

Some Lacunary Sequence Spaces of Invariant Means Defined by Musielak-Orlicz Functions on 2-Norm Space

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Received 11 July 2014; revised 15 August 2014; accepted 25 August 2014

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Abstract

The purpose of this paper is to introduce and study some sequence spaces which are defined by combining the concepts of sequences of Musielak-Orlicz functions, invariant means and lacunary convergence on 2-norm space. We establish some inclusion relations between these spaces under some conditions.

Keywords

Invariant Means, Musielak-Orlicz Functions, 2-Norm Space, Lacunary Sequence

1. Introduction

Let ω be the set of all sequences of real numbers ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with $(x_k) \in \mathbb{R}$ or \mathbb{C} the usual norm $\|x\| = \sup_k |x_k|$, where $k \in \mathbb{N} = 1, 2, 3, \dots$, the positive integers.

The idea of difference sequence spaces was first introduced by Kizmaz [1] and then the concept was generalized by Et and Çolak [2]. Later on Et and Esi [3] extended the difference sequence spaces to the sequence spaces:

$$X(\Delta_v^m) = \{x = (x_k) : (\Delta_v^m x) \in X\},$$

for $X = \ell_\infty, c$ and c_0 , where $v = (v_k)$ be any fixed sequence of non zero complex numbers and $(\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$.

The generalized difference operator has the following binomial representation,

$$\Delta_v^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} v_{k+i} x_{k+i}, \text{ for all } k \in \mathbb{N}.$$

The sequence spaces $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ and $\Delta_v^m(c_0)$ are Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^m |v_i x_i| + \|\Delta_v^m x\|_\infty.$$

The concept of 2-normed space was initially introduced by Gähler [4] as an interesting linear generalization of normed linear space which was subsequently studied by many others [5] [6]). Recently a lot of activities have started to study summability, sequence spaces and related topics in these linear spaces [7] [8]).

Let X be a real vector space of dimension d , where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot, \cdot\|: X \times X \rightarrow \mathbb{R}$ which satisfies:

- 1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
- 2) $\|x, y\| = \|y, x\|$,
- 3) $\|\alpha x, y\| = |\alpha| \|x, y\|, \alpha \in \mathbb{R}$,
- 4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot, \cdot\|)$ is called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\|$ = the area of parallelogram spanned by the vectors x and y , which may be given explicitly by the formula

$$\|x_1, x_2\|_E = \text{abs} \left(\begin{vmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{vmatrix} \right).$$

Then clearly $(X, \|\cdot, \cdot\|)$ is 2-normed space. Recall that $(X, \|\cdot, \cdot\|)$ is a 2-Banach space if every Cauchy sequence in X is convergent to some $x \in X$.

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or σ -mean if and only if

- 1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has, $x_n \geq 0$ for all n ,
- 2) $\phi(e) = 1, e = (1, 1, 1, \dots)$,
- 3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

If $x = (x_k)$, where $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \left\{ x \in \ell_\infty : \lim_k t_{kn}(x) = l, \text{ uniformly in } n \right\}$$

$l = \sigma\text{-}\lim x$, where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1(n)} + x_{\sigma^2(n)} + \dots + x_{\sigma^k(n)}}{k+1} \quad [9].$$

In the case σ is the translation mapping $n \rightarrow n+1$, σ -mean is often called a Banach limit and V_σ the set of bounded sequences of all whose invariant means are equal is the set of almost convergent sequence [10].

By Lacunary sequence $\theta = (k_r), r = 0, 1, 2, \dots$ where $k_0 = 0$ we mean an increasing sequence of non negative integers $h_r = (k_r - k_{r-1}) \rightarrow \infty (r \rightarrow \infty)$. The intervals determined by θ are denoted by $I_r = [k_{r-1} - k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r . The space of lacunary strongly convergent sequence N_θ was defined by Freedman *et al.* [11] as follows:

$$N_\theta = \left\{ x = (x_i) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k=I_r} |x_k - l| = 0 \text{ for some } l \right\}.$$

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

It is well known that if M is convex function and $M(0) = 0$ then $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 \leq \lambda \leq 1$.

Lindenstrauss and Tzafriri [12] used the idea of Orlicz function and defined the sequence space which was called an Orlicz sequence space ℓ_M such as

$$\ell_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which was a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

which was called an Orlicz sequence space. The ℓ_M was closely related to the space ℓ_p which was an Orlicz sequence space with $M(t) = |t|^p$, for $1 \leq p < \infty$. Later the Orlicz sequence spaces were investigated by Prashar and Choudhry [13], Maddox [14], Tripathy *et al.* [15]-[17] and many others.

A sequence of function $M = (M_k)$ of Orlicz function is called a Musielak-Orlicz function [18] [19]. Also a Musielak-Orlicz function $\Phi = (\Phi_k)$ is called complementary function of a Musielak-Orlicz function M if

$$\Phi_k(t) = \sup \{ |t|s - M_k(s) : s \geq 0 \}, \text{ for } k = 1, 2, 3, \dots$$

For a given Musielak-Orlicz function M , the Musielak-Orlicz sequence space l_M and its subspaces \hat{h}_M are defined as follow:

$$l_M = \{ x = x_k \in \omega : I_M(cx) < \infty, \text{ for some } c > 0 \}$$

$$\hat{h}_M = \{ x = x_k \in \omega : I_M(cx) < \infty, \text{ for all } c > 0 \}$$

where I_M is a convex modular defined by

$$I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in l_M.$$

We consider l_M equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_M\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} (1 + I_M(kx)) : k > 0 \right\}.$$

The main purpose of this paper is to introduce the following sequence spaces and examine some properties of the resulting sequence spaces. Let $M = (M_k)$ be a Musielak-Orlicz function, $(X, \|\cdot, \cdot\|)$ is called a 2-normed space. Let $p = (p_k)$ be any sequences of positive real numbers, for all $k \in \mathbb{N}$ and $u = (u_k)$ such that $u_k \neq 0$ ($k = 1, 2, 3, \dots$). Let s be any real number such that $s \geq 0$. By $\omega(2-X)$ we denote the space of all sequences defined over $(X, \|\cdot, \cdot\|)$. Then we define the following sequence spaces:

$$\begin{aligned} & \left[\omega^\rho, M, p, u, s, \|\cdot, \cdot\| \right]_\sigma^\infty (\Delta_v^m) \\ & = \left\{ x = (x_k) \in \omega(2-X) : \sup_{r, n} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \left[M_k \left(\left\| \frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} < \infty, \rho > 0, s \geq 0 \right\} \end{aligned}$$

$$\begin{aligned} & \left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\| \right]_\sigma (\Delta_v^m) \\ & = \left\{ x = (x_k) \in \omega(2-X) :: \lim_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \left[M_k \left(\left\| \frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} = 0 \text{ for some } l, \rho > 0, s \geq 0 \right\} \\ & \left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\| \right]_\sigma^0 (\Delta_v^m) \\ & = \left\{ x = (x_k) \in \omega(2-X) :: \lim_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \left[M_k \left(\left\| \frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} = 0 \quad \rho > 0, s \geq 0 \right\} \end{aligned}$$

Definition 1. A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalar (α_k) with $|\alpha_k| \leq 1$ [20].

Definition 2. A sequence space E is said to be monotone if it contains the canonical pre-images of all its steps spaces, [20].

Definition 3. If X is a Banach space normed by $\|\cdot\|$, then $\Delta^m(X)$ is also Banach space normed by

$$\|x\|_\Delta = \sum_{k=1}^m |x_k| + f(\Delta^m x).$$

Remark 1. The following inequality will be used throughout the paper. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G$, $D = \max(1, 2^{G-1})$. Then for all $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N}$. We have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}). \tag{1}$$

2. Main Results

Theorem 1. Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real number and $\theta = (k_r)$ be a lacunary sequence. Then $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\| \right]_\sigma^\infty (\Delta_v^m)$, $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\| \right]_\sigma (\Delta_v^m)$ and $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\| \right]_\sigma^0 (\Delta_v^m)$ are linear spaces over the field of complex numbers.

Proof 1. Let $x = (x_k), y = (y_k) \in \left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\| \right]_\sigma^0 (\Delta_v^m)$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some ρ_3 such that,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m(\alpha x_k + \beta y_k))}{\rho_3}, z \right\| \right) \right]^{p_k} = 0, \text{ uniformly in } n.$$

Since $(x_k), (y_k) \in \left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\| \right]_\sigma^0 (\Delta_v^m)$, there exist positive ρ_1, ρ_2 such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} = 0 \text{ uniformly in } n$$

and

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{kn}(\Delta_v^m y_k)}{\rho}, z \right\| \right) \right]^{p_k} = 0 \text{ uniformly in } n.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is non decreasing and convex

$$\begin{aligned}
 & \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m(\alpha x_k + \beta y_k))}{\rho_3}, z \right\| \right) \right]^{p_k} \\
 & \leq \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m(\alpha x_k))}{\rho_3}, z \right\| + \left\| \frac{t_{nk}(\Delta_v^m(\beta y_k))}{\rho_3}, z \right\| \right) \right]^{p_k} \\
 & \leq \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho_1}, z \right\| + \left\| \frac{t_{nk}(\Delta_v^m y_k)}{\rho_2}, z \right\| \right) \right]^{p_k} \\
 & \leq \frac{D}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} + \frac{D}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m y_k)}{\rho}, z \right\| \right) \right]^{p_k} \\
 & \rightarrow 0, \text{ as } r \rightarrow \infty, \text{ uniformly in } n.
 \end{aligned}$$

So that $(\alpha x_k) + (\beta y_k) \in [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma^0]_\sigma(\Delta_v^m)$. This completes the proof. Similarly, we can prove that $[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma]_\sigma(\Delta_v^m)$ and $[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma^\infty]_\sigma(\Delta_v^m)$ are linear spaces.

Theorem 2. Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real number and $\theta = (k_r)$ be a lacunary sequence. Then $[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma^0]_\sigma(\Delta_v^m)$ is a topological linear space totalparanormed by

$$g_\Delta(x) = \sum_{k=1}^m |x_k| + \inf \left\{ \rho^{p_r/H} : \left(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} \right)^{1/H} \leq 1 \text{ for some } \rho, r = 1, 2, \dots \right\}$$

Proof 2. Clearly $g_\Delta(x) = g_\Delta(-x)$. Since $M_k(0) = 0$, for all $k \in \mathbb{N}$. we get $g_\Delta(\bar{\theta}) = 0$, for $x = \bar{\theta}$. Let $x = (x_k)$, $y = (y_k) \in [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma^0]_\sigma(\Delta_v^m)$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_r h_r^{-1} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m(x_k))}{\rho_1}, z \right\| \right) \right]^{p_k} \leq 1 \quad r = 1, 2, 3, \dots$$

and

$$\sup_r h_r^{-1} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m(y_k))}{\rho_2}, z \right\| \right) \right]^{p_k} \leq 1 \quad r = 1, 2, 3, \dots$$

Let $\rho = \rho_1 + \rho_2$, then we have

$$\begin{aligned}
 & \sup_r h_r^{-1} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m(x_k + y_k))}{\rho}, z \right\| \right) \right]^{p_k} \\
 & \leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_r h_r^{-1} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m(x_k))}{\rho_1}, z \right\| \right) \right]^{p_k} \\
 & \quad + \frac{\rho_2}{\rho_1 + \rho_2} \sup_r h_r^{-1} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m(y_k))}{\rho_2}, z \right\| \right) \right]^{p_k} \leq 1.
 \end{aligned}$$

Since $\rho > 0$, we have

$$\begin{aligned}
 &g_{\Delta}(x+y) \\
 &= \sum_{k=1}^m |x_k + y_k| + \inf \left\{ \rho^{p_r/H} : \left(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk} \Delta_v^m(x_k + y_k)}{\rho}, z \right\| \right) \right]^{p_k} \right)^{1/H} \leq 1 \text{ for some } \rho > 0, r = 1, 2, \dots \right\} \\
 &\leq \sum_{k=1}^m |x_k| + \inf \left\{ \rho_1^{p_r/H} : \left(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk} \Delta_v^m(x_k)}{\rho_1}, z \right\| \right) \right]^{p_k} \right)^{1/H} \leq 1 \text{ for some } \rho_1 > 0, r = 1, 2, \dots \right\} \\
 &\quad + \sum_{k=1}^m |y_k| + \inf \left\{ \rho_2^{p_r/H} : \left(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk} \Delta_v^m(y_k)}{\rho_2}, z \right\| \right) \right]^{p_k} \right)^{1/H} \leq 1 \text{ for some } \rho_2 > 0, r = 1, 2, \dots \right\}.
 \end{aligned}$$

$$g_{\Delta}(x+y) \leq g_{\Delta}(x) + g_{\Delta}(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be a given non zero scalar in \mathbb{C} . Then the continuity of the product follows from the following expression.

$$\begin{aligned}
 g_{\Delta}(\lambda x) &= \sum_{k=1}^m |\lambda x_k| + \inf \left\{ \rho^{p_r/H} : \left(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk} \Delta_v^m(\lambda x_k)}{\rho}, z \right\| \right) \right]^{p_k} \right)^{1/H} \leq 1 \text{ for some } \rho > 0, r = 1, 2, \dots \right\} \\
 &= \lambda \sum_{k=1}^m |x_k| + \inf \left\{ (|\lambda| \zeta)^{p_r/H} : \left(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk} \Delta_v^m(x_k)}{\zeta}, z \right\| \right) \right]^{p_k} \right)^{1/H} \leq 1 \text{ for some } \zeta > 0, r = 1, 2, \dots \right\}
 \end{aligned}$$

where $\zeta = \frac{\rho}{|\lambda|} > 0$. Since $|\lambda|^{p_r} \leq \max(1, |\lambda|)^{\sup p_r}$,

$$\begin{aligned}
 &g_{\Delta}(\lambda x) = \max(1, |\lambda|)^{\sup p_r} \\
 &\quad + \inf \left\{ \rho^{p_r/H} : \left(\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} \right)^{1/H} \leq 1, \text{ for some } \rho > 0, r = 1, 2, \dots \right\}.
 \end{aligned}$$

This completes the proof of this theorem.

Theorem 3. Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real number and $\theta = (k_r)$ be a lacunary sequence. Then

$$[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma]_\sigma^\infty(\Delta_v^m) \subset [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma]_\sigma^0(\Delta_v^m) \subset [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma]_\sigma^0(\Delta_v^m).$$

Proof 3. The inclusion $[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma]_\sigma^0(\Delta_v^m) \subset [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma]_\sigma^0(\Delta_v^m)$ is obvious. Let $x_k \in [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma]_\sigma^0(\Delta_v^m)$. Then there exists some positive number ρ_1 such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k - l e)}{\rho_1}, z \right\| \right) \right]^{p_k} \rightarrow 0$$

as $r \rightarrow \infty$, uniformly in n . Define $\rho = 2\rho_1$. Since M_k is non decreasing and convex for all $k \in \mathbb{N}$, we have

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} \\ & \leq \frac{D}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k - le)}{\rho_1}, z \right\| \right) \right]^{p_k} \\ & \quad + \frac{D}{h_r} \sum_{k \in I_r} \left[M_k \left(\left\| \frac{le}{\rho_1}, z \right\| \right) \right]^{p_k} \\ & \leq \frac{D}{h_r} \sum_{k \in I_r} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k - le)}{\rho_1}, z \right\| \right) \right]^{p_k} \\ & \quad + D \max \left\{ 1, \left[M \left(\left\| \frac{le}{\rho_1}, z \right\| \right) \right]^G \right\} \end{aligned}$$

where $G = \sup_k (p_k)$, $D = \max(1, 2^G - 1)$ by (1).

Thus $x_k \in [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma(\Delta_v^m)$.

Theorem 4. Let $M = (M_k)$ be a Musielak-Orlicz functions. If $\sup_k [M_k(z)]^{p_k} < \infty$ for all $z > 0$, then $[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma(\Delta_v^m) \subset [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma^\infty(\Delta_v^m)$.

Proof 4. Let $x_k \in [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma(\Delta_v^m)$ by using (1), we have

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} \\ & \leq \frac{D}{h_r} \sum_{k \in I_r} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k - le)}{\rho}, z \right\| \right) \right]^{p_k} \\ & \quad + \frac{D}{h_r} \sum_{k \in I_r} \left[M_k \left(\left\| \frac{le}{\rho}, z \right\| \right) \right]^{p_k}. \end{aligned}$$

Since $\sup_k [M(z)]^{p_k} < \infty$, we can take the $\sup_k [M(z)]^{p_k} = K$. Hence we can get $x_k \in [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma(\Delta_v^m)$.

This complete the proof.

Theorem 5. Let $m \geq 1$ be fixed integer. Then the following statements are equivalent:

- 1) $[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma^\infty(\Delta_v^{m-1}) \subset [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma^\infty(\Delta_v^m)$,
- 2) $[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma(\Delta_v^{m-1}) \subset [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma(\Delta_v^m)$,
- 3) $[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma^0(\Delta_v^{m-1}) \subset [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma^0(\Delta_v^m)$.

Proof 5. Let $x_k \in [\omega^\theta, M, p, u, s, \|\cdot, \cdot\|]_\sigma^0(\Delta_v^{m-1})$. Then there exist $\rho > 0$ such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} \rightarrow 0.$$

Since M_k is non decreasing and convex, we have

$$\begin{aligned}
 & \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{2\rho}, z \right\| \right) \right]^{p_k} \\
 &= \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})}{2\rho}, z \right\| \right) \right]^{p_k} \\
 &\leq \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^{m-1} x_k)}{2\rho}, z \right\| \right) \right]^{p_k} + \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^{m-1} x_{k+1})}{2\rho}, z \right\| \right) \right]^{p_k} \\
 &\leq \frac{D}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^{m-1} x_k)}{\rho}, z \right\| \right) \right]^{p_k} + \frac{D}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^{m-1} x_{k+1})}{\rho}, z \right\| \right) \right]^{p_k}.
 \end{aligned}$$

Taking $\lim_{r \rightarrow \infty}$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} = 0,$$

i.e. $x_k \in \left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_0^0 (\Delta_v^{m-1})$. The rest of these cases can be proved in similar way.

Theorem 6. Let $M = (M_k)$ and $T = (T_k)$ be two Musielak-Orlicz functions. Then we have

- 1) $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^\infty (\Delta_v^m) \cap \left[\omega^\theta, T, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^\infty (\Delta_v^m) \subset \left[\omega^\theta, M + T, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^\infty (\Delta_v^m)$.
- 2) $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0 (\Delta_v^m) \cap \left[\omega^\theta, T, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0 (\Delta_v^m) \subset \left[\omega^\theta, M + T, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0 (\Delta_v^m)$.
- 3) $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0 (\Delta_v^m) \cap \left[\omega^\theta, T, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0 (\Delta_v^m) \subset \left[\omega^\theta, M + T, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0 (\Delta_v^m)$.

Proof 6. Let $x_k \in \left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^\infty (\Delta_v^m) \cap \left[\omega^\theta, T, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^\infty (\Delta_v^m)$. Then

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} < \infty$$

and

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[T_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} < \infty$$

uniformly in n . We have

$$\left[(M_k + T_k) \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} \leq D \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} + D \left[T_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k}$$

by (1). Applying $\sum_{k \in I_r}$ and multiplying by $u_k, \frac{1}{h_r}$ and k^{-s} both side of this inequality, we get

$$\begin{aligned}
 & \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[(M_k + T_k) \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} \\
 & \leq \frac{D}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k} + \frac{D}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[T_k \left(\left\| \frac{t_{nk}(\Delta_v^m x_k)}{\rho}, z \right\| \right) \right]^{p_k}
 \end{aligned}$$

uniformly in n . This completes the proof 2) and 3) can be proved similar to 1).

Theorem 7. 1) The sequence spaces $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^\infty$ and $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0$ are solid and hence they are monotone.

2) The space $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma$ is not monotone and neither solid nor perfect.

Proof 7. We give the proof for $\left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0$. Let $x_k \in \left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0$ and (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$\frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(\alpha_k x_k)}{\rho}, z \right\| \right) \right]^{p_k} \leq \frac{1}{h_r} \sum_{k \in I_r} u_k k^{-s} \left[M_k \left(\left\| \frac{t_{nk}(x_k)}{\rho}, z \right\| \right) \right]^{p_k} \rightarrow 0$$

($r \rightarrow \infty$), uniformly in n . Hence $(\alpha_k x_k) \in \left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0$ for all sequence of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $x_k \in \left[\omega^\theta, M, p, u, s, \|\cdot, \cdot\|_\sigma \right]_\sigma^0$. The spaces are monotone follows from the remark (1).

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