$. where we write $R = R^1 \otimes R^2 = r^1 \otimes r^2 = T^1 \otimes T^2$. By [3], we can obtain the following results.

Proposition 2.1. For any quasitriangular weak Hopf algebra (L, R), we have

$$(\varepsilon_s \otimes id)(R) = \Delta(1), \quad (id \otimes \varepsilon_s)(R) = (S_L^{-1} \otimes id)\Delta^{op}(1),$$

$$(\varepsilon_t \otimes id)(R) = \Delta^{op}(1), \quad (id \otimes \varepsilon_t)(R) = (S_L^{-1} \otimes id)\Delta(1),$$

$$(S_L \otimes id)(R) = (id \otimes S_L^{-1})(R) = \bar{R},$$

$$(S_L \otimes S_L)(R) = R, \quad (\varepsilon \otimes id)(R) = (id \otimes R) = 1.$$

3. Weak Hopf Algebras in the Yetter-Drinfeld Module Category

Let L be a quasitriangular weak Hopf algebra with a bijective antipode S_L. Suppose H is a weak Hopf algebra in ${}_L\mathscr{M}$. Paper [7] show that H is also a weak Hopf algebra in ${}_L^L\mathscr{YD}$ with a left L-coaction via $\sigma_H : H \to L \otimes H, h \mapsto h^{-1} \otimes h^0 = R^2 \otimes R^1 \to h$. Bing-liang and Shuan-hong introduce the definition of Weak Hopf algebra in the braided monoidal category ${}_L^L\mathscr{YD}$ in [6]. Moreover they have showed that if H is a finite-dimensional weak Hopf algebra in ${}_L^L\mathscr{YD}$, then its dual H^* is a weak Hopf algebra in ${}_L^L\mathscr{YD}$.

Definition 3. Let (L, R) be a quasitriangular weak Hopf algebra. An object $H \in {}_L^L\mathscr{YD}$ is called a weak bialgebra in this category if it is both an algebra and a coalgebra satisfying the following conditions:

1) Δ and ε are not necessarily unit-preserving, such that

$$\Delta(xy) = x_1\left(R^2 \to y_1\right) \otimes \left(R^1 \to x_2\right) y_2,$$

$$\varepsilon(xyz) = \varepsilon(xy_1)\varepsilon(y_2 z) = \varepsilon\left(x\left(R^2 \to y_2\right)\right)\varepsilon\left(\left(R^1 \to y_1\right)z\right),$$

$$\Delta^2(1) = 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes \left(R^2 \to 1'_1\right)\left(R^1 \to 1_2\right) \otimes 1'_2.$$

2) H is a left L-module algebra and left L-module coalgebra if H is a left L-module via $l \otimes x \mapsto l \to x$ such that

$$l \to xy = \left(l_1 \to x\right)\left(l_2 \to y\right),\ l \to 1 = \varepsilon_t(l) \to 1,\ x, y \in H, l \in L,$$

$$\Delta(l \to x) = \left(l_1 \to x_1\right) \otimes \left(l_2 \to x_2\right), \varepsilon_s(l) \to x = x_1 \varepsilon\left(l \to x_2\right).$$

3) H is a left L-comodule algebra and left L-comodule coalgebra if H is a left L-comodule via $x \mapsto \sigma_H(x) = x^{-1} \otimes x^0 = R^2 \otimes R^1 \to x$ such that

$$\sigma_H(xy) = x^{-1}y^{-1} \otimes x^0 y^0 = R^2 r^2 \otimes \left(R^1 \to x\right)\left(r^1 \to y\right),$$

$$R^2 \otimes x\left(R^1 \to 1\right) = \varepsilon_s\left(R^2\right) \otimes \left(R^1 \to x\right),$$

$$R^2 \otimes \left(R^1 \to x\right)_1 \otimes \left(R^1 \to x\right)_2 = R^2 r^2 \otimes R^1 \to x_1 \otimes r^1 \to x_2,$$

$$\varepsilon\left(R^1 \to x\right) R^2 = \varepsilon\left(R^1 \to x\right)\varepsilon_t\left(R^2\right).$$

4) Furthermore, H is called a weak Hopf algebra in ${}_L^l\mathscr{YD}$ if there exists an antipode $S : H \to H$ (here S is left L-linear and left L-colinear *i.e.*, S is a morphism in the category of ${}_L^L\mathscr{YD}$) satisfying

$$x_1 S\left(x_2\right) = \varepsilon\left(\left(R^2 \to 1_1\right)\left(R^1 \to x\right)\right)1_2,$$

$$S\left(x_1\right)x_2 = 1_1 \varepsilon\left(\left(R^2 \to x\right)\left(R^1 \to 1_2\right)\right),$$

$$S\left(x_1\right)x_2 S\left(x_3\right) = S(x),\ \forall x \in H.$$

Similar to the definition of weak Hopf algebra, we denote $\varepsilon_t(x) = x_1 S\left(x_2\right)$, $\varepsilon_s(x) = S\left(x_1\right)x_2$. If $x = 1$ one can obtain $\varepsilon_t(1) = \varepsilon_s(1) = 1$. According to the definitions of $\varepsilon_t, \varepsilon_s$ one obtains explicit expressions for these coproducts

$$\Delta\left(\varepsilon_t(x)\right) = \varepsilon_t(x)1_1 \otimes 1_2, \quad \Delta\left(\varepsilon(x)\right) = 1_1 \otimes 1_2 \varepsilon_s(x).$$

Paper [7] give the following results:

Proposition 3.1. Suppose H is a weak Hopf algebra in ${}_L^L\mathscr{YD}$. For all $x \in H$ we have the identities

$$x_1 \otimes \varepsilon_s\left(x_2\right) = x 1_1 \otimes S\left(1_2\right), \quad \varepsilon_t\left(x_1\right) \otimes x_2 = S\left(1_1\right) \otimes 1_2 x.$$

Since a weak Hopf algebra H in the weak Yetter-Drinfeld categories ${}_L^L\mathscr{YD}$ is both algebra and coalgebra, one can consider modules and comodules over H. As in the theory of Hopf algebras, an H-Hopf module is an H-module which is also an H-comodule such that these two structures are compatible (the action "commutes" with coaction):

Definition 4. Let H be a weak Hopf algebra in ${}_L^L\mathscr{YD}$. A right H-Hopf module M in ${}_L^L\mathscr{YD}$ is an object $M \in {}_L^L\mathscr{YD}$ such that it is both a right H-module $\varphi_M : M \otimes H \to M$ and a right H-comodule via

$\rho_M : M \to M \otimes H, \rho_M(m) = m_0 \otimes m_1$ and the following equations hold for $m \in M, h \in H, l \in L$:

1) $\rho_M(mh) = m_0\left(R^2 \to h_1\right) \otimes \left(R^1 \to m_1\right) h_2$,

2) $l \to (mh) = (l_1 \to m)(l_2 \to h)$,

3) $R^2 \otimes \left(R^1 \to m\right)_0 \otimes \left(R^1 \to m\right)_1 = R^2 r^2 \otimes \left(R^1 \to m_0\right) \otimes \left(r^1 \to m_1\right)$,

4) $\sigma_M(mh) = R^2 r^2 \otimes \left(R^1 \to m\right)\left(r^1 \to h\right)$,

5) $\rho_M(l \to m) = (l_1 \to m_0)(l_2 \to m_1)$.

We remark that $M \otimes_t H$ is a right H-module by $(m \otimes h)x = m\left(R^2 \to x_1\right) \otimes \left(R^1 \to h\right) x_2$ and a right H-comodule $\rho_{M\otimes H}(m \otimes h) = m_0 \otimes R^2 \to h_1 \otimes \left(R^1 \to m_1\right) h_2$. The condition (1) means that the H-comodule structure $\rho_M : M \to M \otimes H$ is H-linear, or equivalently the H-module structure map $\varphi_M : M \otimes H \to M$ is H-colinear. Also, (4) (resp. (2)) $\Leftrightarrow$ φ_M is L-colinear (resp. L-linear); (3)(resp. (5)) $\Leftrightarrow$ ρ_M is L-colinear (resp. L-linear).

Example 3.2. H itself is a right H-Hopf module (in ${}_L^L\mathcal{YD}$) in the natural way. If V is an object in ${}_L^L\mathcal{YD}$, then so is $V \otimes_t H$ by $l \to (v \otimes h) = (l_1 \to v) \otimes (l_2 \to h)$ and $\sigma_{V\otimes H}(v \otimes h) = R^2 r^2 \otimes \left(R^1 \to v\right) \otimes \left(r^1 \to h\right)$. It is also both a right H-module and a right H-comodule by $(v \otimes h)x = v \otimes hx$ and $\rho_{V\otimes H}(v \otimes h) = v \otimes h_1 \otimes h_2$. One easily checks that $V \otimes_t H$ is an right H-Hopf module.

when H is a weak Hopf algebra in ${}_L\mathcal{M}$ and M a right H-Hopf module in ${}_L\mathcal{M}$, we prove the Fundamental Theorem 3.3 [7]. Furthermore we will show $M^{coH} = \left\{m \in M \mid \rho_M(m) = m1_1 \otimes 1_2\right\}$ is a L-subcomodule of M.

Applying $R^2 \otimes R^1 \to 1 = R^2 \otimes \varepsilon_t\left(R^1\right) \to 1 = 1_{(1)} \otimes \left(1_{(2)} \to 1\right)$ we obtain

$$\begin{aligned} R^2 \otimes \left(R^1 \to 1\right)_1 \otimes \left(R^1 \to 1\right)_2 &= 1_{(1)} \otimes \left(1_{(2)} \to 1_1\right) \otimes 1_2 \\ &= 1_{(1)} \otimes \left(\varepsilon_t\left(1_{(2)}\right) \to 1_1\right) \otimes 1_2 \\ &= 1_{(1)} \otimes \left(1_{(2)} \to 1\right) 1_1 \otimes 1_2 \\ &= R^2 \otimes \left(R^1 \to 1\right) 1_1 \otimes 1_2. \end{aligned}$$

For $n \in M^{coH}$ we do a calculation:

$$\begin{aligned} n^{-1} \otimes \left(n^0\right)_1 \otimes \left(n^0\right)_2 &= R^2 \otimes \left(R^1 \to n\right)_0 \otimes \left(R^1 \to n\right)_1 \\ &= R^2 r^2 \otimes \left(R^1 \to n1_1\right) \otimes \left(r^1 \to 1_2\right) \\ &= R^2 T^2 r^2 \otimes \left(R^1 \to n\right)\left(T^1 \to 1_1\right) \otimes \left(r^1 \to 1_2\right) \\ &= R^2 r^2 \otimes \left(R^1 \to n\right)\left(r^1 \to 1\right)_1 \otimes \left(r^1 \to 1\right)_2 \\ &= R^2 r^2 \otimes \left(R^1 \to n\right)\left(r^1 \to 1\right) 1_1 \otimes 1_2 \\ &= R^2 \otimes \left(R^1 \to n\right) 1_1 \otimes 1_2 \\ &= n^{-1} \otimes n^0 1_1 \otimes 1_2. \end{aligned}$$

This implies that $\sigma(n) = n^{-1} \otimes n^0 \in L \otimes M^{coH}$. So $M^{coH} \in {}_L^L\mathcal{YD}$.

It is clearly to prove F is a left L-colinear by the following equation

$$\sigma(F(n \otimes h)) = \sigma(nh) = n1_{(1)} R^2 \otimes 1_{(2)}\left(R^1 \to h\right) = (id \otimes F)\sigma(n \otimes h).$$

Furthermore we can obtain the Structure Theorem for right H-Hopf modules in the category of Yetter-Drinfeld modules.

Theorem 3.3. If H is a weak Hopf algebra in ${}_L^L\mathcal{YD}$ and M is a right H-Hopf module in ${}_L^L\mathcal{YD}$, M^{coH} is defined as above. Then

1) Let $P(m) = m_0 S(m_1), m \in M$. Then $P(m) \in M^{coH}$. If $n \in M^{coH}$ and $h \in H$, Then $\rho_M(nh) = nh_1 \otimes h_2$ and $P(nh) = n\varepsilon_t(h)$.

2) The map $F : M^{coH} \otimes_t H \to M, F(n \otimes h) = nh$ is an isomorphism of Hopf modules. The inverse map is

given by $G(m) = P(m_0)m_1$.

4. Fundamental Theorem for H^* in ${}_L^L\mathcal{YD}$

In [4] H^* has the contragredient left L-module structure by

$$(l \to f)(h) = f(S_L(l) \to h), \quad l \in L,\ f \in H^*, h \in H.$$

Since H is a finite-dimensional left L-comodule, H^* has the transposed right L-comodule structure and so it becomes a left L-comodule via

$$\sigma_{H^*} : H^* \to L \otimes H^*, \sigma_{H^*}(f) = f^{-1} \otimes f^0 = R^2 \otimes R^1 \to f.$$

i.e. $f^0(h) f^{-1} = f(h^0) S_L^{-1}(h^{-1}) = f(R^1 \to h) S^{-1}(R^2), h \in H$. Now assume that H is finite-dimensional. We will show that H^* becomes a right H-Hopf module in ${}_L^L\mathcal{YD}$. First H^* is a right H-module by

$$(fh)(x) = f(hx), \quad f \in H^*, \ h \in H.$$

Second, H^* is a right H-comodule using the identification $\theta_H : H^* \otimes H \cong Hom(H, H)$, $\theta_H(f \otimes h)(x) = f(R^2 \to x)(R^1 \to h)$ as follows:

$$\rho_{H^*} : H^* \to Hom(H, H) \cong H^* \otimes H, \rho_{H^*}(f)(x) = f(x_1) S(x_2).$$

That is $\rho_{H^*}(f) = f_0 \otimes f_1$ means

$$f(x_1) S(x_2) = f_0(f_1^{-1} \to x) f_1^0 = f_0(R^2 \to x)(R^1 \to f_1), \quad x \in H.$$

Proposition 4.1. H^* is a right H-comodule by $\theta_{H \otimes H}$.

Proof. Now for $f \in H^*, x \in H$, we have

$$\begin{aligned}
&\theta_{H\otimes H}((f_0)_0 \otimes (f_0)_1 \otimes f_1)(x) \\
&= (f_0)_0 (r^2 \to (R^2 \to x))(r^1 \to (f_0)_1) \otimes R^1 \to f_1, \\
&= f_0((R^2)_1 \to x_1) S((R^2)_2 \to x_2) \otimes R^1 \to f_1, \\
&= f(x_1)(R^2 \to S(x_3)) \otimes R^1 \to S(x_2), \\
&= \Delta_H(f(x_1) S(x_2)), \\
&= f_0(R^2 \to x)((R^1)_1 \to (f_1)_1) \otimes ((R^1)_2 \to (f_1)_2), \\
&= f_0(R^2 r^2 \to x)(R^1 \to (f_1)_1) \otimes (r^1 \to (f_1)_2), \\
&= \theta_{H\otimes H}(f_0 \otimes (f_1)_1 \otimes (f_1)_2)(x).
\end{aligned}$$

It implies that $(f_0)_0 \otimes (f_0)_1 \otimes f_1 = f_0 \otimes (f_1)_1 \otimes (f_1)_2$.

Accord to $\rho_{H^*}(f) \in {}_L^L\mathcal{YD}$ we have $(1_{(1)} \to f_0) \otimes (1_{(2)} \to f_1) = f_0 \otimes f_1$. Applying the equality $\varepsilon(R^1 \to x) R^2 = \varepsilon(R^1 \to x) \varepsilon_t(R^2)$ we obtain

$$\begin{aligned}
f(x) &= f_0(R^2 \to x) \varepsilon(R^1 \to f_1) \\
&= f_0(\varepsilon_t(R^2) \to x) \varepsilon(R^1 \to f_1) \\
&= f_0(S_L(1_{(1)}) \to x) \varepsilon(1_{(2)} \to f_1) \\
&= (1_{(1)} \to f_0)(x) \varepsilon(1_{(2)} \to f_1), \\
&= (f_0 \varepsilon(f_1))(x).
\end{aligned}$$

Hence $(id \otimes \varepsilon)\rho_{H^*}(f) = f$. Thus H^* becomes a right H-comodule.

Theorem 4.2. With the notation as above, then H^* is a right H-Hopf module in ${}_L^L\mathcal{YD}$. Moreover,

$$\left(H^*\right)^{coH}=\left\{f\in H^*\mid f\left(x_1\right)S\left(x_2\right)=f\left(1_1x\right)1_2,\ x\in H\right\}.$$

Proof. Now we prove that H^* is a right H-Hopf module. First we will show that $\rho(fh)=f_0\left(R^2\rightarrow h_1\right)\otimes\left(R^1\rightarrow f_1\right)h_2$. Since for $x\in H$,

$$\begin{aligned}
&\theta_H\left(f_0\left(R^2\rightarrow h_1\right)\otimes\left(R^1\rightarrow f_1\right)h_2\right)(x)\\
&=f_0\left(\left(R^2\rightarrow h_1\right)\left(r^2T^2\rightarrow x\right)\right)\left(r^1R^1\rightarrow f_1\right)\left(T^1\rightarrow h_2\right)\\
&=f_0\left(R^2\rightarrow h_1\left(T^2\rightarrow x\right)\right)\left(R^1\rightarrow f_1\right)\left(T^1\rightarrow h_2\right)\\
&=f\left(\left(h_1\left(T^2\rightarrow x\right)\right)_1\right)S\left(\left(h_1\left(T^2\rightarrow x\right)\right)_2\right)\left(T^1\rightarrow h_2\right)\\
&=f\left(h_1\left(\left(R^2\right)_1\rightarrow x_1\right)\right)\left(\left(R^2\right)_2\rightarrow S\left(x_2\right)\right)\left(R^1\rightarrow\varepsilon_s\left(h_2\right)\right)\\
&=f\left(h1_1\left(R^2\rightarrow x_1\right)\right)\left(r^2\rightarrow S\left(x_2\right)\right)\left(r^1\rightarrow\left(R^1\rightarrow S\left(1_2\right)\right)\right)\\
&=\left(f\cdot h\right)\left(1_1\left(R^2\rightarrow x_1\right)\right)S\left(\left(R^1\rightarrow 1_2\right)x_2\right)=\left[\left(f\cdot h\right)_0\right]\otimes\left[\left(f\cdot h\right)_1\right](x).
\end{aligned}$$

Next we want to check $\left(l\rightarrow fh\right)=\left(l_1\rightarrow f\right)\left(l_2\rightarrow h\right)$ for $l\in L, h\in H, f\in H^*$. Since for $x\in H$

$$\begin{aligned}
&\left(\left(l_1\rightarrow f\right)\left(l_2\rightarrow h\right)\right)(x)\\
&=\left(l_1\rightarrow f\right)\left(\left(l_2\rightarrow h\right)x\right)\\
&=f\left(S_L\left(l_1\right)\rightarrow\left(\left(l_2\rightarrow h\right)x\right)\right)\\
&=f\left(\left(S_L\left(l_2\right)l_3\rightarrow h\right)\left(S_L\left(l_1\right)\rightarrow x\right)\right)\\
&=f\left(\left(\varepsilon_s\left(l_2\right)\rightarrow h\right)\left(S_L\left(l_1\right)\rightarrow x\right)\right)\\
&=f\left(\left(S_L\left(1_{(2)}\right)\rightarrow h\right)\left(S_L\left(l1_{(1)}\right)\rightarrow x\right)\right)\\
&=f\left(\left(1_{(1)}\rightarrow h\right)\left(1_{(2)}\rightarrow\left(S_L\left(l\right)\rightarrow x\right)\right)\right)\\
&=\left(fh\right)\left(S_L\left(l\right)\rightarrow x\right)=\left(l\rightarrow fh\right)(x).
\end{aligned}$$

Applying the equality $R^1S\left(l_2\right)\otimes S\left(l_1\right)R^2=l_1R^1\otimes R^2S\left(l_2\right)$ for $x\in H$

$$\begin{aligned}
&R^2r^2\otimes\theta_H\left(\left(R^1\rightarrow f_0\right)\otimes\left(r^1\rightarrow f_1\right)\right)(x)\\
&=R^2f_0\left(S\left(\left(R^1\right)_1\right)r^2\rightarrow x\right)\left(r^1\left(\left(R^1\right)_2\right)\rightarrow f_1\right)\\
&=R^2f_0\left(r^2\rightarrow\left(S\left(\left(R^1\right)_2\right)\rightarrow x\right)\right)\left(\left(R^1\right)_1r^1\rightarrow f_1\right)\\
&=R^2f\left(\left(S\left(\left(R^1\right)_2\right)\rightarrow x\right)_1\right)\left(\left(R^1\right)_1\rightarrow S\left(\left(S\left(\left(R^1\right)_2\right)\rightarrow x\right)_2\right)\right)\\
&=R^2f\left(S\left(\left(R^1\right)_3\right)\rightarrow x_1\right)\left(\left(R^1\right)_1S\left(\left(R^1\right)_2\right)\rightarrow S\left(x_2\right)\right)\\
&=R^2f\left(S\left(\left(R^1\right)_2\right)\rightarrow x_1\right)\left(\varepsilon_t\left(\left(R^1\right)_1\right)\rightarrow S\left(x_2\right)\right)\\
&=R^2f\left(S\left(R^1\right)S_L\left(1_{(2)}\right)\rightarrow x_1\right)\left(S_L\left(1_{(1)}\right)\rightarrow S\left(x_2\right)\right)\\
&=R^2\left(R^1\rightarrow f\right)\left(x_1\right)S\left(x_2\right)\\
&=R^2\otimes\theta_H\left(\left(R^1\rightarrow f\right)_0\otimes\left(R^1\rightarrow f\right)_1\right)(x).
\end{aligned}$$

It implies that $R^2 \otimes \left(R^1 \to f\right)_0 \otimes \left(R^1 \to f\right)_1 = R^2 r^2 \otimes R^1 \to f_0 \otimes r^1 \to f_1,\ \ f \in H^*$. Using the equality $l \to fh = \left(l_1 \to f\right)\left(l_2 \to h\right)$ we compute

$$\begin{aligned} f^{-1}h^{-1} \otimes f^0 h^0 &= R^2 r^2 \otimes \left(R^1 \to f\right)\left(r^1 \to h\right) \\ &= R^2 \otimes \left(R^1 \to fh\right) \\ &= (fh)^{-1} \otimes (fh)^0 \\ &= \sigma(fh). \end{aligned}$$

Finally we show that $\rho_{H^*}(l \to f) = l_1 \to f_0 \otimes l_2 \to f_1$. Since for $l \in L,\ f \in H^*,\ x \in H$

$$\begin{aligned} &\theta_H\left(\left(l_1 \to f_0\right) \otimes \left(l_2 \to f_1\right)\right)(x) \\ &= \left(l_1 \to f_0\right)\left(R^2 \to x\right)\left(R^1 l_2 \to f_1\right) \\ &= f_0\left(S_L\left(l_1\right) R^2 \to x\right)\left(R^1 l_2 \to f_1\right) \\ &= f_0\left(R^2 S_L\left(l_2\right) \to x\right)\left(l_1 R^1 \to f_1\right) \\ &= f_0\left(R^2 \to \left(S_L\left(l_2\right) \to x\right)\right)\left(l_1 \to \left(R^1 \to f_1\right)\right) \\ &= f\left(\left(S_L\left(l_3\right) \to x\right)_1\right)\left(l_1 \to S\left(\left(S_L\left(l_2\right) \to x\right)_2\right)\right) \\ &= f\left(S_L\left(l_3\right) \to x_1\right)\left(l_1 \to S\left(S_L\left(l_2\right) \to x_2\right)\right) \\ &= f\left(S_L\left(l_3\right) \to x_1\right)\left(l_1 S_L\left(l_2\right) \to S\left(x_2\right)\right) \\ &= f\left(S_L\left(l_2\right) \to x_1\right)\left(\varepsilon_t\left(l_1\right) \to S\left(x_2\right)\right) \\ &= f\left(S_L\left(1_{(2)} l\right) \to x_1\right)\left(S_L\left(1_{(1)}\right) \to S\left(x_2\right)\right) \\ &= f\left(S_L(l) \to \left(1_{(1)} \to x_1\right)\right) S\left(1_{(2)} \to x_2\right) \\ &= f\left(S_L(l) \to x_1\right) S\left(x_2\right) = (l \to f)\left(x_1\right) S\left(x_2\right) \\ &= \theta_H\left((l \to f)_0 \otimes (l \to f)_1\right)(x). \end{aligned}$$

From all above, H^* is a right H-Hopf module in ${}_L^L\mathcal{YD}$.

Applying Theorem 4.2 we can obtain the following result.

Corollary 4.3. H^* is defined a right H-Hopf module in ${}_L^L\mathcal{YD}$ as above, then $H^{*CoH} \otimes_t H \cong H^*$.

5. Applications

As a consequence the space of coinvariants of the finite dimensional Hopf algebra is free of rank one. This is the case for the weak Hopf algebra in the category of the Yetter-Drinfeld modules.

Theorem 5.1. If H is a finite-dimensional weak Hopf algebra in ${}_L^L\mathcal{YD}$. Then

1) $\dim H^{*CoH} = 1$. $Fix\phi \neq 0 \in H^{*CoH}$.

2) The map $\theta: H \to H^*, \phi \otimes h \to \phi h$ is an right H-module and an right H-comodules isomorphism. In particular H is a Frobenius weak Hopf algebra with Frobenius map ϕ.

3) There exist a right integral t in H, $\chi_L \in Alg(L, k)$ and a group-like elment g_L in L such that for all $x \in H$ $\phi(tx) = \varepsilon_H(x)$

a) $\phi(tx) = \varepsilon_H(x)$,

b) $\phi(l \to x) = \chi_L(l)\phi(x)$,

c) $\phi(x) g_L = \phi\left(R^1 \to x\right) S_L^{-1}\left(R^2\right)$

d) $l \to t = \chi_L(l) t$, for all $l \in L$.

4) The map $\theta: H \to H^*$ is a left L-semilinear and a left L-semicolinear in the sense that for all $h \in H$

$\theta(l \to h) = \chi_L(l_1) l_2 \to \theta(h)$, $\sigma(\theta(h)) = \theta(h)^{-1} \otimes \theta(h)^0 = g_L R^2 \otimes \theta(R^1 \to h)$.

Proof. 1) Since H^{*CoH} is a right H-Hopf module in ${}_L^L\mathcal{M}$, we have $H^{*CoH} \otimes H \cong H^*$, $f \otimes h \to fh$. Since $\dim H^* = \dim H$, it follows that $\dim H^* = 1$.

2) Choose $\phi \neq 0 \in H^{*CoH}$. Then by (1) $\theta : H \to H^*, \phi \otimes h \to \phi h$ is an right H-modules and an right H-comodules. Thus H is Frobenius weak Hopf algebra.

3) a) Since $H \cong H^* (\phi \otimes h \to \phi h)$, there is a unique element t in H such that $\phi t = \varepsilon_H$, *i.e.* $\phi(tx) = \varepsilon_H(x)$. For all $h \in H$ we have $(\phi th)(x) = (\phi t)(hx) = \varepsilon_H(hx) = \varepsilon_H(\varepsilon_s(h)x) = (\phi t \varepsilon_s(h))(x)$. It follows that $th = t\varepsilon_s(h), h \in H$. So t is a right integral in H.

b) We remark that $S_L^{-1}(l) \to \phi \in (H^*)^{CoH}$ for all $l \in L$ from Theorem 3.3. This implies $S_L^{-1} \to \phi = \chi_L(l)\phi$, *i.e.* $\phi(l \to x) = \chi_L(l)\phi(x), x \in H$ for some $\chi_L(l) \in Alg(L, k)$, by $\dim H^{*CoH} = 1$.

c) From Theorem 3.3 we have H^{*CoH} is a right L-comodule, *i.e.* $\sigma(\phi) \in L \otimes (H^*)^{CoH}$. By $\dim H^{*CoH} = 1$ we can obtain $\sigma(\phi) = g_L \otimes \phi$ for some group-like element g_L in L. This implies that $\phi(x) g_L = \phi(R^1 \to x) S^{-1}(R^2)$.

d) Applying $\phi(l \to x) = \chi_L(l)\phi(x)$ we have

$$\begin{aligned}
\phi(l \to t)x &= \phi((l_1 \to t)(l_2 S_L(l_3) \to x)), = \phi(l_1 \to (t(S_L(l_2) \to x))) \\
&= \chi_L(l_1)\varepsilon(S_L(l_2) \to x) = \chi_L(1_{(1)} l)\varepsilon(S_L(1_{(2)}) \to x) \\
&= \chi_L(l)\chi_L(1_{(1)})\varepsilon(S_L(1_{(2)}) \to x) \\
&= \chi_L(l)\phi(1_{(1)} \to t(S_L(1_{(2)}) \to x)) \\
&= \chi_L(l)\phi((1_{(1)} \to t)(1_{(2)} S_L(1_{(3)}) \to x)) \\
&= \chi_L(l)\phi((1_{(1)} \to t)(1_{(2)} \to x)) \\
&= \chi_L(l)\phi(tx) = (\phi\chi_L(l)t)(x),
\end{aligned}$$

This means $l \to t = \chi_L(l)t$, for all $l \in L$.

4) For all $x \in H$ we have

$$\begin{aligned}
\theta(l \to h)(x) &= \phi((l \to h)x) = \phi(l_1 \to (h(S_L(l_2) \to x))), \\
&= \chi_L(l_1)\phi(h(S_L(l_2) \to x)) = \chi_L(l_1)\theta(h)(S_L(l_2) \to x), \\
&= \chi_L(l_1)(l_2 \to \theta(h)(x)) = (\chi_L(l_1) l_2 \to \theta(h))(x).
\end{aligned}$$

This implies $\theta(l \to h) = \chi_L(l_1) l_2 \to \theta(h)$.

$$\sigma(\theta(h)) = (\phi h)^{-1} \otimes (\phi h)^0 = g_L h^{-1} \otimes \phi h^0 = g_L R^2 \otimes \theta(R^1 \to h)$$

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References

[1] Böhm, G., Nill, F. and Szlachányi, K. (1999) Weak Hopf Algebras I. Integral Theory and C^*-Structure. *Journal of Al-*

gebra, **221**, 385-438. http://dx.doi.org/10.1006/jabr.1999.7984

[2] Caenpeel, S, Wang, D.-D. and Yin, Y.-M. (2005) Yetter-Drinfeld Modules over Weak Bialgebras. *Annali Dell'universitia DI Ferrarà*, *Sezione VII-Scienze Matematiche*, **51**, 69-98.

[3] Kadison, L. and Nikshych, D. (2001) Frobenius Extensions and Weak Hopf Algebras. *Journal of Algebra*, **244**, 312-342. http://dx.doi.org/10.1006/jabr.2001.8911

[4] Takeuchi, M. (1998) Hopf Modules in Yetter-Drinfeld Categories. *Communications in Algebra*, **26**, 3057-3070. http://dx.doi.org/10.1080/00927879808826327

[5] Bohm, G. (2000) Doi-Hopf Modules over Weak Hopf Algebras. *Communications in Algebra*, **28**, 4687-4689.

[6] Shen, B.-L. and Wang, S.-H. (2009) Weak Hopf Algebra Duality in Weak Yetter-Drinfeld Categories and Applications. *Internation Electeronic Journal of Algebra*, **6**, 74-94.

[7] Yin, Y.-M. and Zhang, M.-C. (2011) Hopf Modules in the Braided Monoidal Category ${}_{L}M$. *Le Matematiche*, **LXVI**, 81-92.