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A New Approach for Solving Nonlinear Equations by Using of Integer Nonlinear Programming

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Abstract

One of the most important issues in numerical calculations is finding simple roots of nonlinear equations. This topic is one of the oldest challenges in science and engineering. Many important problems in engineering, to achieve the result need to solve a nonlinear equation. Thus, the formulation of a recursive relationship with high order of convergence and low time complexity is very important. This paper provides a modification to the Weerakoon-Fernando and Parhi-Gupta methods. It is shown that, in each iterate, the improved method requires three evaluations of the function and two evaluations of the first derivatives of function. The proposed with the Kou *et al.*, Neta, Parhi-Gupta, Thukral and Mir *et al.* methods have been applied to a collection of 12 test problem. The results show that proposed approach significantly reduces the number of function calls when compared to the above methods. The numerical examples show that the proposed method is more efficiency than other methods in this class, such as sixth-order method of Parhi-Gupta or eighth-order method of Mir *et al.* and Thukral. We show that the order of convergence the proposed method is 9 and also, the modified method has the efficiency of $\sqrt[5]{9}$.

Keywords

Newton Method, Nonlinear Equations, Convergence Theorem, Efficiency Index

1. Introduction

In the real world, many of the complex problems after simplification lead to solving nonlinear problems. Find an approximation of the simple roots of the equations is one of the important problems on this issue. The rapid development of technology has led to different of algorithms. Over time, many algorithms have been developed. In

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this state, one of the ways for comparison of different algorithms is finding of complexity of time and index efficiency of algorithms. MAPLE software is one of the powerful algebraic systems from Maplesoft company, such that in this article it has been used for the calculation. The boundary value problems appearing in kinetic theory of gases, elasticity and other applied areas are reduced to solve these equations. Many optimization problems also lead such equations. Hence, one of the most important problems in numerical analysis is to find a simple root α of a nonlinear equation $f(x)=0$, where $f:\mathbb{D}\subseteq\mathbb{R}\to\mathbb{R}$ for an open interval $\mathbb{D}$ is a scalar function. In this study, in order to find α, we should start with an initial approximation x_0 which is near to the root α and generates successive iterates $\{x_0\}_0^\infty$ converging to simple root α of nonlinear equation $f(x)=0$. In all iteration, the improved method requires three evaluations of the function and two evaluations of the first order derivatives of function. Therefore, the modified method has the efficiency index $\sqrt[5]{9}$. The numerical examples show that, the proposed method has more efficient with respect to the Newton method and other methods in this class. The effectiveness of the modified ninth-order method will be examined by approximation the simple root of a given non-linear equation. The suggested method is comparable to the sixth-order methods [1] [2]; also the eighth-order methods [3] and [4].

In the reminder, we proceed as follows: In Section 2, we recall the basic concepts. The proposed method is described in Section 3. In Section 4, the convergence analysis is carried out to establish the ninth-order of convergence of our method. In Section 5, as is shown in the numerical examples, this method is more efficient than Newton method and other methods of lower or same order. We conclude with some remarks on the presented approaches in Section 6.

2. Several Basic Definitions

Our goal is to find the value of x that satisfies the following equation.

$$f(x)=0, \tag{1}$$

where $f(x)$ is a nonlinear equation. The value of x that satisfies (1) is called a root of $f(x)$ and denoted by α. Therefore, the procedure used of to find x is called root-finding. Let α is a simple root of Equation (1) and $\{x_n\}_{n=1}^\infty$ is a real sequence.

Definition 1. *See* [5]: *The sequence* $\{x_n\}_{n=1}^\infty$ *is said to converge to* α *if*

$$\lim_{n\to\infty}|x_n-\alpha|=0.$$

Furthermore, if there exists positive constant c and p such that:

$$\lim_{n\to\infty}\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p}=c\neq 0$$

we say that $\{x_n\}_{n=1}^\infty$ converges to α of order p. Larger values of p correspond to faster convergence. Let $e_n=x_n-\alpha$ be error in the nth iterate of the method which produces the sequence $\{x_n\}$. The relation

$$e_{n+1}=ce_n^p+O\left(e_n^{p+1}\right)$$

is called the error equation. The value of p is called the order of convergence of method, see [6].

Definition 2. *Let* α *be a root of the function f and suppose that* x_{n+1}, x_n *and* x_{n-1} *are three consecutive iterations closer to the root* α. *The the computational order of convergence* ρ *can be approximated using the formula*:

$$\rho\approx\frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}$$

3. New Proposed Scheme

The new method is based on [2] method. With a simple manipulation, and a new approach to get the following equations.

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ w_n &= z_n - \frac{f(z_n)}{f'(z_n)}\frac{f'(x_n)+f'(y_n)}{3f'(y_n)-f'(x_n)}, \end{aligned} \tag{2}$$

and

$$x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)}. \tag{3}$$

This is four-step method. It is not necessary to compute the first-order derivative at the point w_n since a good approximation can be obtained. In order to approximate $f'(w_n)$ use the linear interpolation on two points $(y_n, f'(y_n))$ and $(z_n, f'(z_n))$, so we have:

$$f'(x) \approx \frac{x-z_n}{y_n-z_n} f'(y_n) + \frac{x-y_n}{z_n-y_n} f'(z_n). \tag{4}$$

Therefore,

$$f'(w_n) \approx \frac{w_n-z_n}{y_n-z_n} f'(y_n) + \frac{w_n-y_n}{z_n-y_n} f'(z_n). \tag{5}$$

Now using Equations (2), we have:

$$\begin{aligned} f'(w_n) = &-\frac{1}{f(x_n)\left(f'(x_n)+f'(y_n)\right)\left(-3f'(y_n)+f'(x_n)\right)} \\ &\times\Big[\left(f(x_n)-f(z_n)\right)f'(x_n)^3 + \left(-6f'(y_n)f(x_n)-f(z_n)f'(y_n)\right)f'(x_n)^2 \\ &+\left(f(z_n)f'(y_n)^2+9f(x_n)f'(y_n)^2\right)f'(x_n)+\left(f(z_n)f'(y_n)^3\right)\Big]. \end{aligned} \tag{6}$$

Substituting the relation of (6) into the relation (3), in this case, we obtain the following formula:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ w_n &= z_n - \frac{f(z_n)}{f'(x_n)}\frac{f'(x_n)+f'(y_n)}{3f'(y_n)-f'(x_n)}, \end{aligned}$$

where,

$$\begin{aligned} f'(w_n) = &-\frac{1}{f(x_n)\left(f'(x_n)+f'(y_n)\right)\left(-3f'(y_n)+f'(x_n)\right)} \\ &\times\Big[\left(f(x_n)-f(z_n)\right)f'(x_n)^3 + \left(-6f'(y_n)f(x_n)-f(z_n)f'(y_n)\right)f'(x_n)^2 \\ &+\left(f(z_n)f'(y_n)^2+9f(x_n)f'(y_n)^2\right)f'(x_n)+\left(f(z_n)f'(y_n)^3\right)\Big]. \end{aligned}$$

Obviously this method requires evaluations of three function f and two derivatives f'.

4. Convergence Analysis

To determine order of convergence of proposed method, we must be solving integer nonlinear programming as follow:

$$\begin{aligned} &\min \quad k \\ &\text{s.t.} \quad e_{n+1} = C \cdot e_n^k + O\left(e_m^{k+1}\right), \\ &\qquad\quad k \in Z, k \geq 0. \end{aligned} \tag{7}$$

where C is a special coefficient of e_n^k. This is equivalent to the bellow theorem, *i.e*. we show that the convergence of the proposed method is of the order of 9.

Theorem 1. *Let* $f:\mathbb{R}\to\mathbb{R}$ *has continuous derivative function and* $\alpha\in\mathbb{D}$ *is a simple root of f. If the initial point* x_0 *is sufficiently close*[1] *to α, then the method defined by* (2) *converges to α in the ninth-order. Furthermore, the error in the method given by* (2) *satisfies the equation*:

$$e_{n+1} = \left[\frac{15}{4}c_2^2c_3^3 + 6c_2^4c_3^2 - 3c_2^6c_3\right]e_n^9 + O\left(e_n^{10}\right). \tag{8}$$

where $e_n = x_n - \alpha$ and $c_j = \dfrac{f^{(j)}(\alpha)}{j!\,f'(\alpha)}$ for $j\in\mathbb{N}$.

Proof. Let $e_n = x_n - \alpha$ be the error term in the iterate x_n. Using Taylor expansion, we have:

$$f(x_n) = f'(\alpha)\left[e_n + \frac{1}{2!}\frac{f''(\alpha)}{f'(\alpha)}e_n^2 + O\left(e_n^3\right)\right] = f'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + O\left(e_n^4\right)\right]. \tag{9}$$

and

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O\left(e_n^4\right)\right]. \tag{10}$$

Quotient relations (9) and (10), gives the following results:

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 + O\left(e_n^5\right).$$

Thus we have

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 - 2\left(c_2^2 - c_3\right)e_n^3 - \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 + O\left(e_n^5\right).$$

Taylor expansion of the function $f'(y_n)$ around the point α to get the following result (*i.e* (11)):

$$f'(y_n) = f'(\alpha)\left[1 + 2c_2^2e_n^2 + 4c_2\left(c_3 - c_2^2\right)e_n^3 + \left(-11c_2^2c_3 + 8c_2^4 + 6c_2c_4\right)e_n^4 + O\left(e_n^5\right)\right]. \tag{11}$$

Substituting (9), (10) and (11) into the z_n section of the Equation (2), we have:

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} = \alpha - \left(-\frac{1}{2}c_3 - c_2^2\right)e_n^3 - \left(-\frac{3}{2}c_2c_3 + 3c_2^3 - c_4\right)e_n^4 + O\left(e_n^5\right). \tag{12}$$

Furthermore, the Taylor expansion of $f(z_n)$ about α is

$$f(z_n) = f'(\alpha)\left[\left(\frac{1}{2}c_3 + c_2^2\right)e_n^3 + \left(\frac{3}{2}c_2c_3 - 3c_2^3 + c_4\right)e_n^4 + O\left(e_n^5\right)\right]. \tag{13}$$

Since from (10), (12) and (13) we get:

$$w_n = z_n - \frac{f(z_n)}{f'(x_n)}\frac{f'(x_n) + f'(y_n)}{3f'(y_n) - f'(x_n)} = \alpha + \left(-\frac{5}{4}c_2c_3^2 - 2c_2^3c_3 + c_2^5\right)e_n^6 + O\left(e_n^7\right). \tag{14}$$

Again, using the Taylor expansion of function $f(w_n)$ about the point α, in this case we have:

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[1] Since the proposed method in this paper is the revised and generalized form of the Newton method, it consist of common problems in Newton method such as proper selection of initial point. In order to solve this problem we can use a number of repetitions of ever-convergence methods as Bisection or False-position.

$$f(w_n) = f'(\alpha)\left[\left(-\frac{5}{4}c_2c_3^2 - 2c_2^3c_3 + c_2^5\right)e_n^6 + O\left(e_n^7\right)\right]. \tag{15}$$

Taylor expansion of the function $f'(w_n)$ around the point α to get the following result (*i.e* (16)):

$$f'(w_n) = f'(\alpha)\left[1 - 3c_2c_3e_n^3 + \left(6c_2^2c_3 - 4c_2c_4 - 6c_3^2\right)e_n^4 + O\left(e_n^5\right)\right]. \tag{16}$$

In this case, using the above result (*i.e* (15), (16) and (14)) and corresponding to the relation (3), we get:

$$x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} = \alpha + \left(\frac{15}{4}c_2^2c_3^3 + 6c_2^4c_3^2 - 3c_2^6c_3\right)e_n^9 + O\left(e_n^{10}\right). \tag{17}$$

Therefore, we have:

$$e_{n+1} = \left[\frac{15}{4}c_2^2c_3^3 + 6c_2^4c_3^2 - 3c_2^6c_3\right]e_n^9 + O\left(e_n^{10}\right).$$

Thus, the ninth order of convergence of the method is established.

Numerical Examples

In order to demonstrate the performance, accuracy and effectiveness of the proposed ninth-order method, we take 12 special nonlinear equation test problems from [2] [7] and [8]. We compare the proposed method with Wang-Liu's third-order method [8], Weerakoon-Fernando and Parhi-Gupta's sixth-order methods [1] [2] and Kou *et al*. and Neta's eight-order methods as [3] and [4], respectively. The computing results displayed in **Tables 1-5**. In every problem we try to seek an approximation x_n of the root α of Equation (1) after n times

Table 1. Comparison of result of proposed method (PM) with Kou and Li (KL) method.

Functions	x_0	n		Run time		NFE	
		PM	KL	PM	KL	PM	KL
$f_1(x)$	1	2	9	0.203	0.031	10	36
	2	2	8	0.156	0.032	10	32
$f_2(x)$	1	3	16	0.390	0.468	15	64
	3	3	9	0.421	0.343	15	36
$f_3(x)$	0.5	11	DIV	0.187	-	55	-
	1.5	3	32	0.125	0.047	15	128
$f_4(x)$	2.5	3	9	0.265	0.047	15	36
	3.5	3	13	0.249	0.016	15	52
$f_5(x)$	3.25	3	68	0.406	0.312	15	272
	3.5	4	DIV	0.484	-	20	-
$f_6(x)$	1.5	3	21	0.156	0.047	15	84
$f_7(x)$	2	4	DIV	0.234	-	20	-
	3	2	4	0.249	0.125	10	16
$f_8(x)$	3.5	5	163	0.203	0.125	25	652
	4.5	7	433	0.156	0.343	35	1732
$f_9(x)$	1	3	DIV	0.296	-	15	-
$f_{10}(x)$	3	3	10	0.468	0.312	15	40
	1.2	2	6	0.515	0.156	10	24
$f_{11}(x)$	−0.85	10	DIV	0.827	-	50	-
	0	3	DIV	0.577	-	15	-
$f_{12}(x)$	0.8	4	23	0.171	0.031	20	92
	0	5	51	0.281	0.016	25	204

Table 2. Comparison of result of proposed method (PM) with Parhi and Gupta (PG) method.

Functions	x_0	n		Run time		NFE	
		PM	PG	PM	PG	PM	PG
$f_1(x)$	1	2	2	0.203	0.031	10	8
	2	2	2	0.156	0.093	10	8
$f_2(x)$	1	3	3	0.390	0.218	15	12
	3	3	3	0.421	0.312	15	12
$f_3(x)$	0.5	11	11	0.187	0.124	55	44
	1.5	3	3	0.125	0.031	15	12
$f_4(x)$	2.5	3	3	0.265	0.094	15	12
	3.5	3	3	0.249	0.078	15	12
$f_5(x)$	3.25	3	3	0.406	0.250	15	12
	3.5	4	4	0.484	0.265	20	16
$f_6(x)$	1.5	3	3	0.156	0.047	15	12
$f_7(x)$	2	4	4	0.234	0.172	20	16
	3	2	2	0.249	0.141	10	8
$f_8(x)$	3.5	5	3	0.203	0.125	25	12
	4.5	7	7	0.156	0.140	35	28
$f_9(x)$	1	3	3	0.296	0.063	15	12
$f_{10}(x)$	3	3	3	0.468	0.312	15	12
	1.2	2	2	0.515	0.249	10	8
$f_{11}(x)$	−0.85	10	15	0.827	0.656	50	60
	0	3	3	0.577	0.250	15	12
$f_{12}(x)$	0.8	4	4	0.171	0.078	20	16
	0	5	5	0.281	0.062	25	20

Table 3. Comparison of result of proposed method (PM) with Neta (NM) method.

Functions	x_0	n		Run time		NFE	
		PM	NM	PM	NM	PM	NM
$f_1(x)$	1	2	2	0.203	0.109	10	8
	2	2	2	0.156	0.156	10	8
$f_2(x)$	1	3	3	0.390	0.312	15	12
	3	3	3	0.421	0.374	15	12
$f_3(x)$	0.5	11	DIV	0.187	-	55	-
	1.5	3	3	0.125	0.094	15	12
$f_4(x)$	2.5	3	2	0.265	0.124	15	8
	3.5	3	3	0.249	0.093	15	12
$f_5(x)$	3.25	3	3	0.406	0.390	15	12
	3.5	4	5	0.484	0.437	20	20
$f_6(x)$	1.5	3	3	0.156	0.141	15	12
$f_7(x)$	2	4	DIV	0.234	-	20	-
	3	2	2	0.249	0.141	10	8
$f_8(x)$	3.5	5	5	0.203	0.093	25	20
	4.5	7	7	0.156	0.125	35	28
$f_9(x)$	1	3	3	0.296	0.125	15	12
$f_{10}(x)$	3	3	3	0.468	0.390	15	12
	1.2	2	2	0.515	0.359	10	8
$f_{11}(x)$	−0.85	10	3	0.827	0.421	50	12
	0	3	2	0.577	0.437	15	8
$f_{12}(x)$	0.8	4	3	0.171	0.141	20	12
	0	5	4	0.281	0.156	25	16

Table 4. Comparison of result of proposed method (PM) with Thaukral (TM) method.

Functions	x_0	n		Run time		NFE	
		PM	TM	PM	TM	PM	TM
$f_1(x)$	1	2	3	0.203	0.125	10	12
	2	2	3	0.156	0.141	10	12
$f_2(x)$	1	3	DIV	0.390	-	15	-
	3	3	3	0.421	0.281	15	12
$f_3(x)$	0.5	11	DIV	0.187	-	55	-
	1.5	3	4	0.125	0.140	15	16
$f_4(x)$	2.5	3	3	0.265	0.109	15	12
	3.5	3	DIV	0.249	-	15	-
$f_5(x)$	3.25	3	4	0.406	0.374	15	12
	3.5	4	6	0.484	0.484	20	24
$f_6(x)$	1.5	3	122	0.156	0.655	15	488
$f_7(x)$	2	4	DIV	0.234	-	20	-
	3	2	2	0.249	0.219	10	8
$f_8(x)$	3.5	5	7	0.203	0.156	25	28
	4.5	7	12	0.156	0.218	35	48
$f_9(x)$	1	3	DIV	0.296	-	15	-
$f_{10}(x)$	3	3	3	0.468	0.560	15	12
	1.2	2	3	0.515	0.421	10	12
$f_{11}(x)$	−0.85	10	DIV	0.827	-	50	-
	0	3	3	0.577	0.375	15	12
$f_{12}(x)$	0.8	4	DIV	0.171	-	20	-
	0	5	7	0.281	0.125	25	28

Table 5. Comparison of result of proposed method (PM) with Mir (MM) method.

Functions	x_0	n		Run time		NFE	
		PM	MM	PM	MM	PM	MM
$f_1(x)$	1	2	2	0.203	0.125	10	10
	2	2	2	0.156	0.140	10	10
$f_2(x)$	1	3	2	0.390	0.297	15	10
	3	3	2	0.421	0.218	15	10
$f_3(x)$	0.5	11	DIV	0.187	-	55	-
	1.5	3	3	0.125	0.109	15	15
$f_4(x)$	2.5	3	2	0.265	0.094	15	10
	3.5	3	DIV	0.249	-	15	-
$f_5(x)$	3.25	3	3	0.406	0.437	15	15
	3.5	4	4	0.484	0.437	20	20
$f_6(x)$	1.5	3	2	0.156	0.188	15	10
$f_7(x)$	2	4	DIV	0.234	-	20	-
	3	2	DIV	0.249	-	10	-
$f_8(x)$	3.5	5	5	0.203	0.125	25	25
	4.5	7	8	0.156	0.203	35	40
$f_9(x)$	1	3	2	0.296	0.109	15	10
$f_{10}(x)$	3	3	3	0.468	0.296	15	15
	1.2	2	2	0.515	0.484	10	10
$f_{11}(x)$	−0.85	10	DIV	0.827	-	50	-
	0	3	DIV	0.577	-	15	-
$f_{12}(x)$	0.8	4	3	0.171	0.218	20	15
	0	5	5	0.281	0.172	25	20

iteration. In this paper, the stoping criterion is $\epsilon = \left|f\left(x_{n+1}\right)\right| + \left|x_{n+1} - \alpha\right| < 10^{-20}$. the Run time and the Number of function evaluations (NFE) are also given in **Tables 1-5**. "DIV" in the tables implies that the corresponding method is diverges. Furthermore, a comparison of the rate of convergence of the proposed method and Kou-Li method [8] for function $f_8(x)$ at point $x_0 = 4.5$ is shown in **Figure 1**. The comparison is clearly marked on **Figure 1**. It should be noted that, Numerical computations reported here have been carried out in the MAPLE 18 environment. The results show that the speed of convergence in all methods discussed in this article, are depends on proper selection of the initial point. For example, in **Table 1** for $f_7(x)$, choose an initial point $x_0 = 2$ is leading to the divergence of Kou *et al*. method, whereas the choose the initial point $x_0 = 3$ in the same method is leading to the coverage to the simple root α, see **Table 1**. In all examples, it is evident that the proposed approach, for any initial point is coverage to simple root of α.

The test functions are listed as follows:

$$f_1(x) = x^3 + 4x^2 - 10, \quad \alpha = 1.36523001341409684576,$$

$$f_2(x) = \sin^2 x - x^2 + 1, \quad \alpha = 1.40449164821534122604,$$

$$f_3(x) = x^{10} - 1, \quad \alpha = 1,$$

$$f_4(x) = (x-1)^3 - 1, \quad \alpha = 2,$$

$$f_5(x) = \mathrm{e}^{x^2+7x-30} - 1, \quad \alpha = 3,$$

$$f_6(x) = x^3 - 10, \quad \alpha = 2.15443469003188362176,$$

$$f_7(x) = x^2 - \mathrm{e}^x + 3x + 2, \quad \alpha = 2.99223487205393686509,$$

$$f_8(x) = (x-2)^{23} - 1, \quad \alpha = 3,$$

$$f_9(x) = \arctan(x), \quad \alpha = 0,$$

$$f_{10}(x) = \sin(x-1) + (x-1)^2, \quad \alpha = 1,$$

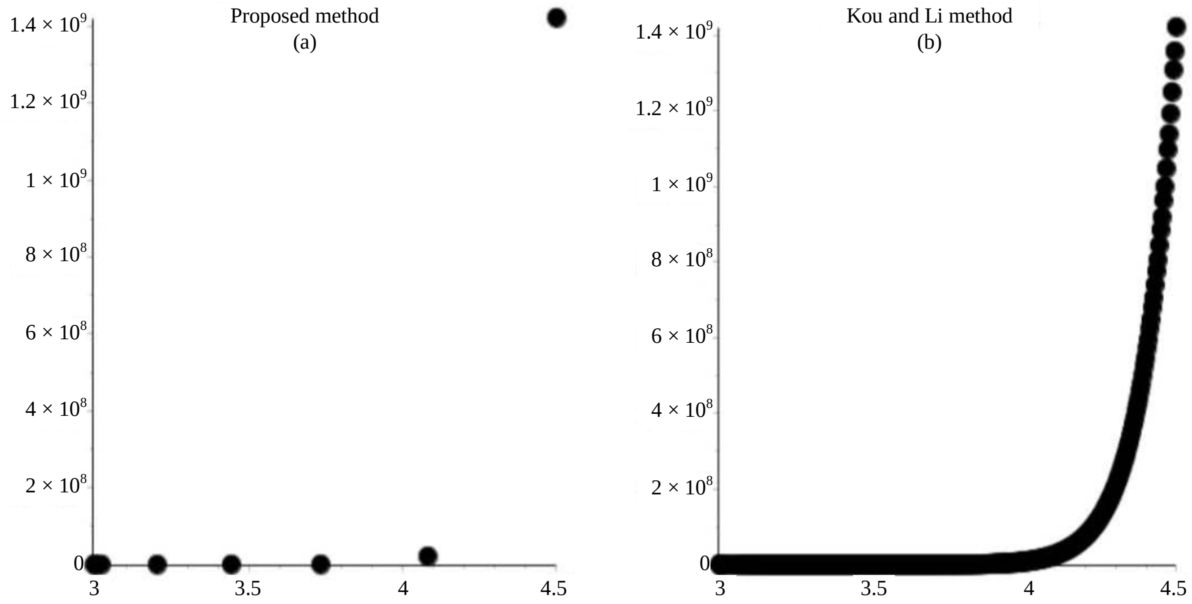

Figure 1. A comparison of the rate of convergence of the proposed method and Kou *et al*. method for function $f_8(x)$ at point $x_0 = 4.5$. The proposed and Kou *et al*. methods converged to the simple root $\alpha = 3$ in 3 and 433 iteration, respectively. This show that the PM method is vary faster with respect to the Kou *et al*. method. More details are given in **Table 1**.

$$f_{11}(x) = \cos(x) - x, \quad \alpha = 0.73908513321516064166,$$

$$f_{12}(x) = \prod_{m=0}^{4}\left(x - (1 + 0.1m)\right), \quad \alpha = 1.$$

One can easily see from **Tables 1-5** that our method behaves either similarly or better then the compared methods. The results show that the new method has advantages over the Kou *et al*. [8] method and the eight-order method as Thukral [4] method. Also, the new method have iteration stabilities to the original iteration value and behave either similarly or better than the methods compared. All numerical results are in accordance with the theory and the basic advantage of the variants of Newton's method based on means or integration methods that they do not require the computation of second- or higher-order derivatives although they are of ninth order.

5. Conclusion

In numerical analysis, many methods produce sequences of real numbers, for example the iterative schemes for solving $f(x) = 0$. Sometimes, the convergence of these sequences is slow and their utility in solving practical problems, quite limited. Convergence acceleration methods try to transform a slowly converging sequence into a fast convergent one. Accordingly in this work, a new method has developed. In this study, a new ninth-order method to solve nonlinear equations has been proposed. From numerical examples, it has been observed that the proposed method convergence quickly toward root α is compared to lower order methods. In addition, in practical terms, the method is noticeable. Also, the above-mentioned ninth-order method requires the evaluation of three functions and two first derivatives of the function. Therefore, the new method has the efficiency index $\sqrt[5]{9}$. Unlike the other methods, the proposed method converges well when the initial point x_0 is at the boot sides of root α. This is obviously clear understood from the examples.

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