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On a Max-Type Difference System

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Abstract

In this paper, we show that every well-defined solution of the max-type system of difference equations $x_{n+1}=\max\left\{\frac{\beta}{x_n},y_{n-1}\right\}$, $y_{n+1}=\max\left\{\frac{\beta}{y_n},x_{n+1}\right\}$, $n\in\mathbb{N}_0$ **is eventually periodic with period four.**

Keywords

Periodic Solution, Max-Type Difference System

1. Introduction

Max-type difference equations and max-type difference systems have been wisely applied in biology, computer science and automatic control systems and so on. There has been great interest in studying these equations in recent years.

For example, Briden *et al*. [1] investigated the periodicity character of the solution of the max-type difference equation

$$x_{n+1}=\max\left\{\frac{1}{x_n},\frac{A_n}{x_{n-1}}\right\},\quad n\in\mathbb{N}_0$$

Xiao Qian *et al*. [2] showed that the solution of the max-type difference equation

$$x_{n+1}=\max\left\{\frac{\beta}{x_n},x_{n-1}\right\},\quad n\in\mathbb{N}_0$$

is periodic with period two.

W. Q. Ji *et al*. [3] showed that the solution of the max-type difference system

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$$\begin{cases} x_{n+1} = \max\left\{\dfrac{\beta}{y_n}, x_{n-1}\right\} \\ y_{n+1} = \max\left\{\dfrac{\beta}{x_n}, y_{n-1}\right\} \end{cases}, \quad n \in \mathbb{N}_0$$

is periodic with period two.

In addition, E. M. Elasyed, Stevo Stević and others investigated some periodic max-type difference equations and periodic max-type difference systems in [4]-[7].

In this paper we show that every solution of the following max-type difference system

$$\begin{cases} x_{n+1} = \max\left\{\dfrac{\beta}{x_n}, y_{n-1}\right\} \\ y_{n+1} = \max\left\{\dfrac{\beta}{y_n}, x_{n-1}\right\} \end{cases}, \quad n = 0,1,\cdots \tag{1}$$

where the initial conditions x_{-1}, x_0, y_{-1}, y_0 are arbitrary non-zero real numbers and $n \in \mathbb{R}$, is periodic with period four.

Remark 1. Note that if $\beta = 0$, then System (1) becomes $x_{n+1} = y_{n-1}$, $x_{n-1} = y_{n+1}$, from which it follows that $x_n = x_{n+4}$, $y_n = y_{n+4}$ and every solution is periodic with period four.

2. Some Lemmas

Lemma 1 *Assume that* $\{x_n, y_n\}_{n=-1}^{\infty}$ *is a solution of System* (1) *and there exists a* $k_0 \in \{-1,0,1,2,\cdots\}$ *such that*

$$x_{k_0} = y_{k_0+2},\ x_{k_0+1} = y_{k_0+3},\ y_{k_0} = x_{k_0+2},\ y_{k_0+1} = x_{k_0+3} \tag{2}$$

Then every solution is periodic with period four.

Proof Frist, we will prove that

$$\begin{aligned} &x_{k_0} = x_{k_0+4m},\ x_{k_0+1} = x_{k_0+4m+1},\ x_{k_0+2} = x_{k_0+4m+2},\ x_{k_0+3} = x_{k_0+4m+3}; \\ &y_{k_0} = y_{k_0+4m},\ y_{k_0+1} = y_{k_0+4m+1},\ y_{k_0+2} = y_{k_0+4m+2},\ y_{k_0+3} = y_{k_0+4m+3}. \end{aligned} \tag{3}$$

where $m \in \mathbb{N}$, from which the lemma follows.

Now, we use the method of induction. For $m = 1$, Equation (3) becomes the following equations

$$\begin{aligned} &x_{k_0} = x_{k_0+4},\ x_{k_0+1} = x_{k_0+5},\ x_{k_0+2} = x_{k_0+6},\ x_{k_0+3} = x_{k_0+7}; \\ &y_{k_0} = y_{k_0+4},\ y_{k_0+1} = y_{k_0+5},\ y_{k_0+2} = y_{k_0+6},\ y_{k_0+3} = y_{k_0+7}. \end{aligned} \tag{4}$$

By System (1) and Equation (2), we obtain that

$$\begin{cases} x_{k_0+4} = \max\left\{\dfrac{\beta}{x_{k_0+3}}, y_{k_0+2}\right\} = \max\left\{\dfrac{\beta}{y_{k_0+1}}, x_{k_0}\right\} = y_{k_0+2} = x_{k_0}, \\ y_{k_0+4} = \max\left\{\dfrac{\beta}{y_{k_0+3}}, x_{k_0+2}\right\} = \max\left\{\dfrac{\beta}{x_{k_0+1}}, y_{k_0}\right\} = x_{k_0+2} = y_{k_0}, \end{cases}$$

$$\begin{cases} x_{k_0+5} = \max\left\{\dfrac{\beta}{x_{k_0+4}}, y_{k_0+3}\right\} = \max\left\{\dfrac{\beta}{y_{k_0+2}}, x_{k_0+1}\right\} = y_{k_0+3} = x_{k_0+1}, \\ y_{k_0+5} = \max\left\{\dfrac{\beta}{y_{k_0+4}}, x_{k_0+3}\right\} = \max\left\{\dfrac{\beta}{x_{k_0+2}}, y_{k_0+1}\right\} = x_{k_0+3} = y_{k_0+1}, \end{cases}$$

$$\begin{cases} x_{k_0+6} = \max\left\{\dfrac{\beta}{x_{k_0+5}}, y_{k_0+4}\right\} = \max\left\{\dfrac{\beta}{x_{k_0+1}}, y_{k_0}\right\} = x_{k_0+2}, \\ y_{k_0+6} = \max\left\{\dfrac{\beta}{y_{k_0+5}}, x_{k_0+4}\right\} = \max\left\{\dfrac{\beta}{y_{k_0+1}}, x_{k_0}\right\} = y_{k_0+2}, \end{cases}$$

$$\begin{cases} x_{k_0+7} = \max\left\{\dfrac{\beta}{x_{k_0+6}}, y_{k_0+5}\right\} = \max\left\{\dfrac{\beta}{x_{k_0+2}}, y_{k_0+1}\right\} = x_{k_0+3}, \\ y_{k_0+7} = \max\left\{\dfrac{\beta}{y_{k_0+6}}, x_{k_0+5}\right\} = \max\left\{\dfrac{\beta}{y_{k_0+2}}, x_{k_0+1}\right\} = y_{k_0+3}. \end{cases}$$

From which, Equation (4) holds.

Assume Equation (3) holds for $1 \le m \le m_0$, and by using System (1) and Equation (2), we obtain that

$$\begin{cases} x_{k_0+4(m_0+1)} = \max\left\{\dfrac{\beta}{x_{k_0+4m_0+3}}, y_{k_0+4m_0+2}\right\} = \max\left\{\dfrac{\beta}{x_{k_0+3}}, y_{k_0+2}\right\} = x_{k_0+4} = x_{k_0}, \\ y_{k_0+4(m_0+1)} = \max\left\{\dfrac{\beta}{y_{k_0+4m_0+3}}, x_{k_0+4m_0+2}\right\} = \max\left\{\dfrac{\beta}{y_{k_0+3}}, x_{k_0+2}\right\} = y_{k_0+4} = y_{k_0}, \end{cases}$$

$$\begin{cases} x_{k_0+1+4(m_0+1)} = \max\left\{\dfrac{\beta}{x_{k_0+4m_0+4}}, y_{k_0+4m_0+3}\right\} = \max\left\{\dfrac{\beta}{x_{k_0+4}}, y_{k_0+3}\right\} = x_{k_0+5} = x_{k_0+1}, \\ y_{k_0+1+4(m_0+1)} = \max\left\{\dfrac{\beta}{y_{k_0+4m_0+4}}, x_{k_0+4m_0+3}\right\} = \max\left\{\dfrac{\beta}{y_{k_0+4}}, x_{k_0+3}\right\} = y_{k_0+5} = y_{k_0+1}, \end{cases}$$

$$\begin{cases} x_{k_0+2+4(m_0+1)} = \max\left\{\dfrac{\beta}{x_{k_0+4m_0+5}}, y_{k_0+4m_0+4}\right\} = \max\left\{\dfrac{\beta}{x_{k_0+5}}, y_{k_0+4}\right\} = x_{k_0+6} = x_{k_0+2}, \\ y_{k_0+2+4(m_0+1)} = \max\left\{\dfrac{\beta}{y_{k_0+4m_0+5}}, x_{k_0+4m_0+4}\right\} = \max\left\{\dfrac{\beta}{y_{k_0+5}}, x_{k_0+4}\right\} = y_{k_0+6} = y_{k_0+2}, \end{cases}$$

$$\begin{cases} x_{k_0+3+4(m_0+1)} = \max\left\{\dfrac{\beta}{x_{k_0+4m_0+6}}, y_{k_0+4m_0+5}\right\} = \max\left\{\dfrac{\beta}{x_{k_0+6}}, y_{k_0+5}\right\} = x_{k_0+7} = x_{k_0+3}, \\ y_{k_0+3+4(m_0+1)} = \max\left\{\dfrac{\beta}{y_{k_0+4m_0+6}}, x_{k_0+4m_0+5}\right\} = \max\left\{\dfrac{\beta}{y_{k_0+6}}, x_{k_0+5}\right\} = y_{k_0+7} = y_{k_0+3}. \end{cases}$$

So we complete the proof.

Lemma 2 *Assume that* $\beta > 0$. *Then every solution of System* (1) *is positive if initial conditions satisfy one of the following conditions* $x_{-1} > 0$ *or* $x_0 > 0$ *or* $y_{-1} > 0$ *or* $y_0 > 0$.

Proof Without loss of generality, we assume that $x_{-1} > 0$ and from System (1) we have

$$y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} > 0, \quad y_2 = \max\left\{\frac{\beta}{y_1}, x_0\right\} > 0;$$

$$x_3 = \max\left\{\frac{\beta}{x_2}, y_1\right\} > 0, \quad x_4 = \max\left\{\frac{\beta}{x_3}, y_2\right\} > 0.$$

By using the method of induction, we have

$$x_n, y_n > 0, \quad n \ge 3$$

Similarly, when $x_0 > 0$ or $y_{-1} > 0$ or $y_0 > 0$, there exists an $n_0 \in \mathbb{N}_0$ such that

$$x_n,\ y_n > 0, \quad n \ge n_0$$

The proof is completed.

Lemma 3 *Assume that* $\beta > 0$. *Then every solution of System* (1) *with positive initial conditions is periodic with period four.*

Proof By System (1), we obtain that

$$x_1 = \max\left\{\frac{\beta}{x_0}, y_{-1}\right\} > 0, \quad y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} > 0$$

Let $p = \max\{y_0, x_0\}$, $q = \max\left\{\frac{\beta}{y_{-1}}, y_0\right\}$, $s = \max\left\{\frac{\beta}{x_{-1}}, x_0\right\}$ and there are four cases which need to be discussed.

Case 1. $\frac{\beta}{x_0} \ge y_{-1}$, $\frac{\beta}{y_0} \ge x_{-1}$. We have

$$x_1 = \max\left\{\frac{\beta}{x_0}, y_{-1}\right\} = \frac{\beta}{x_0}, \quad y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} = \frac{\beta}{y_0};$$

$$x_2 = \max\left\{\frac{\beta}{x_1}, y_0\right\} = \max\{x_0, y_0\} = p, \quad y_2 = \max\left\{\frac{\beta}{y_1}, x_0\right\} = \max\{y_0, x_0\} = p;$$

$$x_3 = \max\left\{\frac{\beta}{x_2}, y_1\right\} = \max\left\{\frac{\beta}{p}, \frac{\beta}{y_0}\right\} = \frac{\beta}{y_0} = y_1, \quad y_3 = \max\left\{\frac{\beta}{y_2}, x_1\right\} = \max\left\{\frac{\beta}{p}, \frac{\beta}{x_0}\right\} = \frac{\beta}{x_0} = x_1;$$

$$x_4 = \max\left\{\frac{\beta}{x_3}, y_2\right\} = \max\{y_0, p\} = p = y_2, \quad y_4 = \max\left\{\frac{\beta}{y_3}, x_2\right\} = \max\{x_0, p\} = p = x_2.$$

Hence, $x_1 = y_3$, $x_2 = y_4$, $y_1 = x_3$, $y_2 = x_4$. And by Lemma 1, we have that the solution is periodic with period four. Moreover, we have

$$x_{4n+1} = y_{4n+3} = x_1 = \frac{\beta}{x_0}, \quad y_{4n+1} = x_{4n+3} = y_1 = \frac{\beta}{y_0};$$

$$x_{4n+2} = y_{4n+4} = x_2 = p, \quad y_{4n+2} = x_{4n+4} = y_2 = p,$$

where $n \in \mathbb{N}_0$, and the solution has the following form

$$\{x_n\}_{n=-1}^{\infty} = \left\{x_{-1}, x_0, \frac{\beta}{x_0}, p, \frac{\beta}{y_0}, p, \frac{\beta}{x_0}, p, \frac{\beta}{y_0}, p, \cdots\right\},$$

$$\{y_n\}_{n=-1}^{\infty} = \left\{y_{-1}, y_0, \frac{\beta}{y_0}, p, \frac{\beta}{x_0}, p, \frac{\beta}{y_0}, p, \frac{\beta}{x_0}, p, \cdots\right\}.$$

Case 2. $\frac{\beta}{x_0} < y_{-1}$, $\frac{\beta}{y_0} \ge x_{-1}$. We have

$$x_1 = \max\left\{\frac{\beta}{x_0}, y_{-1}\right\} = y_{-1}, \quad y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} = \frac{\beta}{y_0};$$

$$x_2 = \max\left\{\frac{\beta}{x_1}, y_0\right\} = \max\left\{\frac{\beta}{y_{-1}}, y_0\right\} = q, \quad y_2 = \max\left\{\frac{\beta}{y_1}, x_0\right\} = \max\{y_0, x_0\} = p;$$

$$x_3 = \max\left\{\frac{\beta}{x_2}, y_1\right\} = \max\left\{\frac{\beta}{q}, \frac{\beta}{y_0}\right\} = \frac{\beta}{y_0} = y_1, \quad y_3 = \max\left\{\frac{\beta}{y_2}, x_1\right\} = \max\left\{\frac{\beta}{p}, y_{-1}\right\} = y_{-1} = x_1;$$

$$x_4 = \max\left\{\frac{\beta}{x_3}, y_2\right\} = \max\{y_0, p\} = p = y_2, \quad y_4 = \max\left\{\frac{\beta}{y_3}, x_2\right\} = \max\left\{\frac{\beta}{y_{-1}}, q\right\} = q = x_2.$$

Hence, $x_1 = y_3,\ x_2 = y_4,\ y_1 = x_3,\ y_2 = x_4$. And by Lemma 1, we have that the solution is periodic with period four. Moreover, we have

$$x_{4n+1} = y_{4n+3} = x_1 = y_{-1}, \quad y_{4n+1} = x_{4n+3} = y_1 = \frac{\beta}{y_0};$$

$$x_{4n+2} = y_{4n+4} = x_2 = q, \quad y_{4n+2} = x_{4n+4} = y_2 = p,$$

where $n \in \mathbb{N}_0$, and the solution has the following form

$$\{x_n\}_{n=-1}^{\infty} = \left\{x_{-1}, x_0, y_{-1}, q, \frac{\beta}{y_0}, p, y_{-1}, q, \frac{\beta}{y_0}, p, \cdots\right\},$$

$$\{y_n\}_{n=-1}^{\infty} = \left\{y_{-1}, y_0, \frac{\beta}{y_0}, p, y_{-1}, q, \frac{\beta}{y_0}, p, y_{-1}, q, \cdots\right\}.$$

Case 3. $\frac{\beta}{x_0} \ge y_{-1},\ \frac{\beta}{y_0} < x_{-1}$. We have

$$x_1 = \max\left\{\frac{\beta}{x_0}, y_{-1}\right\} = \frac{\beta}{x_0}, \quad y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} = x_{-1};$$

$$x_2 = \max\left\{\frac{\beta}{x_1}, y_0\right\} = \max\{x_0, y_0\} = p, \quad y_2 = \max\left\{\frac{\beta}{y_1}, x_0\right\} = \max\left\{\frac{\beta}{x_{-1}}, x_0\right\} = s;$$

$$x_3 = \max\left\{\frac{\beta}{x_2}, y_1\right\} = \max\left\{\frac{\beta}{p}, x_{-1}\right\} = x_{-1} = y_1, \quad y_3 = \max\left\{\frac{\beta}{y_2}, x_1\right\} = \max\left\{\frac{\beta}{s}, \frac{\beta}{x_0}\right\} = \frac{\beta}{x_0} = x_1;$$

$$x_4 = \max\left\{\frac{\beta}{x_3}, y_2\right\} = \max\left\{\frac{\beta}{x_{-1}}, s\right\} = s = y_2, \quad y_4 = \max\left\{\frac{\beta}{y_3}, x_2\right\} = \max\{x_0, p\} = p = x_2.$$

Hence, $x_1 = y_3,\ x_2 = y_4,\ y_1 = x_3,\ y_2 = x_4$. And by Lemma 1, we have that the solution is periodic with period four. Moreover, we have

$$x_{4n+1} = y_{4n+3} = x_1 = \frac{\beta}{x_0}, \quad y_{4n+1} = x_{4n+3} = y_1 = x_{-1};$$

$$x_{4n+2} = y_{4n+4} = x_2 = p, \quad y_{4n+2} = x_{4n+4} = y_2 = s,$$

where $n \in \mathbb{N}_0$, and the solution has the following form

$$\{x_n\}_{n=-1}^{\infty} = \left\{x_{-1}, x_0, \frac{\beta}{x_0}, p, x_{-1}, s, \frac{\beta}{x_0}, p, x_{-1}, s, \cdots\right\},$$

$$\{y_n\}_{n=-1}^{\infty} = \left\{y_{-1}, y_0, x_{-1}, s, \frac{\beta}{x_0}, p, x_{-1}, s, \frac{\beta}{x_0}, p, \cdots\right\}.$$

Case 4. $\frac{\beta}{x_0} < y_{-1},\ \frac{\beta}{y_0} < x_{-1}$. We have

$$x_1 = \max\left\{\frac{\beta}{x_0}, y_{-1}\right\} = y_{-1}, \quad y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} = x_{-1};$$

$$x_2 = \max\left\{\frac{\beta}{x_1}, y_0\right\} = \max\left\{\frac{\beta}{y_{-1}}, y_0\right\} = q, \quad y_2 = \max\left\{\frac{\beta}{y_1}, x_0\right\} = \max\left\{\frac{\beta}{x_{-1}}, x_0\right\} = s;$$

$$x_3 = \max\left\{\frac{\beta}{x_2}, y_1\right\} = \max\left\{\frac{\beta}{q}, x_{-1}\right\} = x_{-1} = y_1, \quad y_3 = \max\left\{\frac{\beta}{y_2}, x_1\right\} = \max\left\{\frac{\beta}{s}, y_{-1}\right\} = y_{-1} = x_1;$$

$$x_4 = \max\left\{\frac{\beta}{x_3}, y_2\right\} = \max\left\{\frac{\beta}{x_{-1}}, s\right\} = s = y_2, \quad y_4 = \max\left\{\frac{\beta}{y_3}, x_2\right\} = \max\left\{\frac{\beta}{y_{-1}}, q\right\} = q = x_2.$$

Hence, $x_1 = y_3,\ x_2 = y_4,\ y_1 = x_3,\ y_2 = x_4$. And by Lemma 1, we have that the solution is periodic with period four. Moreover, we have

$$x_{4n+1} = y_{4n+3} = x_1 = y_{-1}, \quad y_{4n+1} = x_{4n+3} = y_1 = x_{-1};$$
$$x_{4n+2} = y_{4n+4} = x_2 = q, \quad y_{4n+2} = x_{4n+4} = y_2 = s,$$

where $n \in \mathbb{N}_0$, and the solution has the following form

$$\{x_n\}_{n=-1}^{\infty} = \{x_{-1}, x_0, y_{-1}, q, x_{-1}, s, y_{-1}, q, x_{-1}, s, \cdots\},$$
$$\{y_n\}_{n=-1}^{\infty} = \{y_{-1}, y_0, x_{-1}, s, y_{-1}, q, x_{-1}, s, y_{-1}, q, \cdots\}.$$

So we complete the proof.

Lemma 4 *Assume that* $\beta > 0$. *Then every solution of System* (1) *with negative initial conditions is periodic with period four.*

Proof Since $x_{-1},\ x_0,\ y_{-1},\ y_0 < 0$ and $\beta > 0$, by induction we have $x_n < 0,\ y_n < 0$. If we use the change $z_n = -x_n,\ w_n = -y_n$ and System (1) can be rewritten as follows

$$\begin{cases} z_{n+1} = \min\left\{\dfrac{\beta}{z_n}, w_{n-1}\right\}, \\ w_{n+1} = \min\left\{\dfrac{\beta}{w_n}, z_{n-1}\right\}. \end{cases} \tag{5}$$

where $z_n,\ w_n > 0,\ n = -1, 0, 1, \cdots$.

Now, we will prove that every solution of System (5) with positive initial conditions is periodic with period four.

Let $p' = \min\{z_0, w_0\}$, $q' = \min\left\{\dfrac{\beta}{w_{-1}}, w_0\right\}$, $s' = \min\left\{\dfrac{\beta}{z_{-1}}, z_0\right\}$. Similar to the proof of Lemma (3), there are four cases which need to be discussed.

Case 1. $\dfrac{\beta}{z_0} < w_{-1},\ \dfrac{\beta}{w_0} < z_{-1}$. We obtain that

$$\{z_n\}_{n=-1}^{\infty} = \left\{z_{-1}, z_0, \frac{\beta}{z_0}, p', \frac{\beta}{w_0}, p', \frac{\beta}{z_0}, p', \frac{\beta}{w_0}, p', \cdots\right\},$$
$$\{w_n\}_{n=-1}^{\infty} = \left\{w_{-1}, w_0, \frac{\beta}{w_0}, p', \frac{\beta}{z_0}, p', \frac{\beta}{w_0}, p', \frac{\beta}{z_0}, p', \cdots\right\}.$$

Case 2. $\dfrac{\beta}{z_0} > w_{-1},\ \dfrac{\beta}{w_0} < z_{-1}$. We obtain that

$$\{z_n\}_{n=-1}^{\infty} = \left\{z_{-1}, z_0, w_{-1}, q', \frac{\beta}{w_0}, p', w_{-1}, q', \frac{\beta}{w_0}, p', \cdots\right\},$$
$$\{w_n\}_{n=-1}^{\infty} = \left\{w_{-1}, w_0, \frac{\beta}{w_0}, p', w_{-1}, q', \frac{\beta}{w_0}, p', w_{-1}, q', \cdots\right\}.$$

Case 3. $\dfrac{\beta}{z_0} < w_{-1},\ \dfrac{\beta}{w_0} > z_{-1}$. We obtain that

$$\{z_n\}_{n=-1}^{\infty} = \left\{z_{-1}, z_0, \frac{\beta}{z_0}, p', z_{-1}, s', \frac{\beta}{z_0}, p', z_{-1}, s', \cdots\right\},$$
$$\{w_n\}_{n=-1}^{\infty} = \left\{w_{-1}, w_0, z_{-1}, s', \frac{\beta}{z_0}, p', z_{-1}, s', \frac{\beta}{z_0}, p', \cdots\right\}.$$

Case 4. $\frac{\beta}{z_0} > w_{-1}$, $\frac{\beta}{w_0} > z_{-1}$. We obtain that

$$\{z_n\}_{n=-1}^{\infty} = \{z_{-1}, z_0, w_{-1}, q', z_{-1}, s', w_{-1}, q', z_{-1}, s', \cdots\},$$
$$\{w_n\}_{n=-1}^{\infty} = \{w_{-1}, w_0, z_{-1}, s', w_{-1}, q', z_{-1}, s', w_{-1}, q', \cdots\}.$$

So we complete the proof.

Lemma 5 *Assume that* $\beta < 0$. *Then every solution of System* (1) *is periodic with period four if initial conditions satisfy one of the following conditions*

$$(a)\ x_{-1},\ x_0,\ y_{-1},\ y_0 > 0; \quad (b)\ x_{-1},\ x_0,\ y_{-1},\ y_0 < 0;$$
$$(c)\ x_{-1},\ y_0 > 0,\ x_0,\ y_{-1} < 0; \quad (d)\ x_{-1},\ y_0 < 0,\ x_0,\ y_{-1} > 0.$$

Proof (a) If $x_{-1}, x_0, y_{-1}, y_0 > 0$, by using System (1), we know that there is only one case which needs to be discussed. That is

$$\frac{\beta}{x_0} < y_{-1}, \quad \frac{\beta}{y_0} < x_{-1}$$

Then we have

$$x_1 = \max\left\{\frac{\beta}{x_0}, y_{-1}\right\} = y_{-1} > 0, \quad y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} = x_{-1} > 0;$$
$$x_2 = \max\left\{\frac{\beta}{x_1}, y_0\right\} = y_0 > 0, \quad y_2 = \max\left\{\frac{\beta}{y_1}, x_0\right\} = x_0 > 0;$$
$$x_3 = \max\left\{\frac{\beta}{x_2}, y_1\right\} = y_1 = x_{-1} > 0, \quad y_3 = \max\left\{\frac{\beta}{y_2}, x_1\right\} = x_1 = y_{-1} > 0;$$
$$x_4 = \max\left\{\frac{\beta}{x_3}, y_2\right\} = y_2 = x_0 > 0, \quad y_4 = \max\left\{\frac{\beta}{y_3}, x_2\right\} = x_2 = y_0 > 0.$$

Hence, $x_1 = y_3, x_2 = y_4, y_1 = x_3, y_2 = x_4$. And by Lemma 1, we have that the solution is periodic with period four.

The proof of case $(b)(c)(d)$ is similar to the proof of case (a), so we omit it. Then, the proof is completed.

Lemma 6 *Assume that* $\beta < 0$. *Then every solution of System* (1) *is periodic with period four if initial conditions satisfy one of the following conditions*

$$(e)\ x_{-1} < 0,\ x_0,\ y_{-1},\ y_0 > 0; \quad (f)\ x_0 < 0,\ x_{-1},\ y_{-1},\ y_0 > 0;$$
$$(j)\ y_{-1} < 0,\ x_{-1},\ x_0,\ y_0 > 0; \quad (h)\ y_0 < 0,\ x_{-1},\ x_0,\ y_{-1} > 0;$$
$$(i)\ x_{-1} > 0,\ x_0,\ y_{-1},\ y_0 < 0; \quad (j)\ x_0 > 0,\ x_{-1},\ y_{-1},\ y_0 < 0;$$
$$(k)\ y_{-1} < 0,\ x_{-1},\ x_0,\ y_0 > 0; \quad (l)\ y_0 > 0,\ x_{-1},\ x_0,\ y_{-1} < 0.$$

Proof (e) If $x_{-1} < 0, x_0, y_{-1}, y_0 > 0$, by using System (1), we know that $\frac{\beta}{x_0} < y_{-1}$ so there are two cases which need to be discussed. That is

$$(e1)\ \frac{\beta}{y_0} \geq x_{-1}, \quad (e2)\ \frac{\beta}{y_0} < x_{-1}$$

Case 1. $(e1)$ $\frac{\beta}{y_0} \geq x_{-1}$. We have

$$x_1 = \max\left\{\frac{\beta}{x_0}, y_{-1}\right\} = y_{-1} > 0, \quad y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} = \frac{\beta}{y_0} < 0;$$

$$x_2 = \max\left\{\frac{\beta}{x_1}, y_0\right\} = \max\left\{\frac{\beta}{y_{-1}}, y_0\right\} = y_0 > 0, \quad y_2 = \max\left\{\frac{\beta}{y_1}, x_0\right\} = \max\left\{y_0, x_0\right\} = p > 0;$$

$$x_3 = \max\left\{\frac{\beta}{x_2}, y_1\right\} = \max\left\{\frac{\beta}{y_0}, \frac{\beta}{y_0}\right\} = \frac{\beta}{y_0} = y_1 < 0, \quad y_3 = \max\left\{\frac{\beta}{y_2}, x_1\right\} = \max\left\{\frac{\beta}{p}, y_{-1}\right\} = y_{-1} = x_1 > 0;$$

$$x_4 = \max\left\{\frac{\beta}{x_3}, y_2\right\} = \max\left\{y_0, p\right\} = p = y_2 > 0, \quad y_4 = \max\left\{\frac{\beta}{y_3}, x_2\right\} = \max\left\{\frac{\beta}{y_{-1}}, y_0\right\} = y_0 = x_2 > 0.$$

Hence, $x_1 = y_3,\ x_2 = y_4,\ y_1 = x_3,\ y_2 = x_4$. And by Lemma 1, we have that the solution is periodic with period four.

Case 2. $(e2)\ \frac{\beta}{y_0} < x_{-1}$. We have

$$x_1 = \max\left\{\frac{\beta}{x_0}, y_{-1}\right\} = y_{-1} > 0, \quad y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} = x_{-1} < 0;$$

$$x_2 = \max\left\{\frac{\beta}{x_1}, y_0\right\} = \max\left\{\frac{\beta}{y_{-1}}, y_0\right\} = y_0 > 0, \quad y_2 = \max\left\{\frac{\beta}{y_1}, x_0\right\} = \max\left\{\frac{\beta}{x_{-1}}, x_0\right\} = s > 0;$$

$$x_3 = \max\left\{\frac{\beta}{x_2}, y_1\right\} = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} = x_{-1} = y_1 < 0, \quad y_3 = \max\left\{\frac{\beta}{y_2}, x_1\right\} = \max\left\{\frac{\beta}{s}, y_{-1}\right\} = y_{-1} = x_1 > 0;$$

$$x_4 = \max\left\{\frac{\beta}{x_3}, y_2\right\} = \max\left\{\frac{\beta}{x_{-1}}, s\right\} = s = y_2 > 0, \quad y_4 = \max\left\{\frac{\beta}{y_3}, x_2\right\} = \max\left\{\frac{\beta}{y_{-1}}, y_0\right\} = y_0 = x_2 > 0.$$

Hence, $x_1 = y_3,\ x_2 = y_4,\ y_1 = x_3,\ y_2 = x_4$. And by Lemma 1, we have that the solution is periodic with period four.

The proof of case $(f)-(l)$ is similar to the proof of case (e), so we omit it. Then, the proof is completed.

Lemma 7 *Assume that* $\beta < 0$. *Then every solution of System* (1) *is periodic with period four if initial conditions satisfy one of the following conditions*

$$(m)\ x_{-1},\ x_0 > 0,\ y_{-1},\ y_0 < 0; (n)\ x_{-1},\ x_0 < 0,\ y_{-1},\ y_0 > 0; (0)\ x_{-1},\ y_{-1} > 0,\ x_0,\ y_0 < 0; (p)\ x_{-1},\ y_{-1} < 0,\ x_0,\ y_0 > 0.$$

Proof (m) If $x_{-1},\ x_0 > 0,\ y_{-1},\ y_0 < 0$, by using System (1), we know that there are four cases which need to be discussed. That is

$$(m1)\ \frac{\beta}{x_0} \ge y_{-1},\ \frac{\beta}{y_0} \ge x_{-1}; (m2)\ \frac{\beta}{x_0} < y_{-1},\ \frac{\beta}{y_0} \ge x_{-1}; (m3)\ \frac{\beta}{x_0} \ge y_{-1},\ \frac{\beta}{y_0} < x_{-1}; (m4)\ \frac{\beta}{x_0} < y_{-1},\ \frac{\beta}{y_0} < x_{-1}.$$

Case 1. $(m1)\ \frac{\beta}{x_0} \ge y_{-1},\ \frac{\beta}{y_0} \ge x_{-1}$. We have

$$x_1 = \max\left\{\frac{\beta}{x_0}, y_{-1}\right\} = \frac{\beta}{x_0} < 0, \quad y_1 = \max\left\{\frac{\beta}{y_0}, x_{-1}\right\} = \frac{\beta}{y_0} > 0;$$

$$x_2 = \max\left\{\frac{\beta}{x_1}, y_0\right\} = \max\left\{x_0, y_0\right\} = x_0 > 0, \quad y_2 = \max\left\{\frac{\beta}{y_1}, x_0\right\} = \max\left\{y_0, x_0\right\} = x_0 > 0;$$

$$x_3 = \max\left\{\frac{\beta}{x_2}, y_1\right\} = \max\left\{\frac{\beta}{x_0}, \frac{\beta}{y_0}\right\} = \frac{\beta}{y_0} = y_1 > 0, \quad y_3 = \max\left\{\frac{\beta}{y_2}, x_1\right\} = \max\left\{\frac{\beta}{x_0}, \frac{\beta}{x_0}\right\} = \frac{\beta}{x_0} = x_1 < 0;$$

$$x_4 = \max\left\{\frac{\beta}{x_3}, y_2\right\} = \max\left\{y_0, x_0\right\} = x_0 = y_2 > 0, \quad y_4 = \max\left\{\frac{\beta}{y_3}, x_2\right\} = \max\left\{x_0, x_0\right\} = x_0 = x_2 > 0.$$

Hence, $x_1 = y_3,\ x_2 = y_4,\ y_1 = x_3,\ y_2 = x_4$. And by Lemma 1, we have that the solution is periodic with period four.

The proof of case $(m2)(m3)(m4)$ is similar to the proof of case $(b)(c)(d)$ in Lemma 3, so we omit it and case (m) is completed.

Similarly, the proof of case $(n)(o)(p)$ is similar to the proof of case (m), so we omit it.

3. Main Results

By using Lemma 2 and Lemma 3, we obtain the following result.

Theorem 1 *Assume that* $\beta > 0$. *Then every solution of System* (1) *is periodic with period four if initial conditions satisfies one of the following conditions* $x_{-1} > 0$ *or* $x_0 > 0$ *or* $y_{-1} > 0$ *or* $y_0 > 0$.

By using Theorem 1 and Lemma 4, we obtain the following result.

Theorem 2 *Assume that* $\beta > 0$. *Then every well-defined solution of System* (1) *is periodic with period four*.

By using Lemma 5, Lemma 6 and Lemma 7, we obtain the following result.

Theorem 3 *Assume that* $\beta < 0$. *Then every well-defined solution of System* (1) *is periodic with period four*.

By using Theorem 2 and Theorem 3, we obtain the following result.

Theorem 4 *Assume that* $\beta \in \Re$. *Then every well-defined solution of System* (1) *is periodic with period four*.

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