

Extended Reciprocity to Include Active Circuits

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Abstract

The reciprocity to include active circuits is quite new. This article intends to generalize the subject of covering active circuits with all kinds of dependent sources and I/O ports. First, we start by specifying how this reciprocity responds with a variety of component representations, for example, impedances and admittances. The purpose is to construct a reciprocal for any active circuit so that it produces the same transfer function when the input and output ports are swapped. There is no limit on the number of output ports in this development, but for multiple input ports, this reciprocity works only when the input sources are related. This presentation does not address general multi-ports cases, however. One way to deal with the issue is to consider I/Os pair by pair and then use superposition to manage the combinations.

Keywords

Admittance Method, Analog Circuits, Nullors and Nullor Circuits, Sum of Tree Products, Reciprocity, Transfer Functions

1. Introduction

Limited reciprocity of passive circuits is developed long before [1]-[9]. The development is not limited only to circuit theory but also in other fields such as electromagnetics and microwaves. In fact, reciprocity of electrical circuits is a special case of electromagnetics, going way back to Lorentz reciprocity. What we are going to propose in this presentation is to make a general approach to circuits, both passive and active. It is going to be an extended reciprocity for circuits with all types of dependent sources as well as all types of I/O ports. The active components in circuits typically consist of four types of controlled sources: Voltage Controlled Voltage Sources (VCVS or e), Current Controlled Current Sources (CCCS or f), Voltage Controlled Current Sources (VCCS or g), and Current Controlled Voltage Sources (CCVS or h). The pair of I/O ports can also be one of the four types:

Input Voltage and Output Voltage (VIVO or e), Input Current and Output Current (CICO or f), Input Current and Output Voltage (CIVO or g), and Input Voltage and Output Current (VICO or h). Similar, but limited, approaches are also reported in literature [6]-[9] for active circuits. Typical I/O ports are considered on pair at a time here. Dealing with I/O ports, particularly with more than one pair, is also investigated in [10]. However, this presentation takes the lead and combines all cases into consideration and applies them to the reciprocal issue.

2. Reciprocal of Passive and Active Circuits

Reciprocity in linear circuits is defined as a property that cross-relates voltages and currents at two ports in a circuit, where the position of ports (input and output) is interchanged between the circuit and its reciprocal. It states as the following. Given circuit N , assume a current I_1 applied to a port P_1 generates a voltage V_2 at port P_2 . Now, if we apply the same current I_1 to P_2 in N the same voltage V_2 is produced at P_1 . Similarly, assume a voltage V_1 applied to port P_1 in N which generates a current I_2 at another port P_2 . With reciprocity, if voltage V_1 is applied to P_2 it produces a current I_2 at P_1 .

For proof we can simply consider the *Nodal Admittance Matrix* (NAM) of circuit N which is symmetrical about the main diagonal. Evidently, we can always reduce the NAM to a 2×2 matrix containing only ports P_1 and P_2 . The new NAM is still symmetrical, and the reciprocity is quite apparent. Here is the reduced NAM with only I/O ports, P_1 and P_2 present.

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (1)$$

For $I_2 = 0$ we get

$$\frac{V_2}{I_1} = \frac{Y_{21}}{Y_{21}Y_{12} - Y_{11}Y_{22}} \quad (2)$$

For the reciprocal circuit we interchange the inputs and for $I_1 = 0$ we get

$$\frac{V_1}{I_2} = \frac{Y_{12}}{Y_{21}Y_{12} - Y_{11}Y_{22}} \quad (3)$$

Now, since $Y_{21} = Y_{12}$ for passive circuits we get

$$\frac{V_1}{I_2} = \frac{V_2}{I_1} \quad (4)$$

which proves reciprocity in passive circuits. This simply shows that a passive circuit is its own reciprocal. Whereas, as we will see, the reciprocal of an active circuit is different from the original circuit. In addition, the NAM method hardly works for active circuits when we include a variety of dependent sources in the circuit and apply the reciprocity to different types of I/O ports. Therefore, we need to take a different path, but before describing a new path let us start with a generalized reciprocity theorem first.

Theorem 1: General Reciprocity For an LTI active circuit N_i with all four types

of controlled sources and assuming a pair of I/O ports of any type, we can always find another LTI active circuit N_2 , as the reciprocal of N_1 , with the following properties.

Assume an excitation signal E_1 applied to a port P_1 in N_1 generates an output response R_1 in another port P_2 resulting in the transfer function $F_1 = R_1/E_1$. Next, let us assume N_2 to be the reciprocal of N_1 with P_2 as its input port and P_1 its output port. Then an excitation signal E_2 applied to port P_2 in N_2 is generating an output response R_2 in port P_1 so that we get the new transfer function $F_2 = R_2/E_2$. Provided that E_2 and R_2 are selected according to the rules (discussed later) we get $F_2 = F_1$. This also proves that, if N_2 is the reciprocal of N_1 then N_1 is also the reciprocal of N_2 .

Starting from the last step, it is evident that, if N_2 is the reciprocal of N_1 then, if we apply the same procedure on N_2 we get to N_1 , the reciprocal of N_2 . To prove the theorem itself, we will try to construct the *reciprocal* circuit N_2 , but before that, let us consider some simple cases first.

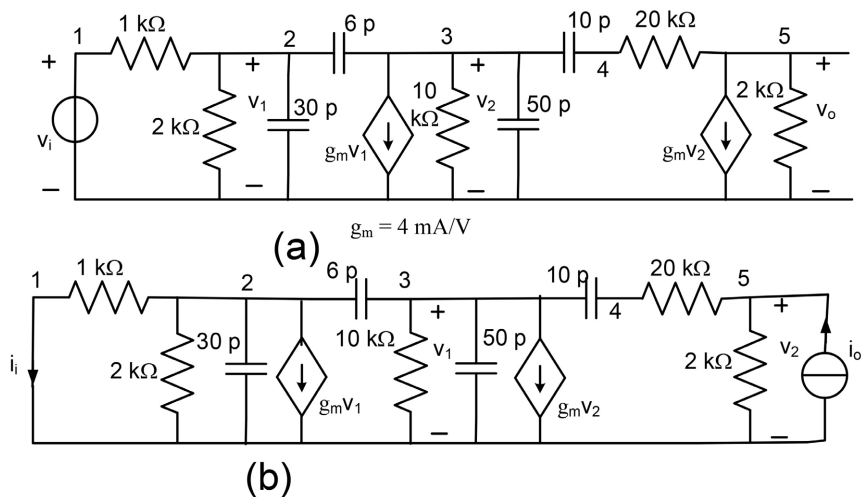


Figure 1. (a) Linear model of a two-stage amplifier; (b) The reciprocal circuit.

Example 1 Let us take a two-stage amplifier with its linear equivalent circuit shown in **Figure 1(a)**. The circuit has two VCCS devices and a pair of I/O ports with an input voltage V_i and an output voltage V_o . The reciprocal circuit of the amplifier is also constructed and given in **Figure 1(b)**. As we can see, the passive elements in the reciprocal stay unchanged. The VCCS devices are turned around, so that in each device the controlling element and the controlled element are swapped. However, in the case of the I/O ports, not only are the input and output ports swapped but the voltages are changed to currents as well. So, the voltage gain V_o/V_i in the amplifier is matched with the current gain I_i/I_o in the reciprocal circuit. After the buildup, both circuits are simulated for the AC analysis, and the frequency responses of the gains (V_o/V_i and I_i/I_o) are plotted in **Figure 2**¹. The s-expanded transfer functions of both the circuit and its reciprocal are also given in

¹One plot is slightly shifted for clarity purposes.

(5), where the frequency f is in MHz scale. Notice that the TF has three poles and two zeros, and this can be easily verified from the output plots. Finally, the Amplifier bandwidth is given as $BW = 93 \text{ KHz}$, and the gain bandwidth product is $GB = 4.88 \text{ dB-MHz}$.

$$\frac{V_o}{V_i} = \frac{I_i}{I_o} = \frac{640 + 125s + 0.19s^2}{3 + 5.6s + 0.6s^2 + 8.7 * 10^{-3} s^3} \tag{5}$$

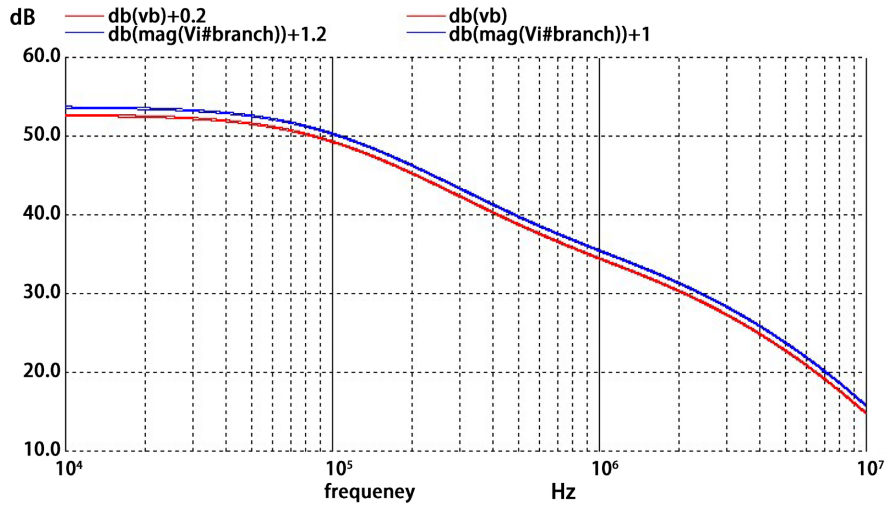


Figure 2. Frequency responses of the two-stage amplifier and its reciprocal.

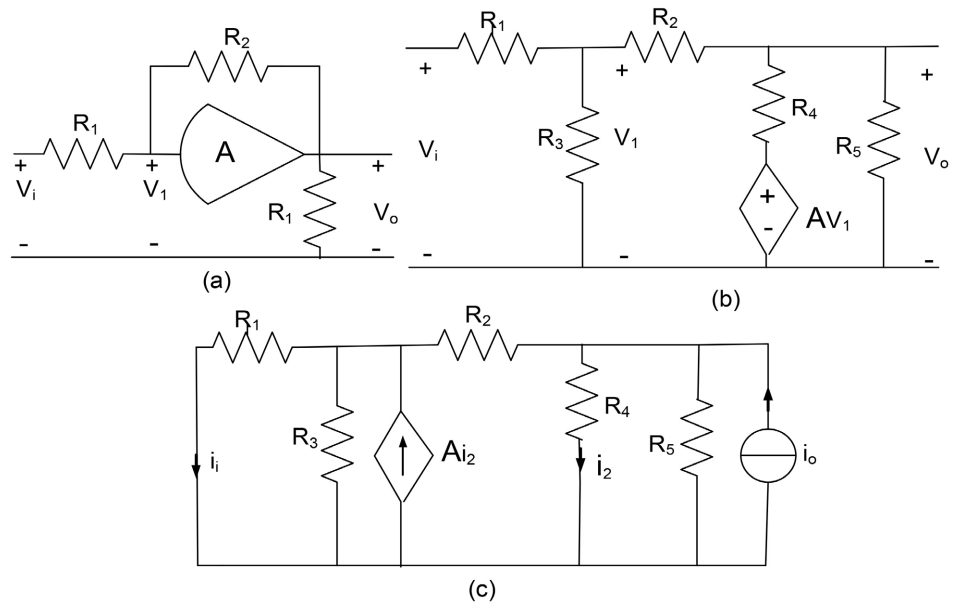


Figure 3. (a) A simplified amplifier with feedback; (b) The linearized model; (c) The reciprocal circuit.

Example 2 Consider an amplifier with feedback, as shown in Figure 3(a). The linearized circuit of the amplifier is shown in Figure 3(b). To construct the reciprocal, we follow the same rules we did in example 1, that is, the passive circuit

stays unchanged while the active components and the I/O ports swap their elements while considering the types. Namely, the VCVS in the amplifier is replaced with a CCCS, and the VIVO is replaced with a CICO ports, as shown in **Figure 3(c)**. After simulating the amplifier and its reciprocal we get the gains as $F_1 = V_i/V_o = -29.3$ for the amplifier, and $F_2 = I_i/I_o = -29.3$ for the reciprocal circuit, as expected.

Transforming Active Devices into Equivalent Nullors

Now, before attempting for a general proof of Theorem 1 we need to choose a suitable platform to get the transfer functions computed. Using the circuit NAM is limited to admittances and VCCS devices. One way to stay in the NAM platform for general cases with other types of circuit components and controlled sources is to move from the node admittance representation, $GV = I$, to a more general form

$$AV = BI \quad (5)$$

where A and B are $n \times n$ circuit matrices, and A becomes the circuit NAM, G , when B becomes a unity matrix. However, in our discussion here we skip this method and adopt a more generalized platform, dealing with graph representation of the circuit using topological formula. A major characteristic of this platform is the ability to replace active devices and I/O ports of all types with nullors. This is an important step, making an active circuit entirely transferred into a full nullor circuit. A comprehensive explanation and the development of the method is given in recent articles [11]-[13]. The main advantages of this procedure are as follows:

1) The method combines all types of controlled sources and I/O ports in the circuit into a bunch of nullors. Therefore, no matter whether we are after the determinant of the circuit NAM or any of its cofactors, the problem is reduced to finding the Sum of the Tree Products (STP) of the corresponding nullor circuit.

2) The 2-graphs (I and V) representation of the circuit, and the 2-tree representations traditionally required for the cofactors' evaluations are removed, making the circuit analysis much simpler.

To start with, we briefly go through introducing nullors and nullor circuits.

3. Nullors and Nullor Circuits

A *nullor* consists of a nullator and a norator. A nullor is either three-terminal or four-terminal. In a three-terminal, the nullator and the norator have a common node. A nullor also carries a sign depending on the direction in the actual component it represents. In a three-terminal the sign is positive if the two arrows merge at the common node or depart. Otherwise, the sign is negative. For four-terminal nullors we are undecided until the process gets through, and they change to three-terminal nullors [10]. A nullor also carries a value. The value of a nullor is 1. When it replaces a controlled device the gain of the controlled source is multiplied with the *coefficient-multiplier* p_k , for the sub-circuit k . For example, if a nullor replaces a VCVS device with $e = 20 \text{ V/V}$, then the nullor selected is of e type with, and the value $e = 20 \text{ V/V}$ is multiplied by p_k . Another property of a nullor is its adaptability

when removed, as illustrated in **Figure 4**. This can be stated as follows. If a nullor, which has replaced an active device or a pair of I/O ports, is removed from a circuit the status of the circuit will be the same as if the original device or the pair of I/O ports is removed [12].

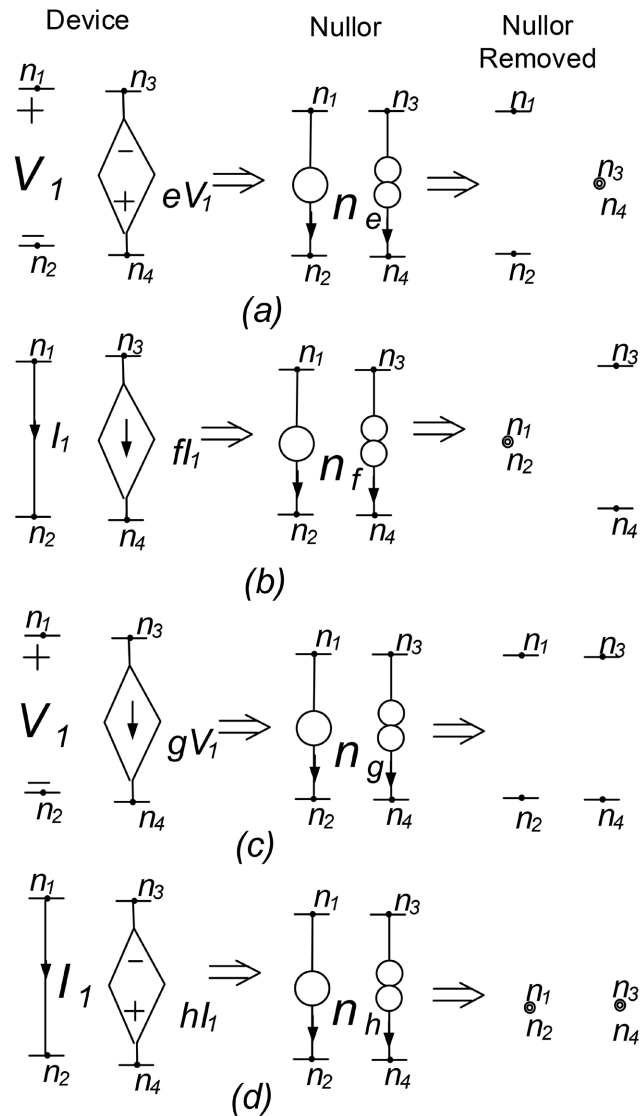


Figure 4. Different stages of four types of controlled sources: dependent sources, nullor equivalent, and the active device is removed from the circuit, namely VCVS (e); CCCS (f); VCCS (g); CCVS (h).

When all active devices and all I/O ports in a circuit are replaced with nullors the circuit is called a *nullor circuit*. That is, a nullor circuit is built up of passive elements and nullors only. Next, when all the passive elements are eliminated through the *Admittance Method (AM)*² the remaining nullor circuit is called *all-nullor circuit*. Now, we need to explain how this nullor circuit is built.

²For brevity, the AM method is not discussed here. A full description of the method can be found in [11].

3.1. Transforming Active Devices into Equivalent Nullors

To explain, we start with a VCVS case. **Figure 4(a)** shows a VCVS in different stages: a controlled source, a nullor equivalent, and when the device is removed from the circuit. In this process we have removed the coefficient e of the controlled source and stored it separately as a *multiplier*, making the nullor magnitude equal to 1. Other devices are likewise transformed into nullors, as shown in **Figure 4**. Notice that there is similarity between different controlled sources (**Figures 4(a)-(d)**) as far as the nullor models are concerned, except, they are classified as e, f, g, and h types, associated with the same types of controlled sources. The differences between them, however, appear when they are removed from the circuit. This is shown in the last column of **Figure 4**. This concludes the transformation of active devices into equivalent nullors. Now we need to do the same for I/O ports.

3.2. Transforming I/O Ports into Equivalent Nullors

As discussed in [12], in topological circuit analysis, a pair of I/O ports can be replaced with a controlled source of its own type. For example, a VIVO I/O ports is replaced with an e -type, *i.e.*, a VCVS device. Similarly, a CICO I/O ports is replaced with a f -type (CCCS), a CIVO I/O ports is replaced with a g -type (VCCS), and a VICO I/O ports is replaced with a h -type (CCVS). Apparently, the controlled sources are then transformed into nullors. **Figure 5** shows the entire process, converting I/O ports to controlled sources, and then controlled sources to nullors. Finally, the last column in the figure shows how the terminals behave when a port is removed.

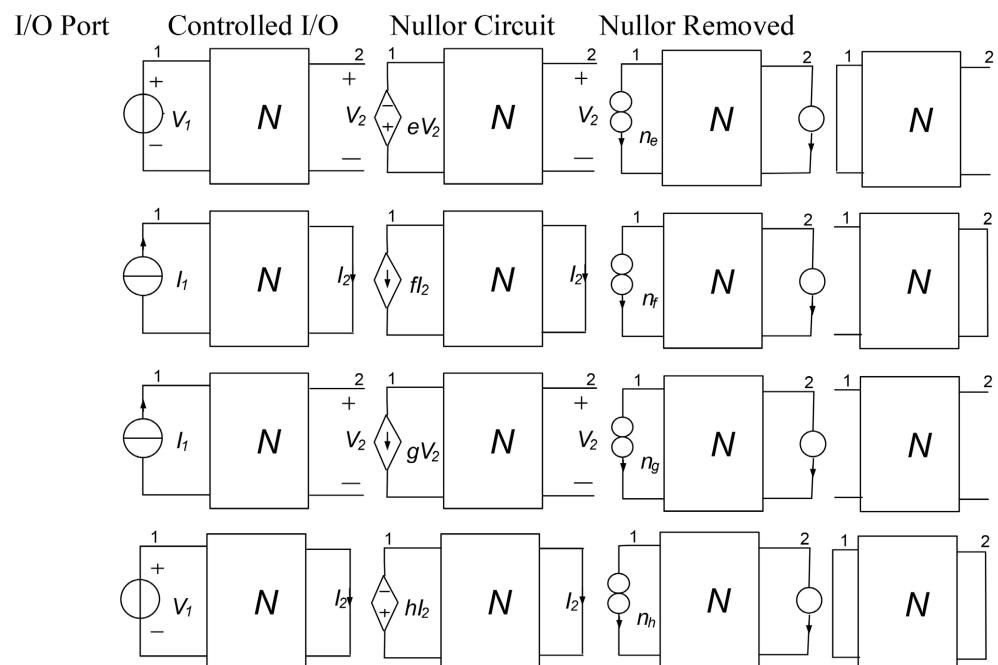


Figure 5. Different stages of four types of I/O ports: Input source applied, a controlled source equivalent, a nullor equivalent, and when the pair of I/O ports is removed from the circuit. I/O types: VIVO (e), CICO (f), CIVO (g), VICO (h).

Evidently, after replacing both controlled sources and I/O ports with nullors we get a full nullor circuit. The job is now to find a reciprocal for the nullor circuit, which is going to be another nullor circuit. Then to complete the journey, we need to go back and reconstruct the actual (active) reciprocal circuit from its nullor equivalent. To proceed, we can simply refer to **Figure 4** for the active devices and **Figure 5** for the I/O ports. The process steps are demonstrated in **Figure 6**. So, the only job left to be done is to construct the reciprocal of a nullor circuit. Later, we will of course show that we can skip the middle stages, given in **Figure 6**, and go directly constructing the reciprocal circuit from the main circuit.

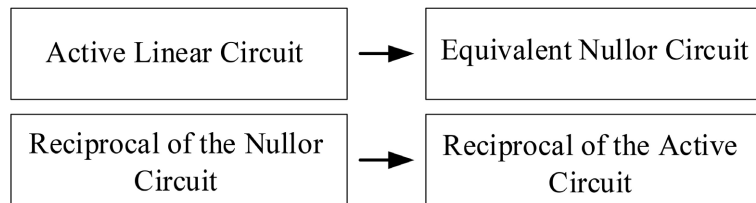


Figure 6. The sequence of steps constructing a reciprocal circuit.

4. Reciprocal of a Nullor Circuit

To create the reciprocal of a nullor circuit we first need to know what we expect from a reciprocal circuit. As stated in Theorem 1, both the circuit and its reciprocal must produce the same transfer function with the I/O ports swapped. This means the same circuit NAM determinant and cofactors. In topological formula this is translated into having the same STP, and one way this can happen is when for each nullor in the circuit the elements (nullator and norator) are swapped. There is, however, one condition attached to this and that is: when one or more nullors are removed from both, the circuit and its reciprocal, they must remain identical, except for the nullor elements being swapped. This means that some types of nullors must be replaced with other types when moving from the main circuit to its reciprocal. **Figure 7** shows how the nullors might change their types when moving from the circuit to its reciprocal. Notice that nullors of types g (VCCS and CIVO ports) and h (CCVS and VICO ports) stay unchanged, whereas types e (VCVS and VIVO ports) and f (CCCS and CICO ports) get interchanged.

Finally, to construct the reciprocal of an active circuit we need to take the following steps.

- 1) Copy the passive circuit without any changes.
- 2) Swap controlled sources with their corresponding controlling elements. Do the same with I/O ports.
- 3) In doing part 2, make device exchanges if needed, according to **Figure 7**, and do the same with the I/O ports.

Notice in **Figure 7** that the second column of nullors and the reciprocals are shown to be identical.

Appendix B provides more details on the computation of the determinants, cofactors, and the transfer functions for both the circuit and its reciprocal. To

clarify, let us take the following example.

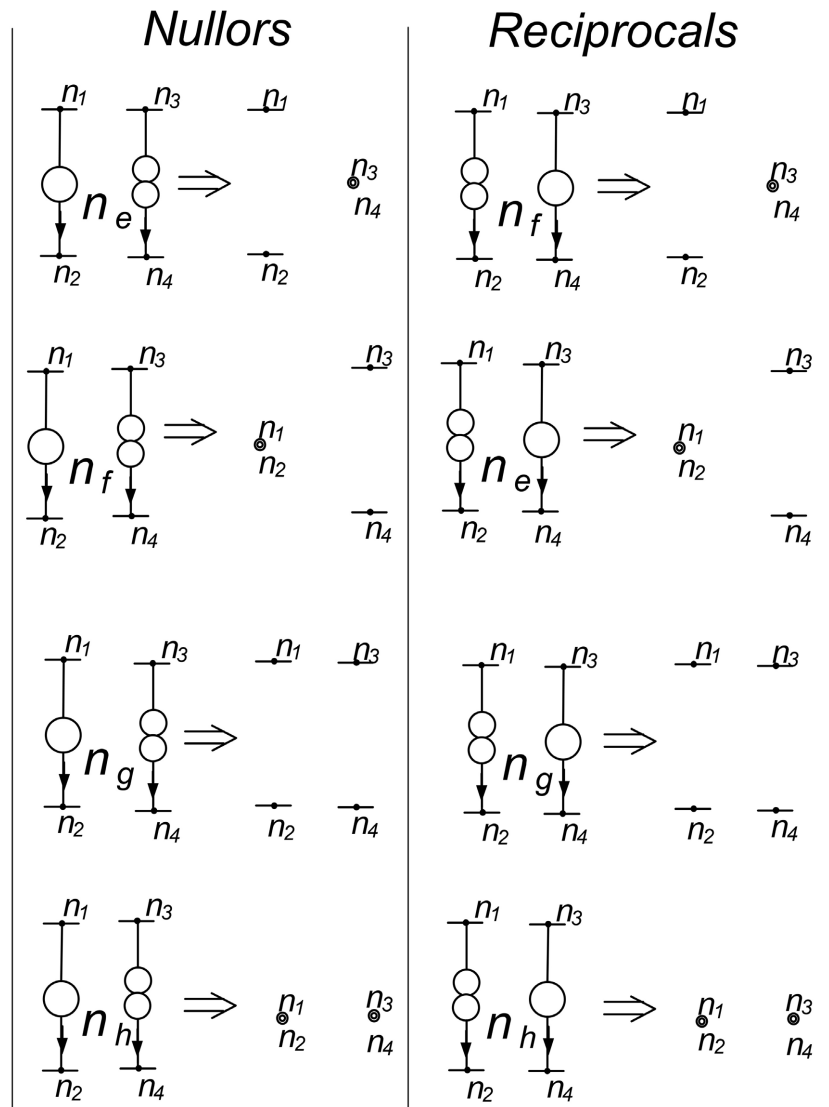


Figure 7. Interchanging active devices, as well as I/O ports, between the reciprocal circuits.

Example 3 Consider an active circuit represented in its nullor equivalent graph, shown in **Figure 8(a)**. The circuit consists of four active devices with one (e_4) replacing a pair of I/O ports. Two devices are of h type and the other two are of e type. To read a nullor, for example, h_3 , the nullator is specified by h_3 and the corresponding norator by 3. Referring to **Figure 7**, it follows that the h type nullors remain unchanged in the reciprocal, whereas the e types are replaced with f devices. The reciprocal is shown in **Figure 8(b)**. So, the points to notice here are as follows: a) the passive circuit remains unchanged with both y and r present in the circuit, b) while the original circuit has two e and two h types of active devices, the reciprocal has two f and two h types of devices. The circuits are simulated and the responses of the two are identical (not shown), as expected.

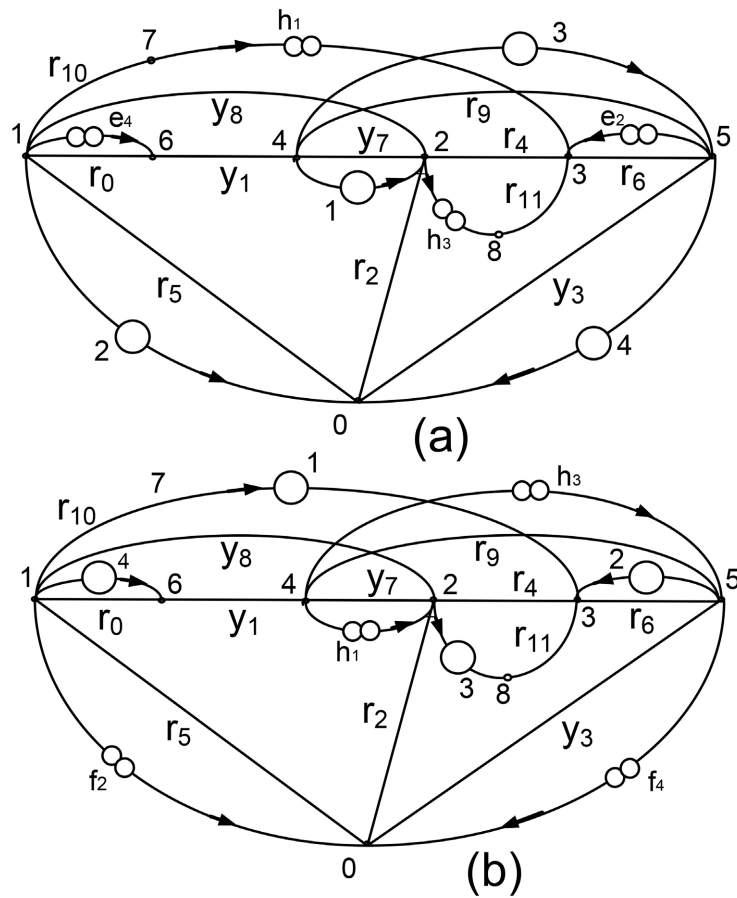


Figure 8. Reciprocal active circuits represented in graph.

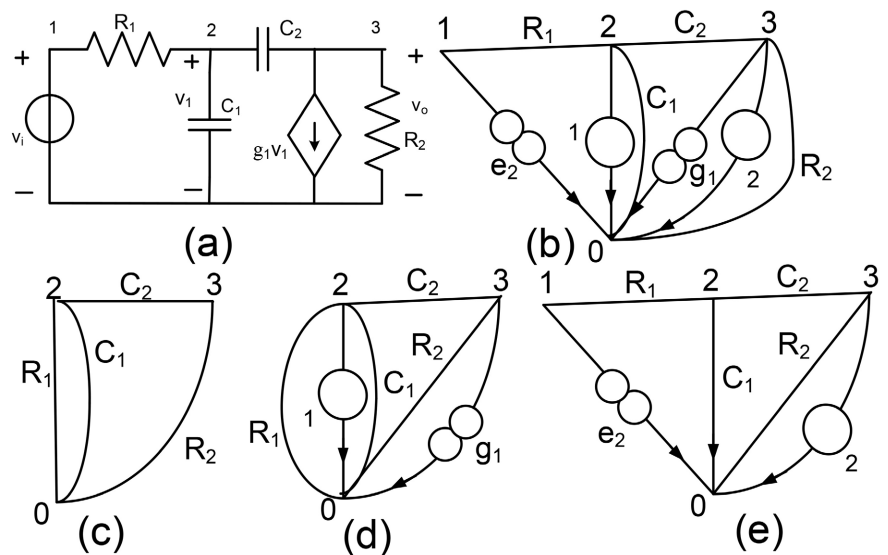


Figure 9. (a) A linearized high frequency MOS amplifier; (b), (c), (d), and (e) the graph representation of the sub-circuits of the amplifier.

Example 4 In another example, we assume a MOS amplifier for high frequencies. Figure 9(a) shows the linearized equivalent circuit of the amplifier with the

component values: $R_1 = 5 \text{ KOhms}$, $R_2 = 10 \text{ KOhms}$, $C_1 = 10 \text{ pF}$, $C_2 = 1 \text{ pF}$, and $g_1 = 2 \text{ mA/V}$. The purpose here is to get the voltage gain $A_v = V_o/V_i$ in s-expanded format through the topological formula by using STP methodology. **Figure 9(b)** is the graphic representation of the amplifier with nullors replacing the dependent source as well as the I/O ports. Notice that the I/O port is of e (VIVO) type with the coefficient $e_2 = 1$ (see **Appendix A**). Because of the two nullors we need to have four sub-circuits with the graphs given in **Figures 9(b)-(e)**. Two graphs, (c) and (d) are to find the determinant and the other two are for getting the specific cofactor. Let us assume that T_c , T_d , T_e and T_b denote the STPs associated with the graphs of **Figures 9(c)-(e)**, and (b) respectively. Then we find

$$T_c = 5 * 10^{-16} * s^2 + 65 * 10^{-9} * s + 1 \quad (6)$$

$$T_d = 5 * 10^{-8} * g_1 * s \quad (7)$$

$$T_e = 10^{-8} * s \quad (8)$$

$$T_b = -10 * g_1 \quad (9)$$

The desire transfer function (gain) is found as

$$A_v = \frac{V_o}{V_i} = \frac{T_e + T_b}{T_c + T_d} = \frac{10^{-8} * s - 10 * g_1}{5 * 10^{-16} * s^2 + 65 * 10^{-9} * s + 1 + 5 * 10^{-8} * g_1 * s}$$

$$A_v = \frac{T_e + T_b}{T_c + T_d} = \frac{10^{-8} * s - 20}{5 * 10^{-16} * s^2 + 165 * 10^{-9} * s + 1} \quad (10)$$

Note that for low frequency we get the expected value

$$A_v = -R_2 * g_1 = -20 \text{ V/V} . \quad (11)$$

5. Reciprocity in Modern Active Devices

As was shown, reciprocity works for any nullor circuit, matching the transfer functions between a nullor circuit and its reciprocal. Apparently, reciprocity must also work for modern active devices [12] [14] since they are represented in nullor circuits. Within those reported, we select a CCII+ circuit to work with.

CCII+ **Figure 10(a)** is a CCII+ in nullor representation [14]. The model circuit shows two pairs of nullors that must always be treated together. **Figure 10(b)** shows a conceptual CCII+ connected as an inverted integrator with the voltage transfer function given as.

$$\frac{V_o}{V_x} = -\frac{R_o}{R_x} \frac{1}{sR_2C_1} \quad (12)$$

We now try to create the reciprocal of the integrator. To do this, we need to make two types of changes. First, swap the nullator and norator in each nullor, and second, get the transfer function to represent the ratio between the current in the input port I_x and the current source I_o connected to the output port, as shown in **Figure 10(c)**. The current transfer function is simply found as.

$$\frac{I_x}{I_o} = -\frac{R_o}{R_x} \frac{1}{sR_2C_1} \quad (13)$$

As we expect, the two transfer functions are identical.

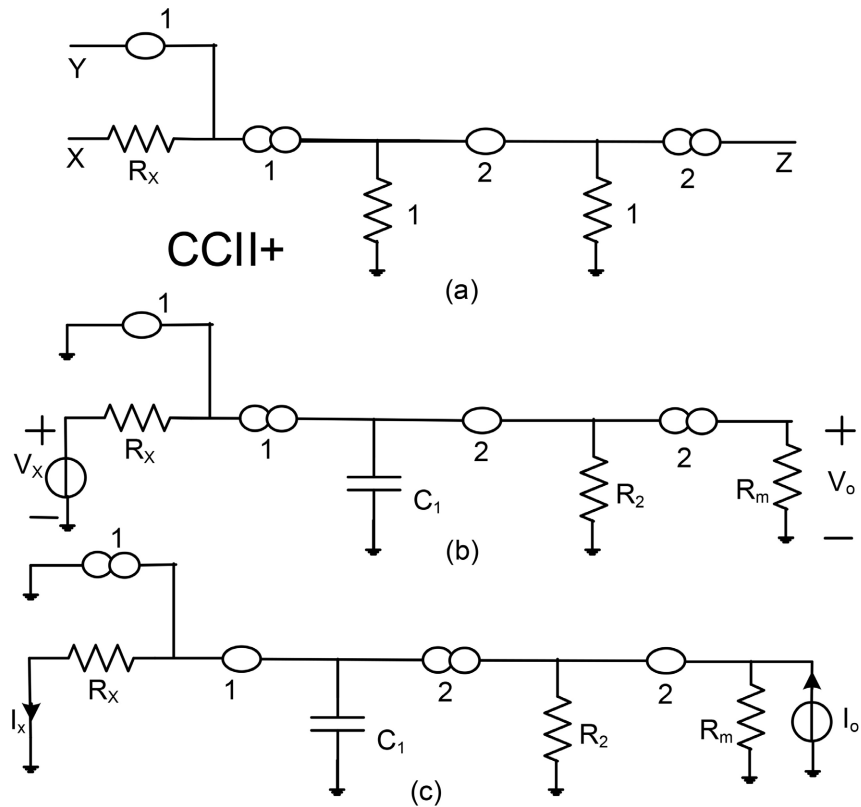


Figure 10. (a) A nullor model of a CCII+ active device, and (b) A current controlled voltage source, an application.

6. Reciprocal for Circuits with Multiple I/O Ports

Up to now we have established reciprocity rules for circuits with one pair of I/O ports. The idea that a pair of I/O ports can be converted to a controlled source, and then to a nullor, was the key factor in uniformity of active circuits, consisting of passive and active components. It so happens that this conversion cannot generally continue for multiple I/O ports, which comes from the fact that, despite dependent sources, pairs of I/O ports are not exclusively related. What it means is that, for an input port P_1 , any other port (or node) in the circuit can be paired as an I/O port.

We usually have no problem with multiple output ports if there is only one source of input. In this situation, the input can be paired separately with each output port. Then, how does the reciprocal circuit look like? Considering the superposition for linear circuits the reciprocal circuit is simply constructed exactly like what we did for a single pair of I/O ports, except the single input source, in the main circuit, repeatedly appears at the multiple output ports, and the input port shows the summation of the outputs in the original circuit. For example, let us assume that we are stimulating a circuit N_1 with an input current $I_i = 10$ mA and we get two output responses, $V_{o1} = 3$ V at port P_1 and $V_{o2} = 5$ V at port P_2 . We

then construct N_2 , the reciprocal of N_1 . Now, if we apply $I_{o1} = 10$ mA and $I_{o2} = 10$ mA to the two output ports P_1 and P_2 we will receive a voltage of $V_i = 3 + 5 = 8$ V at the input port. Note that if the output ports are different in types (I or V) we must refer to **Figure 7** and make the right selections. Let us take another example. Assume we stimulate a circuit N_1 with an input current X_i applied to the input port. Then we get two output responses, one output voltage X_2 at port P_1 and the other one an output current X_3 at port P_2 . We then construct N_2 , the reciprocal of N_1 . Now, if we apply a current X_i at P_1 and a voltage X_i at port P_3 we receive a current X_3 at the input port in the reciprocal circuit, so that $X_3 = X_1 + X_2$. This is shown on **Figure 11(a)** and the reciprocal in **Figure 11(b)**. Example 5 is another case of one input and two outputs.

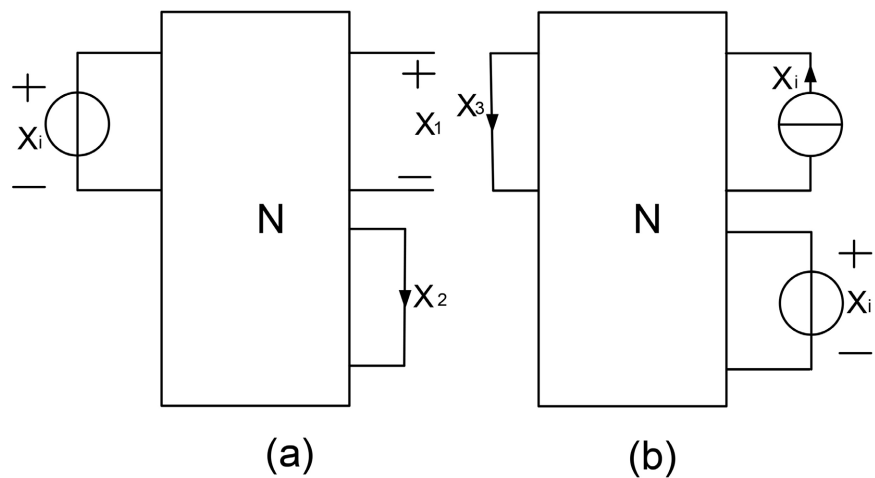


Figure 11. (a) Circuit with one input and two outputs; (b) The reciprocal circuit.

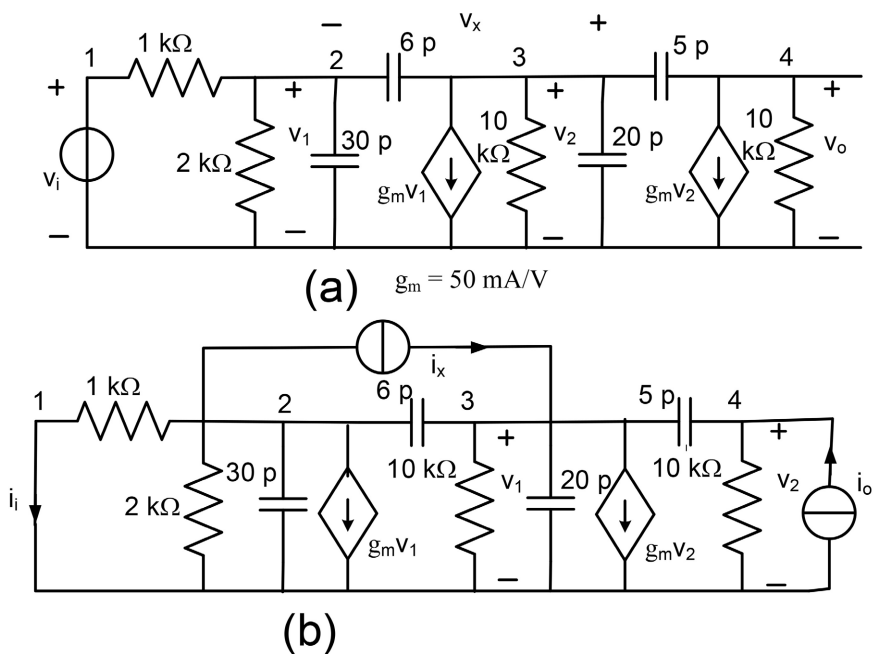


Figure 12. (a) Linear model of a two-stage amplifier; (b) The reciprocal circuit.

Example 5 Consider the same two-stage amplifier given in *Example 1* with its linear equivalent circuit shown in **Figure 12(a)**. The difference here is that we assume two output V_o and V_x here, still with a single input V_i . The reciprocal circuit is given in **Figure 12(b)**, where I_o and I_x are both equal to V_i in terms of the values. The circuit and its reciprocal are simulated, and the resulting plots for the transfer functions are given in **Figure 13**. In **Figure 13** one plot represents V_{out}/V_i where $V_{out} = V_o + V_x$, and the second plot shows I_i/I_o , with $I_x = I_o$.

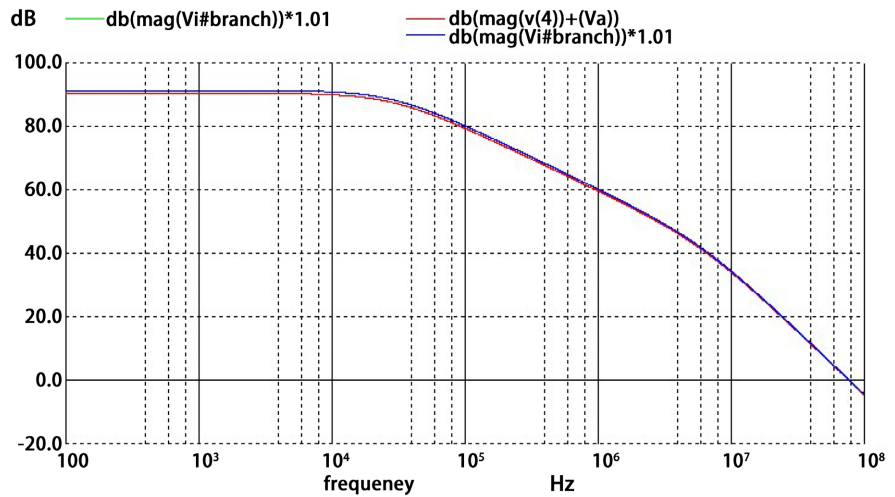


Figure 13. Reciprocal Frequency responses of the two-stage amplifier and its reciprocal.

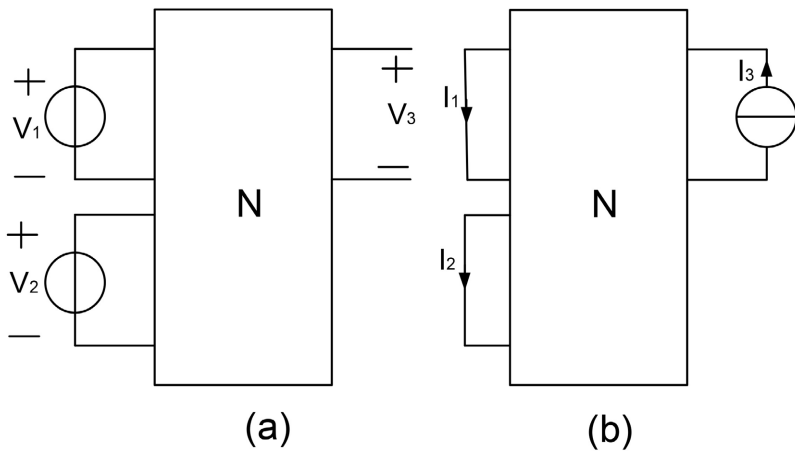


Figure 14. (a) Circuit with two input sources and one output. (b) Reciprocal circuit, applying one source at the output port and getting response from the combinations of ports P_1 and P_2 .

In the case of multiple inputs and a single output, just like the previous case, we can pair each input with the output port separately and repeat it for the other inputs. In fact, this case precisely matches the previous case, the only difference is that we need to exchange the circuit with its reciprocal. For multiple input sources that are proportional we may utilize the superposition applicable to linear circuits. To explain this, let us start with a circuit N with two input voltage sources V_1 and

V_2 , and an output port V_3 applied to ports P_1 , P_2 , and P_3 respectively, as shown in **Figure 14(a)**. Now, let us assume that with only V_1 applied, a voltage $V_{3,1}$ resulted at P_3 , and similarly with only V_2 applied a voltage $V_{3,2}$ has resulted at P_3 . So, we are facing two transfer functions $A_1 = V_{3,1}/V_1$, and $A_2 = V_{3,2}/V_2$. Next consider the proportionality between V_1 and V_2 such that $V_2 = k^* V_1$, where k can be any multiplier, real or complex. Then with both V_1 and V_2 applied to the circuit we can get the output as

$$V_3 = V_{3,1} + V_{3,2} = (A_1 + k^* A_2) * V_1 = A_3 * V_1 \quad (14)$$

where, $A_3 = A_1 + k^* A_2$ can be considered as the pseudo-transfer function between the three ports, P_1 , P_2 , and P_3 .

How then is the reciprocal circuit constructed for this case? As it turns out, the *core* for the reciprocal circuit stays the same. **Figure 14(b)** shows the reciprocal circuit of N with an input current I_3 at port P_3 and two output ports P_1 and P_2 , representing output currents I_1 and I_2 . Now, if a current $I_3 = V_1$ (in value) at P_3 generates two output currents $I_{1,3}$ and $I_{2,3}$ at P_1 and P_2 then according to Equation (14) we get

$$I_{1,3} + k^* I_{2,3} = V_3 \quad (\text{in value}) \quad (15)$$

This, of course, can be conditionally extended to n ports if a) we consider one output at a time, and b) The input sources are proportional.

The situation must be similar for any number of input sources. For example, let us consider a circuit N with n input ports P_1, P_2, \dots, P_n and an output port P_o . Further, assume that current sources I_1, I_2, \dots, I_n (this, of course, can be mixed current and voltages) are assigned to ports P_1, P_2, \dots, P_n respectively, and let $I_m = k_m^* I_1$ for all $m = 2, 3, \dots, n$, where k_m can be any arbitrary multiplier. When all the input sources are applied simultaneously, they collectively generate an output voltage V_o at port P_o . Now, if N_r is the reciprocal circuit of N then removing all n input sources and instead applying a current source I_1 to the output port P_o in the circuit will generate n output voltages $V_{1,o}, V_{2,o}, \dots, V_{n,o}$ at ports P_1, P_2, \dots, P_n . Then, the reciprocity rule states that.

$$\sum_{m=1}^n k_m^* V_{m,o} = V_o, \text{ for } k_1 = 1. \quad (16)$$

Notice that Equation (16) also works for the case of having only one input source.

One may realize that this solution for multiple input sources still is not comprehensive. The major drawback is the way we have adapted the multiplier k_m . There are usually unrelated sources applied to a circuit in multiple input cases. For example, signals with carriers, modulated signals in communication, or mixed signals are not necessarily related by a multiplier.

In general, the reciprocal circuit studied in this discussion is mainly for a single pair of I/O ports. In case of multiple I/O ports, however, we can adopt the superposition for linear circuits and handle the case but only if the inputs are proportionally related. In case we have multiple and unrelated input sources the suggestion is to

use one I/O pair at a time. Any other possible solution to the problem is beyond the scope of this presentation.

To clarify the case, let us consider the active circuit shown on **Figure 13(a)**. The circuit has two inputs, a voltage source V_1 and a current source I_2 . It also has an output voltage V_3 , or multiple outputs if desired.

Example 6 Consider an active circuit N with two input sources: a voltage source V_1 at port P_1 and a current source I_2 at P_2 . The overall response V_3 is received at Port P_3 , as shown in **Figure 15(a)**. The reciprocal circuit N_r is generated and displayed in **Figure 15(b)**. According to the I/O rules, as shown in **Figure 14**, the reciprocal circuit N_r has only one input source I_3 at Port P_3 and two output ports I_1 and V_2 at P_1 and P_2 . Both N and N_r circuits are simulated by SPICE and the results are displayed in **Figure 16** along with the SPICE listing of both circuits.

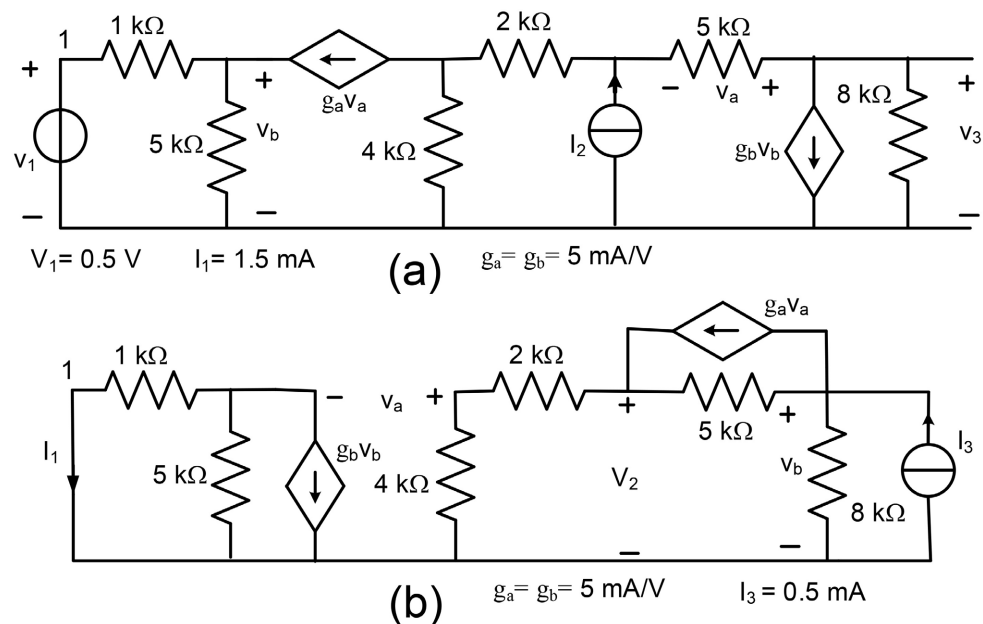


Figure 15. (a) An active circuit N with two input sources and an output response. (b) The reciprocal circuit N_r with one input source and two output responses.

Notice that the response $V_3 = 10.76307 \text{ V}$ in circuit N is equal to the addition of the two responses I_1 (in mA) and V_2 (in V) in reciprocal circuit N_r . Now, since the applied input sources are $V = 5 \text{ V}$ and $I_2 = 15 \text{ mA}$ then the multiplier factor selected here must be $k = 3$.

Generalized reciprocity without nullor equivalents

Overall, we can summarize the reciprocity issue in an algorithm, describing how we can directly construct the reciprocal circuit of a given active circuit without going through the process of forming the nullor equivalent circuit. However, this may raise the following question. If we can directly find the reciprocal of a circuit without going through the nullor circuit, then why did we go through it at first place? The answer to this question is, yes, to get the reciprocal we can go directly to construct the reciprocal circuit without resorting to nullors, as the

following Algorithm offers. The nullor circuits, however, provide a more uniform and simplified process for the circuit analysis that we need to proceed, as explained in Appendices A and B. To show one example regarding the simplification in using nullors is in determining T_n , discussed in Section A, **Appendix B**. So, the bottom line is that, if we are only interested in finding the reciprocal of a circuit, then there is no need to go through converting the circuit to a nullor circuit. In this situation reciprocity is applied to both the dependent sources and I/O ports. But in case we need to test the results, and through the analysis show the proof of the reciprocity, then we offer to convert the circuit into its nullor equivalent. In short, nullors are good for explanation and analysis but in terms of writing efficient programs for reciprocity the conversion to nullors is avoided. I have also avoided using nullors in my program “Circuit Analysis, Simulation, and Design” (CASD).

```

* Reciprocity with two input sources and an output.
.control
destroy all
op
*
print v(5) 3*v(14)+I(v11)
.endc
* Active Circuit  $N$ 
v1 1 0 DC 5
i2 0 4 DC 15
r1 1 2 1
r2 2 0 5
r3 3 0 4
r4 4 3 2
r5 4 5 5
ga 3 2 5 4 5
gb 5 0 2 0 5
* Reciprocal Circuit  $N_r$ 
i3 0 15 DC 5
v11 11 0 DC 0
r1111 12 1
r1212 0 5
r1313 0 4
r1414 13 2
r1514 15 5
gl a 15 14 13 12 5
gl b 12 0 15 0 5
.end

```

(a)

SPICE Response.

```

v(5) = 1.076307e+01
3*v(14)+v11#branch = 1.076307e+01

```

(b)

Figure 16. (a) SPICE circuit listing for both N and N_r circuits. (b) SPICE responses.

Algorithm To find the reciprocal of an active circuit we need to go through the following steps:

- Copy the passive circuit without any changes.
- Swap the controlled sources with their own corresponding controlling elements. Do the same with I/O ports.
- In doing part 2, make device exchanges according to **Figure 7** if needed. Do the same with I/O ports.

7. Practical Aspects of Reciprocity

We consider two cases in practical situations: 1) linear circuits with the actual devices replacing the models, and 2) reciprocity in non-linear circuits.

7.1. Linear Circuits with the Original Devices

If the device models represent the actual devices in practical circuits, one may assume that this is also true for the reciprocal circuit. That is, a reciprocal circuit can be restructured and turned into an actual circuit. However, this is not necessarily true. The problem is that, while the active components change positions in the reciprocal the passive elements remain fixed. Therefore, we can state the following: *the reciprocal of a device model is not necessarily a device model itself unless the passive elements in the model are symmetrical with respect to the I/O ports.*

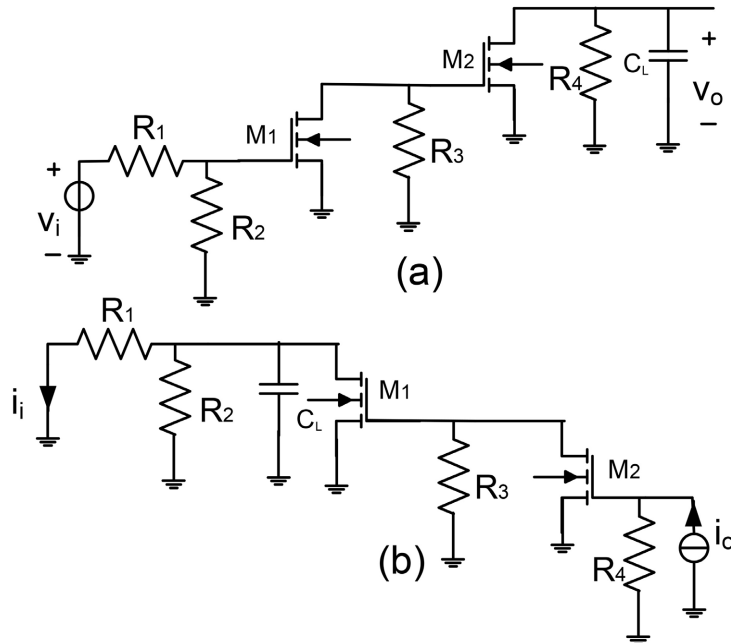


Figure 17. (a) Same as the amplifier in **Figure 12** with MOS devices replacing the models. (b) The reciprocal amplifier circuit.

Nevertheless, we can observe this symmetrical situation in the amplifier given in Example 5. Here, we notice that except for the input resistance the passive

portion of the linearized amplifier is symmetrical. Therefore, it is possible to restructure both the original circuit and its reciprocal to practical circuits, as shown in **Figure 17**.

This example shows, given the right conditions, it is possible to find practical applications for reciprocity. For example, from **Figure 17** we realize that by using the reciprocity we have turned a voltage amplifier into a current amplifier with the same gain and frequency characteristics. Furthermore, this example is an indication that we can directly construct the reciprocal of a circuit without resorting to nullor equivalency.

7.2. Reciprocity in Non-Linear Circuits

Non-linear circuits are not covered in this presentation. However, we can still touch some of the aspects of the topic that are involved with linear circuits, and one such topic is *biasing*.

In fact, reciprocity cannot be generalized for nonlinear circuits without preconditions. This is because reciprocity is a matching process involving two circuits, and during the operation the two circuits must deal with the same location in the nonlinear region.

Here we see the role of biasing. So, basically, for two non-linear circuits to be reciprocal they must be biased so that both experience identical regain of non-linearity during the operation. For example, if we were to show the biasing of the amplifiers in **Figure 16**, we must have biased them so that when we model the transistors the corresponding models in two circuits become identical.

In general, we can propose that *two non-linear circuits can become reciprocal if the non-linear components in two circuits can be paired and each pair with identical non-linear characteristics*.

Evaluation and comparison of the proposed method

In comparison with other reported active circuits reciprocities, the proposed method claims certain features that are new and highly efficient. In specifically the entire procedure is based on several new features.

The method uses topological formulas here rather than Nodal Admittance Matrix. The advantage is the universality of topological methods, whereas NAM, in its original form, is limited to admittances and voltage dependent current sources. The key point is that the component values, admittances or impedances, are given in ratios. This provides much easier data manipulation.

I/O ports are converted into dependent sources of the same kind leaving circuit with only passive and active components. Evidently, no cofactors are needed to obtain a transfer function, only determinants.

The use of Admittance Method instead of tree-enumeration is another feature in this presentation. By applying AM procedure, we gradually and systematically reduce the circuit until the last passive component while the active components remain in place. AM only works on passive elements without the need for 2-graph requirements.

Braking circuit into q sub-circuits, where each sub-circuit provides one term in the final symbolic transfer function. It is shown that AM only works on sub-circuits.

The entire circuit can be turned into a nullor circuit. Although this conversion is not necessary, it helps to further simplify and uniform the circuit.

Comparing the proposed technique with other conventional ones is not simple. For example, the present method goes directly to form the symbolic presentation of transfer functions, which apparently takes more time to find the different coefficients separately, rather than going only for numerical values. One way to measure efficiency and compare the method with others is to eliminate generating sub-circuits. This works quite well in passive circuits, but to use it in active circuits we need to abandon the use of AM operations or go to a *cascade* operation. In *cascade* operation the sub-circuits share portions of AM operations that are common between them. **Figure 18** symbolically shows the operation of an active circuit with three dependent sources. The operations w indicates all remaining AM operations plus finding the determinant of the associated all-nullor circuit, if any. Note that at the end of the chart every sub-circuit appears with its STP, representing a symbolic coefficient in the transfer function.

With applying cascade operation, the efficiency increases substantially without sacrificing accuracy. The procedure is tested for accuracy and efficiency using Circuit Analysis, Simulation, and Design (CASD), a dedicated program.

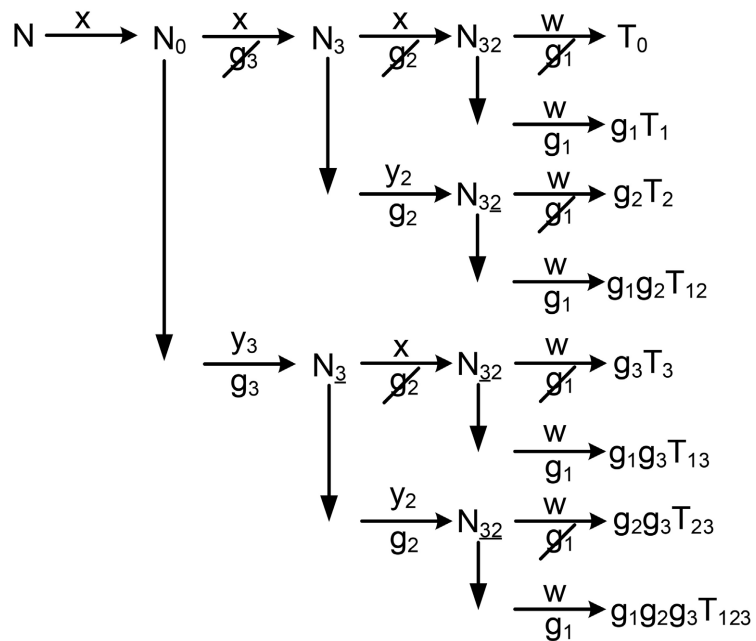


Figure 18. Symbolic representation of a cascaded AM operations.

8. Conclusions

This presentation aims at generalizing the reciprocity for any active circuit with any type of I/O ports. Reciprocity for passive components is already well-established.

For active circuits, particularly when a variety of components are used, still work needs to be done. This work is trying to exactly do this. It demonstrates that for a given linear active circuit N_1 with all four types of controlled sources and assuming an I/O port of any of the four types, there is always a reciprocal circuit N_2 with the following properties. The transfer functions of the two circuits, N_1 and N_2 , are identical, provided that the input and output ports of the two circuits are swapped with the I/O types adjusted. The second task is to realize and construct N_2 , and beyond. To find the reciprocal circuit, we try to simplify and uniform the circuit by using some new techniques. In the first step, we remove the I/O ports by converting them to dependent sources. We then convert the entire circuit to a nullor circuit. Next, we break the circuit into sub-circuits, each for finding the coefficients of a symbolic transfer function. We then reverse the process until we reach the reciprocal circuit with identical symbolic transfer function, or in values.

The cases for multiple I/O ports are also discussed. Multiple outputs can be simply handled by superposition. For multiple inputs, it is shown how the superposition gets involved for related inputs. For non-related inputs, one way to deal with it is to go for one input at a time and hence get multiple reciprocals that are different only in I/O ports.

Finally, an algorithm is provided describing how we can directly construct the reciprocal circuit of a given active circuit without going through the process of forming its nullor equivalent circuit. In short, nullors are recommended for explaining and understanding the reciprocity procedure but in terms of computation and efficient programming the conversion to nullors is avoided.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A

Replacing I/O Ports with Controlled Sources

The idea here is to replace I/O ports with their corresponding controlled sources. The advantage here is that ports are removed all together simplifying the circuit. Hence, we no longer need to deal with I/O ports, cofactors, 2-trees, and other attachments that complicate the circuit analysis and specifically hinders getting transfer functions. We start with a two-port circuit.

2-port circuit Consider a 2-port linear circuit N , shown in **Figure A1(a)**. We can write the transfer admittance y_{21} of N as $y_{21} = I_1 / V_2$, or simply $I_1 = y_{21} * V_2$. Now, assume that I_1 is removed from the 2-port N and instead a VCCS with $g_m = y_{21}$ is replaced it, controlled by the output voltage V_2 , as shown in **Figure A1(b)**. The assumption is that we have initially applied V_2 at the output port to establish the input source $I_1 = g_m V_2$. However, notice that the g_m only appears in the cofactor T_{21} calculation, where $T_{21} = T_o / y_{21}$ and T_o is the circuit determinant. Therefore, if, instead of $g_m = y_{21}$, we assume $g_m = 1$ we obtain $T_{21} = T_{o, g_m=1}$. This means that, for getting the cofactor T_{21} all we need to do is to assign a VCCS with $g_m = 1$ at the I/O port and then go for calculating the NAM determinant.

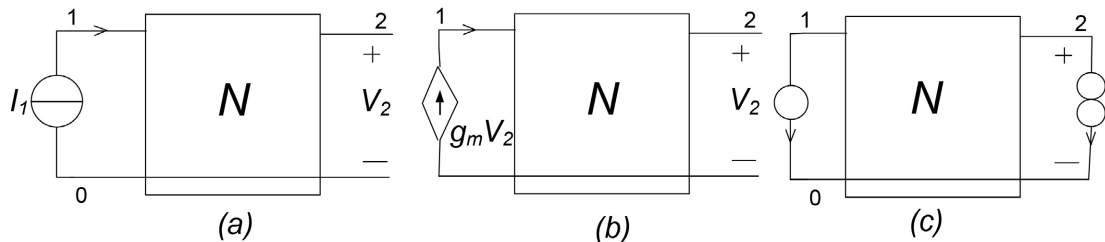


Figure A1. A 2-port circuit to calculate the trans-conductance y_{21} ; (a) Original 2-port; (b) Port replacement with a VCCS; (c) Final replacement with a nullor.

To verify the finding, let us take a single resistance with admittance y . If we connect a current source I_1 to it, as shown in **Figure A2(a)**, an output voltage $V_1 = I_1 / y$ is generated. Now, if we go through STP analysis and assign a nullor to the input and the output ports, like **Figure A1(c)**, we get **Figure A2(b)**, with $g_m = 1$. This simple resistance provides us with the determinant $T_o = y$, and the cofactor $T_{21} = 1$. So, the trans-admittance is $y_{21} = T_o / T_{21} = y$, which is what we expect.

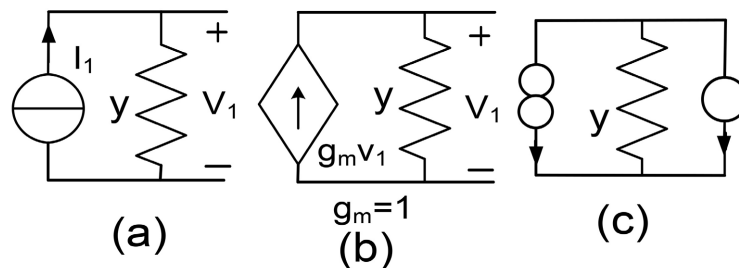


Figure A2. (a) I/O ports assigned to a resistor. (b) Controlled source replacement. (c) Conversion to a nullor.

Next, let us take a simple amplifier circuit shown in **Figure A3(c)**. By using STP procedure on the amplifier we get the circuit (NAM) determinant $T_o = y_1 y_2 + g_1 y_2$. Similarly, when we connect a VCCS with $g_2 = 1$ to the I/O port we get the new (NAM) determinant, which is the cofactor $T_{21} = g_1$. Therefore, the trans-conductance is obtained as.

$$Z_{21} = \frac{T_{21}}{T_o} = \frac{g_1}{y_1 y_2 + g_1 y_2} \tag{1A}$$

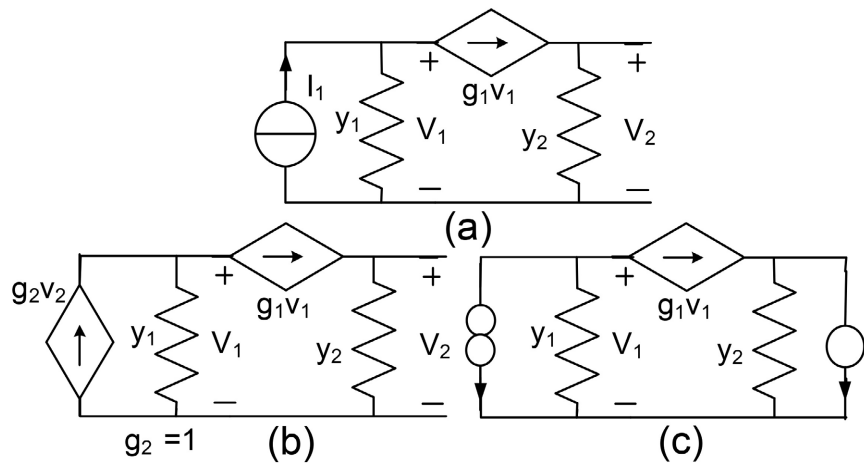


Figure A3. (a) I/O ports assigned to an amplifier circuit. (b) Controlled source replacement. (c) Conversion to a nullor.

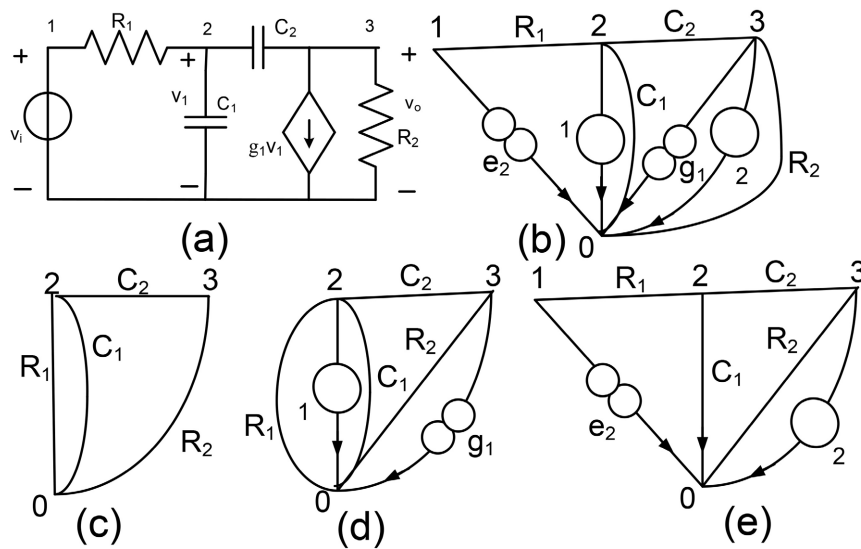


Figure A4. (a) A linearized high frequency MOS amplifier; (b), (c), (d), and (e) the graph representation of the sub-circuits of the amplifier.

In another example, we assume a MOS amplifier for high frequencies. **Figure A4(a)** shows the linearized equivalent circuit of the amplifier with component values: $R_1 = 5 \text{ KOhms}$, $R_2 = 10 \text{ KOhms}$, $C_1 = 10 \text{ pF}$, $C_2 = 1 \text{ pF}$, and $g_1 = 2 \text{ mA/V}$. The purpose is to get the voltage gain $A_v = V_o / V_i$ in s-expanded format through

the topological formula by using STP methodology. **Figure A4(b)** is the graphic representation of the amplifier with nullors replacing the dependent source as well as the I/O ports. Notice that the I/O port is of e (VIVO) type with the coefficient $e_2 = 1$. Because of the two nullors we need to have four sub-circuits with the graphs given in **Figures A4(b)-(e)**. Two graphs, (c) and (d) are to find the determinant and the other two are for getting the specific cofactor. Let us assume that T_c , T_d , T_e and T_b denote the STPs associated with the graphs of **Figures A4(c)-(e)**, and (b) respectively. The we find

$$T_c = 5 * 10^{-16} * s^2 + 65 * 10^{-9} * s + 1 \quad (2A)$$

$$T_d = 5 * 10^{-8} * g_1 * s \quad (3A)$$

$$T_e = 10^{-8} * s \quad (4A)$$

$$T_b = -10 * g_1 \quad (5A)$$

The desire transfer function (gain) is then found as.

$$A_v = \frac{V_o}{V_i} = \frac{T_e + T_b}{T_c + T_d} = \frac{10^{-8} * s - 10 * g_1}{5 * 10^{-16} * s^2 + 65 * 10^{-9} * s + 1 + 5 * 10^{-8} * g_1 * s} \quad (6A)$$

$$A_v = \frac{T_e + T_b}{T_c + T_d} = \frac{10^{-8} * s - 20}{5 * 10^{-16} * s^2 + 165 * 10^{-9} * s + 1} \quad (6A)$$

Note that for low frequency we get the expected value

$$A_v = -R_2 * g_1 = -20 \text{ V/V} . \quad (7A)$$

Appendix B

Admittance Method for Nullor Circuits

Our objective here is to compute the STP of a nullor circuit N_n . To do this, we need to go through three steps. The first is to generate $q = 2^n$ sub-circuits from the main circuit, where n is the number of the circuit nullors (active devices and I/O ports). Each sub-circuit N_k consists of all passive components plus a certain and exclusive number of nullors, including none. The remaining nullors in N_k are removed. In the second step we apply, the newly developed *Admittance Method (AM)* [12] on the passive components of N_k until we reach to a final 2-terminal component with its admittance $y_p = n_p / d_p$, where $n_p = T_p$ representing the determinant of the passive sub-circuit N_k . Consequently, N_k is reached to an all-nullor circuit N_a , and the final step is to find the STP of N_a , T_a . The determinant of the sub-circuit is then computed as $T_k = p_k * T_p * T_a$ where p_k is the *coefficient-multiplier* (later explained). It is important to note that T_p provides the magnitude of T_k whereas T_a makes its sign. This is stated in *Theorem 1B*. Therefore, to find T_k we need to get both T_a and T_p for each sub-circuit, and the final determinant, or cofactor, will be the addition of a certain number of T_k STPs obtained from q sub-circuits. So, for each T_k we need to calculate the corresponding p_k , T_p and T_a .

B1) *Determinant of an all nullor circuit, T_a*

Theorem 1B The STP of an all-nullor circuit T_a is either 0, 1, or -1.

Proof This is because the magnitude of each nullor in the all-nullor circuit is 1, resulting in $|T_a| = 1$ or 0.

Theorem 2B The magnitude $|T_a| = 1$ if and only if the nullator and the norator networks each form a single tree with no loop. Otherwise, $T_a = 0$.

Proof First, each element in the nullator or norator network, representing a controlled or controlling (or an I/O port) element in N_k , must be present in every tree of the STP. Hence, no loop in either nullator or norator network is permissible. Hence, it must be a single tree.

Corollary 1B An all-nullor circuit N_a has a non-zero STP only if 1) the total number of its nodes is equal to the total number of nullors plus one, and 2) any node is incident to at least one nullator and one norator.

We now need to find the sign of an all-nullor circuit, T_a .

Given an all-nullor circuit N_a , first separate the nullator and norator networks from each other and begin eliminating nullors one by one starting from a 3-terminals³. In case the two arrows in the pairing nullator and norator are toward or away from the common node the sign is positive, otherwise it is negative. Next, short-circuit the nullor elements with the common node removed, and then move to the next nullor until all are done. This results in T_a .

For proof, notice that each time we short circuit a nullor, depending on the direction, whether we change the sign of T_a or not. Therefore, the sign of T_a (and T_k) depends on the number of changes we have made in removing all nullors. The operation to find T_a is denoted by z .

B2) *Determinant of the passive portion of a nullor circuit, T_p*

A full description of the AM procedure for removing the entire passive elements in an active circuit is given in [12]. Briefly, the AM procedure uses two *basic* operations: a) *parallel* and *series* operations, and b) *partition* operation [11] [12]. An effective way to do the basic operations is to write admittances in ratios, just like a transfer function. For example, the admittance of a 2-terminal component e_i is given by $y_i = n_i/d_i$, with $n_i = y_i$ and $d_i = 1$. If the element is indicated by its impedance, then $z_i = d_i/n_i$.

Parallel and Series Two passive components e_i and e_j with admittances $y_i = n_i/d_i$ and $y_j = n_j/d_j$ that are in parallel can be replaced with a component e_k with admittance y_k given in (1B). Likewise, the two passive components e_i and e_j in series can be replaced with e_s given in (2B). An exhaustive sequence of *parallel/series (P/S)* operations results in a *P/S free* circuit.

$$y_p = \frac{n_i d_j + n_j d_i}{d_i d_j} = \frac{n_i d_j}{d_i d_j} + \frac{n_j d_i}{d_i d_j}. \quad (1B)$$

$$y_s = \frac{n_i n_j}{n_i d_j + n_j d_i}. \quad (2B)$$

Parallel and Series operations between two passive components are denoted by x .

³In practice, when the passive circuit is eliminated most 4-terminal nullors become 3-terminals. Also removing 3-terminals may change some 4-terminals to 3-terminals.

Partition Given a circuit N , a partitioned circuit $N\{A; B\}$ is obtained from N by removing elements A and short circuiting elements B , and $T\{A; B\}$ refers to the determinant of the NAM (or STP) of $N\{A; B\}$. Now, consider circuit N with its STP as T . If e_i is an element in N with $y_i = n_i/d_i$, then by applying the *partition* procedure we can write

$$T = n_i T\{0; e_i\} + d_i T\{e_i; 0\} \quad (3B)$$

This partitions N into two *sec-circuits*⁴ $N\{0; e_i\}$ and $N\{e_i; 0\}$.

Parallel and Series operations between a passive and a nullor element.

This kind of P/S operation is slightly different from the passive-passive one and is denoted by y . If only one nullor i is involved in the operation it is denoted by y_i .

Theorem 3B Parallel/Series operations with nullors If a passive element e_b , with admittances $y_b = n_b/d_b$, is parallel with a nullor element (a nullator or a norator), e_i is removed and d_i is *preserved*. Likewise, if a passive element e_i is in series with a nullor element, e_i is short-circuited and n_i is *preserved*. Note that, the *preserved* values, such as d_i and n_b , are multiplied with the *coefficient-multiplier* p_k .

Proof Suppose the determinant of the circuit is T . Then, if e_i is in parallel with a nullor element we get $T = n_i T\{e_i; 0\}$, according to (3B). This is because short circuiting e_i creates a nullator or norator self-loop. The same argument applies to e_i if it is in series with a nullor element.

B3) *Determining the coefficient-multiplier p_k , and the passive STP T_p*

Before we start, we must remember that here when we refer to a *circuit*, we mean a *sub-circuit*. The coefficient-multiplier p_k is partially obtained by multiplying all n_b from all those passive elements in series, and all d_b from all those passives elements in parallel with nullors. Another component of p_k comes from the active (controlled sources) coefficients. For example, if a sub-circuit has two VCCSs, g_1 and g_2 , then $g_1 g_2$ is multiplied by p_k .

To find T_p , we first apply a set of P/S operations on N_k to make it P/S free. In case N_k is reduced to a 2-terminal passive component, e_p with admittance $y_p = n_p/d_p$, then $T_p = n_p$, and we are done. However, if N_k is not reduced to a 2-terminal passive component, we need to use the *partition* operation and generate *sec-circuits*. To do this, we find a node m_i that has the lowest degree (counting both passive and nullor elements). We then remove enough passive elements from m_i to make it a series node (either passive-passive or passive-nullor) and start a new set of P/S operations. If we have not reached a final 2-terminal element, we move to another node and do the same. We continue the procedure until we reach to a single passive element e_p with its admittance $y_p = n_p/d_p$, where $n_p = T_p$. This T_p is only the determinant of the passive *sec-circuit*, and we need to repeat the process for all removed passive elements. By multiple use of (3B) and combining T_p for all *sec-circuits* we come up with T_p for a designated sub-circuit, N_k . Therefore, the determinant of the sub-circuit resulted as $T_k = p_k^* T_p^* T_a$. Then by repeating the same procedure for all $q = 2^n$ sub-circuits we come up with q building blocks to

⁴Each time we use *partition* operation removing or short circuiting an element in a sub-circuit we are creating a new *sec-circuit*.

calculate the determinant and the cofactors we need for the transfer functions.

Specifically, let us assume that we need to obtain a circuit determinant T and a cofactor T_{21} , associated with a pair of I/O ports, which is represented by a nullor p . To get both T and T_{21} all we need to do is to partition the q number of sub-circuits into two groups: the first group, from 1 to $q/2$, of the sub-circuits that do not include the nullor p , and the second group from $q/2 + 1$ to q , of the sub-circuits that include the nullor p . Equations (4B) and (5B) explain the procedure.

$$T = \sum_{i=1}^{q/2} p_i T_{pi} T_{ai} \quad (4B)$$

$$T_{21} = \sum_{i=\frac{q}{2}+1}^q p_i T_{pi} T_{ai} \quad (5B)$$

Finally, the transfer function is given in (6B).

$$F = T_{21}/T \quad (6B)$$

This concludes **Appendix B**.