Applied Mathematics, 2025, 16(1), 42-60
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Neural Codes Constructs Based on Combinatorial Design

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How to cite this paper: Huang, J. (2025) Neural Codes Constructs Based on Combinatorial Design. *Applied Mathematics*, **16**, 42-60.
https://doi.org/10.4236/am.2025.161003

Received: November 29, 2024
Accepted: January 20, 2025
Published: January 23, 2025

Abstract

Neuroscience (also known as neurobiology) is a science that studies the structure, function, development, pharmacology and pathology of the nervous system. In recent years, C. Cotardo has introduced coding theory into neuroscience, proposing the concept of combinatorial neural codes. And it was further studied in depth using algebraic methods by C. Curto. In this paper, we construct a class of combinatorial neural codes with special properties based on classical combinatorial structures such as orthogonal Latin rectangle, disjoint Steiner systems, groupable designs and transversal designs. These neural codes have significant weight distribution properties and large minimum distances, and are thus valuable for potential applications in information representation and neuroscience. This study provides new ideas for the construction method and property analysis of combinatorial neural codes, and enriches the study of algebraic coding theory.

Keywords

Combinatorial Neural Codes, Orthogonal Latin Rectangle, Steiner System, Group Divisible Design, Transversal Design

1. Introduction

In 1948, Shannon pioneered the theory of coding in interference channels. Based on Shannon theory, scholars have designed many error correcting codes with efficient decoding algorithms. Information theory has had a big impact on theoretical neuroscience. Mathematical coding theory received a lot of attention more than a decade ago [1] [2], and has been fueled by research on positional cell neurons. O'keefe and Dostrovsky's discovery of positional cells was a major breakthrough. O'keefe was awarded both the 2014 Nobel Prize in Medicine and the Nobel Prize in Physiology. Position cells encode spatial information about an organism's surroundings by firing precisely when the organism is located in the appropriate position field [3] [4]. A positional field is a collection of several convex open sets that may overlap each other, and the firing activity of a group of neurons produces a set of common firing patterns that can be represented as binary vectors or codewords. Each codeword represents a set of neurons stimulated at the same time, and the set of codewords on n neurons C is called a combinatorial neural code [5].

Based on the connection between neuroscience and algebraic coding, one can use coding to study neuroscience [5]-[7]. The earliest work originated in 2013, when C. Curto *et al.* defined a discrete receptive field code and modeled it as a binary code $C \subseteq \{0,1\}^n$. This led to the study of neural coding using algebraic and combinatorial methods [8] [9]. Let the n neurons in the brain be respectively $\{1,2,3,\cdots,n\}$. Each neuron i has its receptive field S_i, $X = \bigcup_{i=1}^{n} S_i$, Per stimulus $x \in X$. We got a code $c(x) = (x_1, x_2, \cdots x_n) \in F_2^n$, among others $1 \le i \le n$. When $x_i \in S_i$, that is x stimulates neuron i, $x_i = 1$, when $x_i \notin S_i$. That is, neuron i is not stimulated by x, $x_i = 0$ of which zero codeword $c(x) = (0,0,\cdots,0)$ is the absence of a stimulus acting on the neuron. Let Σ be the set consisting of K stimulus. A subset C of the K codewords in F_2^n is called a combinatorial neural code (CN code for short) [10]. As shown in **Figure 1**, x is a stimulus and the codeword for x is $c(x) = (00110)$.

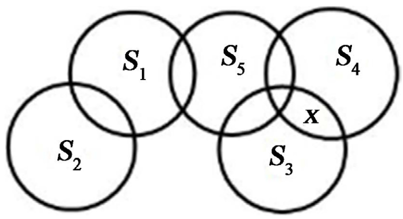

Figure 1. c(x) = (00110).

Stimulus x codewords for $c(x) = (x_1, x_2, \cdots, x_n) \in F_2^n$, but in practice, the received vectors may be $c(x)' = (x_1', x_2', \cdots, x_n') \in F_2^n$. It deviates from $c(x) = (x_1, x_2, \cdots, x_n) \in F_2^n$. That is to say, there is $x_i \in S_i$, but it's not being fired. Or $x_i \notin S_i$, but shows that it is exciting. In reference [10], the following ideal mathematical model is proposed.

1) $x_i = 1$ and $x_i' = 0$ probability of occurrence is $q < \frac{1}{2}$, this means that $x_i \in S_i$ but the probability that neuron i is not stimulated is q.

2) $x_i = 0$ and $x_i' = 1$ probability of occurrence is p and $0 \le p \le q$, this means that $x_i \notin S_i$ but the probability that neuron i is not stimulated is p.

From $c(x)$ to $c(x)'$, consider the binary asymmetric memoryless channel transmission of the following **Figure 2**. If $q = p$, then the channel is a binary symmetric, but in general in neurology $q < p$, the channel is an asymmetric binary channel [11] [12]. Out of a desire to revert to the received codeword $c(x)'$ to the codeword $c(x)$, Therefore, since the 1950's, research on error-correcting codes in binary asymmetric channels has been initiated [12]-[14]. Further proposed new "Matched metrics" for asymmetric $\delta_r(x,y)$ [15] to replace hamming distance, which is not really a distance.

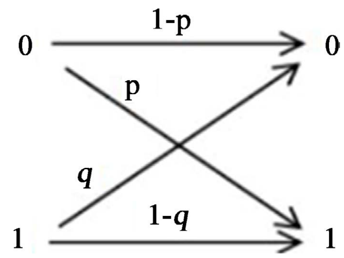

Figure 2. Binary asymmetric memoryless channel.

The aim of this paper is to construct a special class of combinatorial neural codes with special properties using classical combinatorial structures such as Orthogonal Latin rectangle, disjoint Steiner systems, Group Divisible designs, and Transversal designs which have significant weight distribution properties and large minimum distances, and which are of greater value for applications in information representation and neuroscience. Relatively speaking the weight distribution of this paper is very special and the minimum distance is also reached to the maximum, the specific code words of the neural code have indicated, which is more conducive to the research of neuroscience and coding theory.

The paper is structured as follows, and in Section 2, the relevant basics and definitional citations are explained in detail; in Section 3, it is shown that combinatorial neural codes with weight 4 are constructed by orthogonal Latin rectangle, disjoint Steiner systems, Groupable Divisible and a class of combinatorial neural codes are constructed based on the transversal design; and finally, the conclusion of the paper tells about the main conclusions of the paper.

2. Preparatory Knowledge

This section introduces the concepts and properties related to antichain codes [16], it is a combinatorial neural code. The underlying lemmas used to construct the antichain code are described below.

Let $A \subset 2^{[n]}$ be a family of sets, if any distinct $x, y \in A$, with $x \not\subset y$, A is said to be an antichain. For example, layer

$$\binom{[n]}{k} = \{x \subset [n] : |x| = k\}, \tag{1}$$

For all $0 \le k \le n$ form an antichain. From the classical theory of Sperner [17] it is known that the antichain in $2^{[n]}$ is at most $\binom{n}{\frac{n}{2}}$.

A family of vectors $B \subset \{0,1\}^n$ is called a code with distance r, if any two vectors in B have a Hamming distance of at least r. Identifying $\{0,1\}^n$ and

$2^{[n]}$, we call a set family A a distance-r codes if the size of the symmetric difference $x \triangle y$ of any two distinct $x, y \in A$ is always at least r. In the literature [16], for any fixed $r \in N$, if $A \subset 2^{[n]}$ is an antichain and a distance-$(2r+1)$ code, then $|A| = O\left(2^n n^{-r-\frac{1}{2}}\right)$.

Definition 1 [18]: For $y = (y_1, y_2, \cdots, y_n)$ and $x = (x_1, x, \cdots, x_n) \in F_2^n$. Let

$$d_{10}(y,x) = \#\{i \mid 1 \le i \le n, (y_i, x_i) = (1,0)\}$$

$$d_{01}(y,x) = \#\{i \mid 1 \le i \le n, (y_i, x_i) = (0,1)\}$$

The asymmetric discrepancy of y and x is defined as

$$\delta(y,x) = \delta_r(y,x) = r d_{10}(y,x) + d_{01}(y,x) \tag{2}$$

For a combinatorial neural code $C \in F_2^n$, the constituent parameters are $(n, K, \delta(C))$, where size of C is $K = |C| \ge 2$, and the minimum discrepancy is

$$\delta(C) = \delta_r(C) = \min\{\delta_r(c, c') : c, c' \in C, c \ne c'\}$$

For $p = q$, we get $r = 1$, and $\delta_1(y,x) = d_{10}(y,x) + d_{01}(y,x) = d_H(y,x)$ is the Hamming distance. Hence $\delta_1(C) = d_H(C)$, as the minimum Hamming distance of *C*. It is easy to find $\delta(y,x) \ge 0$ and $\delta(y,x) = 0$ if and only if $y = x$. If $p < q$ ($r > 1$) usually $\delta_r(y,x)$ and $\delta_r(x,y)$ are not intended to be the same, but are satisfied by the triangle inequality [4].

According to the definition of combinatorial neural codes, it is easy to know that the antichain code is a class of neural codes, so the antichain code with distance of constitutes a combinatorial neural code with parameters

$$\left(n, |A| = O\left(2^n n^{-r-\frac{1}{2}}\right), \delta_r = 4r+1\right) \tag{3}$$

Definition 2 [19]: A $m \times n$ rectangle G, whose elements consist of a set of s elements such that each of them occurs only once in each row and column is said to be a Latin rectangle; L_1 and L_2 are two Latin rectangles consisting of a basis set of S. G is said to be an orthogonal Latin rectangle if there exists a unique $(i,j) \in m \times n$ for each pair $(x_1, x_2) \in S \times S$ such that $L_1(i,j) = x_1$, $L_2(i,j) = x_2$, then G is said to be an orthogonal Latin rectangle.

Let the family of subsets $\mathcal{F}_V = \{F_1, F_2, \cdots, F_s\}, s \ge 2$ of $[V] = \{0,1,2,\cdots,v\}$, $\mathcal{F}_V$ is said to be k-regular if $\forall i$, $|F_i| = k$, otherwise it is not. If $\forall i, j \in [s]$, $|F_i \cap F_j| \le t-1$ is satisfied, it is called t-intersecting. R. Gabrys *et al.* [19] called this subset family $\mathcal{F}_V = \{F_1, F_2, \cdots, F_s\}$, $s \ge 2$, and the sum of the number of positive and negative labels in each subset is exactly equal to $\{-1, 0, +1\}$ by setting a label L on the elements of $[V]$, some of which are labelled as P_+, and some of which are labelled as P_-, family of sets is the family of extremal balanced sets denoted as (t,k,v), $A(t,k,v)$ denotes the number of subsets satisfying (t,k,v).

Lemma 1 [20]: If $t < k < v$, where $p_+ = |P_+|, p_- = |P_-|, p_+ > \frac{k+1}{2}, v > 2$, then

$$A(t,k,v) < S(t,k,v)$$

where $S(t,k,v)$ denotes the Steiner system: the set X has v points, the set of subsets of X of size k is called a block, and t points of X are in exactly one block. p_+ denotes the number of elements denoted as P_+, p_- denotes the number of elements denoted as P_-

Lemma 2 [20]: Extremal balanced systems based on Latin rectangle constructions. If $\forall v \geq 8$, then

$$A(t,k,v) = \left\lfloor \frac{\left\lfloor \frac{v}{2} \right\rfloor \left\lceil \frac{v}{2} \right\rceil}{2} \right\rfloor. \tag{4}$$

3. Main Results

3.1. Based on Orthogonal Latin Rectangle

Let v be an even, make a token for each element of V as follows

$P_+ = \left\{0,1,\cdots,\frac{v}{2}-1\right\}$, $P_- = \left\{\frac{v}{2},\frac{v}{2}+1,\cdots,v-1\right\}$, $p_+ = |P_+|$, $p_- = |P_-|$ then $v = 2p_+ = 2p_-$, $r_{i,j}$ denotes the element of Latin rectangle, i is the row indicator $i \in P_+$ and j is the column indicator $j \in P_-$. If the element of a Latin rectangle is equal to its row or column indicator, such a point is called a fixed point.

The Latin rectangle constructed in this paper cannot contain fixed points, if they contain fixed points that the elements in the corresponding set have the same contradict the definition of the set, two Latin rectangles without fixed points if they satisfy the definition of orthogonal Latin rectangle, the orthogonal Latin rectangle formed also do not contain fixed points. Latin rectangle without fixed points can form a Steiner system with parameters $(2,3,v)$ and also form an antichain. Their collection forms an antichain code, which results in a combinatorial neural code with special parameters.

The following shows the construction of an antichain based on orthogonal Latin rectangle with parameters $t = 2, k = 4$.

If the first row of the Latin rectangle is all P_+ or P_-, then after the cycle will inevitably appear in an element of row will be equal to its row index or column index, so the first row needs to contain both P_+ and P_-, in order to facilitate the construction of the Latin rectangle is now divided into two sub-rectangle, $\Re$ represents the Latin rectangle, $\Re_{p_+}$ means that all the element of the positive sub-Latin rectangle are all from the P_+, $\Re_{p_-}$ means that all the negative sub-Latin rectangle are all from the P_-, and its rows and columns to do a substitution. It is easy to find that if $\Re_{p_+}$ and $\Re_{p_-}$ do not have fixed points it means that $\Re$ does not have fixed points either, if $r_{i,j} = k \in \Re_{p_+}$ then $i \in P_+$, $j \in P_-$, $\bmod p_+$ for any element, and if $r_{i,j} = k \in \Re_{p_-}$ then $i \in P_-$, $j \in P_+$, $\bmod 2p_+$ for any element.

The row index and column index of Latin moments and the corresponding elements in the above can form a ternary array. For example, $r_{i,j} = k$ can form a ternary array of $\{i, j, k\}$. It is not hard to see that knowing two of these three elements tells us whether $r_{i,j}$ is an element in a positive sub-Latin rectangle or a negative sub-Latin rectangle. For the convenience of representation, the row index i and the corresponding element k form a pair $\{i, k\}$, and the set formed by such a pair of positive sub-Latin rectangles is called the set of positive sub-Latin rectangles, and the set formed by such a pair of negative sub-Latin rectangles is called the set of negative sub-Latin rectangles.

In order for the set to which the Latin rectangle are pairs to satisfy the definition of set and t-intersectivity, the following three conditions are satisfied on top of the above:

Condition1 (Update rule): If $r_{i,j} = k$ then $r_{i-1,j+1} = k-1, i \in P_+, j \in P_-$.

Condition2 (Symmetry condition): If $r_{i,j} = h \in \Re_{p_+}$ then $r_{h,j} = i \in \Re_{p_+}$; if $r_{i,j} = h \in \Re_{p_-}$ then $r_{h,j} = i \in \Re_{p_-}$ (Note the row and column substitution).

Condition3 (Regularity condition): Different pairs $\{i, h\}$ and $\{i', h'\}$ are belonging to positive sub-Latin rectangle negative sub-Latin rectangle then there is $i - i' \neq h - h'$.

Note: Consider condition 1, e.g. $r_{2,0} = 5 \to r_{1,1} = 4$ and it can be found that the elements of the Latin rectangle are completely determined by the first column, so that a good choice of the first column constructs the Latin square; Condition 2, e.g. $r_{0,8} = 7$ and $r_{7,8} = 0$ form the same triad $(0, 7, 8)$. The symmetry condition groups the elements of the Latin rectangle into pairs that result in identical elements in which exactly half of them are identical.

It can be found that the Latin rectangles as long as the construction of the first column in accordance with the update rule can be constructed throughout the Latin rectangle, but the Latin rectangle also to meet the inside of all the elements are not duplicated, so the construction of the first column to meet the first column cannot be duplicated meta, but also note that the peer cannot be duplicated meta, and the condition of the 2 is to meet the requirements of this. For example: if the pairs $\{6, 2\}$ and $\{4, 0\}$ are part of the set of positive sub-Latin rectangle, $r_{6,0} = 2 \to r_{5,3} = 1 \to r_{4,4} = 0$ and $r_{4,0} = 0$, the presence of two 0 in the same rows contradicts the definition of a Latin rectangle. Thus, the regularity condition ensures that there are no same elements in each row.

Construct 1: $p_+ = p_-$ is a multiple of 4 if $v = 4m$.

The positive sub-Latin rectangle set of the first Latin rectangle $\Re_1$ has elements $\{i, -i-1\}$, where $i \in \left[\frac{p_+}{4} - 1\right]$; the negative sub-Latin rectangle set has elements $\{i, -i-1\}$, where $i \in \left[\frac{p_+}{4}, \frac{p_+}{4} + 1, \cdots, \frac{p_+}{2} - 1\right]$;

The positive sub-Latin rectangle of the second Latin rectangle $\Re_2$ has elements $\{i, -i-1\}$, where $i \in \left[\frac{p_+}{4}, \frac{p_+}{4} + 1, \cdots, \frac{p_+}{2} - 1\right]$; the negative sub-Latin rectangle has element $\{i, -i-1\}$, where $i \in \left[\frac{p_+}{4} - 1\right]$.

Orthogonal Latin rectangle $(x_1, x_2) \in v \times v$ are composed and $(i, j) \in p_+ \times p_-$, $\Re_1(i, j) = x_1$, $\Re_2(i, j) = x_2$.

Lemma 3: The Latin rectangle in Construction 1 has no fixed points, nor does the orthogonal Latin rectangle contain fixed points.

Proof: it is only necessary to prove that one Latin rectangle has no fixed point, and the second Latin rectangle is proved using the same steps.

1) There is no such thing as an element appearing twice in the first column, and the subsequent columns are sequentially minus one from the first, so there is also no such thing as an element existing twice, *i.e.*, there are no identical elements in the same column.

2) Let different points $\{i, j\}, \{i', j'\}$ such that $i - i' \neq j - j'$, by construction 1 $j = -i - 1$, $j' = -i' - 1$ thus $i - i' = -i - 1 - (-i' - 1) = -(i - i')$, *i.e.* $i - i' = 0$ or $i - i' = \frac{p_+}{2}$, obviously $i - i' = 0$ needs to be rounded off since i is not the same as i', so $i - i' = \frac{p_+}{2}$. From Construction 1 it follows that if $i, i' \in P_+$,

$\max(i - i') = \frac{p_+}{2} - 2, \min(i - i') = 1$.

If $i, i' \in P_-$,

$$\max i - i' = p_+ - 2, \min(i - i') = \frac{p_+}{2} - 2$$

Neither satisfies $i - i' = \frac{p_+}{2}$, so only one of i and i' belongs to P_- and one to P_+, which contradicts that both have to belong to positive sub-Latin rectangle or both have to belong to negative sub-Latin rectangle, *i.e.* there is no $r_{i,j} = r_{i',j'}$, and so the peers don't have the same element.

3) Let $r_{i,j} = i \rightarrow r_{i+1,j-1} = i + 1 \underset{\dots}{\rightarrow} r_{i+j,0} = i + j$, contradictory to Construct 1. For $r_{i,j} = j$ the case is the same as for $r_{i,j} = i$, *i.e.* it is proved that there is no fixed point.

The two Latin rectangles have no fixed points, *i.e.*, neither do their mutually orthogonal Latin rectangle. #

Example 1: Let $v = 16, p_+ = p_- = 8$, constructed Latin rectangles $\Re_1$ and $\Re_2$ (**Figure 3** and **Figure 4**).

	8	9	10	11	12	13	14	15
0	15	14	3	1	7	5	9	8
1	13	4	2	0	6	8	15	14
2	5	3	1	7	15	14	13	12
3	4	2	0	14	13	12	11	6
4	3	1	13	12	11	10	7	5
5	2	12	11	10	9	0	6	4
6	11	10	9	8	1	7	5	3
7	9	8	15	2	0	6	4	10

Figure 3. $\Re_2$.

	8	9	10	11	12	13	14	15
0	7	5	13	12	11	10	3	1
1	6	12	11	10	9	4	2	0
2	11	10	9	8	5	3	1	7
3	9	8	15	6	4	2	0	10
4	15	14	7	5	3	1	9	8
5	13	0	6	4	2	8	15	14
6	1	7	5	3	15	14	13	12
7	0	6	4	14	13	12	11	2

Figure 4. $\mathfrak{R}_1$.

Constructed orthogonal Latin rectangles (**Figure 5**).

	8	9	10	11	12	13	14	15
0	7 15	5 14	13 3	12 1	11 7	10 5	3 9	1 8
1	6 13	12 4	11 2	10 0	9 6	4 8	2 15	0 14
2	11 5	10 3	9 1	8 7	5 15	3 14	1 13	7 12
3	9 4	8 2	15 0	6 14	4 13	2 12	0 11	10 6
4	15 3	14 1	7 13	5 12	3 11	1 10	9 7	8 5
5	13 2	0 12	6 11	4 10	2 9	8 0	15 6	14 4
6	1 11	7 10	5 9	3 8	15 1	14 7	13 5	12 3
7	0 9	6 8	4 15	14 2	13 0	12 6	11 4	2 10

Figure 5. Orthogonal Latin rectangle.

	8	9	10
0	7 15	5 14	13 3
1	6 13	12 4	
2	11 5		

Figure 6. Rounding off the orthogongal Latin rectangle.

Rounding off the Latin rectangle after repeating 3 elements (**Figure 6**).

The orthogonal Latin rectangle forms six different quaternions:

{0, 7, 8, 15}, {1, 5, 9, 14}, {0, 3, 10, 13}, {1, 6, 8, 13}, {1, 4, 9, 12}, {2, 5, 8, 11}.

These seven different quaternion feet do not contain an inclusion relation, *i.e.*, they form a set that also forms an antichain:

{0, 7, 8, 15}, {1, 5, 9, 14}, {0, 3, 10, 13}, {1, 6, 8, 13}, {1, 4, 9, 12}, {2, 5, 8, 11},

Based on the relationship between antichain codes and combinatorial neural codes, this results in a combinatorial neural code with parameters

$$\left(n = 16, K = 6, \delta_r = 3r + 3\right)$$

and a weight of 4, a character in a binary code

(1000000110000000 1) (0100010001000010) (1001000000100100)
(0100001010000100) (0100100001001000) (0010010010010000).

Consider below the case where $\frac{p_+}{2}$ is odd and $v = 4m$.

Construct 2: $v=4m$, $\frac{p_+}{2}$ is odd.

The elements of the set of positive sub-Latin rectangle of the first Latin rectangle $\Re_1$ are $\{i,-i-1\}$ at $i\in\left[\frac{P_+-2}{4}-1\right]$, at $i\in\left\{\frac{P_+-2}{4},\frac{P_+-2}{4}+1,\cdots,\frac{P_+}{2}-2\right\}$ for $\{i,-i-2\}$; the element of the set of negative sub-Latin rectangle is $\left\{\frac{P_+}{2}-1,-\frac{P_+-2}{4}-1\right\}$ at $i=\frac{P_+}{2}-1$.

The elements of the set of positive sub-Latin rectangle of the second Latin matrix $\Re_2$ are $\left\{\frac{P_+}{2}-1,-\frac{P_+-2}{4}-1\right\}$ at $i=\frac{P_+}{2}-1$; the elements of the set of negative sub-Latin rectangle are $\{i,-i-1\}$ at $i\in\left[\frac{P_+-2}{4}-1\right]$ and at $i\in\left\{\frac{P_+-2}{4},\frac{P_+-2}{4}+1,\cdots,\frac{P_+}{2}-2\right\}$ for $\{i,-i-2\}$.

The orthogonal Latin rectangle $(x_1,x_2)\in v\times v$ are composed and $(i,j)\in p_+\times p_-$, $\Re_1(i,j)=x_1$, $\Re_2(i,j)=x_2$.

Lemma 4: The Latin rectangle in Construction 2 has no fixed points, nor does the orthogonal Latin rectangle contain fixed points.

Proof: It is only necessary to prove that one Latin rectangle has no fixed point, and the second Latin rectangle is proved using the same steps.

1) There is no such thing as an element appearing twice in the first column, and the subsequent columns are in order minus one from the first, so there can be no such thing as an element appearing twice, *i.e.*, there are no identical elements in the same column.

2) Let $\{i,j\}$ and $\{i',j'\}$ be two different elements belonging to the set of positive Latin rectangle, and let $i'-i=j'-j$.

a) When $j=-i-1$, $j'=-i'-1$,

$$i-i'=j-j'=-i-1-(-i'-1)=(i-i'),$$

We get $i-i'=\frac{p_+}{2}$, by construction 2 if

$$i,i'\in\left[\frac{P_+-2}{4}-1\right]\to\max(i-i')=\frac{p_+}{2}-3,\min(i-i')=1$$

so $i-i'\in\left\{1,2,\cdots,\frac{p_+}{2}-3\right\}$, a contradiction.

b) When $j=-i-1$, $j'=-i'-2$,

$$i-i'=j-j'=-i-1-(-i'-2)=-(i-i')+1\to 2(i-i')=1,$$

Obviously not available, since P_+ is even, and in the case of $\bmod P_+$, P_++1 being odd cannot equal $2(i-i')$, a contradiction.

c) When $j=-i-2$, $j'=-i'-1$,

$$i-i'=j-j'=-i-2-(-i'-1)=-(i-i')-1,$$

the same obviously cannot be obtained.

d) When $j=-i-2$, $j'=-i'-2$,

$$i-i'=j-j'=-i-1-(-i'-1)=-(i-i'),$$

We get $i-i'=\frac{p_+}{2}$, by construction 2 it follows that if

$$i,i'\in\left\{\frac{P_+-2}{4},\frac{P_+-2}{4}+1,\cdots,\frac{P_+}{2}-2\right\},\quad i-i'\in\left\{\frac{p_+}{2}+3,\cdots,p_+-2\right\},$$

a contradiction, that satisfies regularity for positive sub-Latin rectangle, discussed below for negative sub-Latin rectangle:

$$i-i'=\frac{P_+}{2}-1-\left(-\frac{P_+-2}{4}-1\right)=\frac{3p_+-2}{4}\quad j-j'=-\frac{P_+-2}{4}-1-\left(\frac{p_+}{2}-1\right)=\frac{2-3p_+}{4}$$

so $i-i'\neq j-j'$, satisfying regularity.

3) Let $r_{i,j}=i\to r_{i+1,j-1}=i+1\underset{\cdots}{\to} r_{i+j,0}=i+j$, contradiction with Construct 2. For $r_{i,j}=j$ the case is the same as for $r_{i,j}=i$, *i.e.* it is proved that there is no fixed point.

There is no fixed point for two Latin rectangle, *i.e.* there is no fixed point for their mutually orthogonal Latin rectangle. #

Example 2: Let $v=20,\frac{p_+}{2}=5$, constructed Latin rectangles $\Re_1$ and $\Re_2$ (**Figure 7** and **Figure 8**).

	10	11	12	13	14	15	16	17	18	19
0	9	7	4	2	17	8	6	14	3	1
1	8	5	3	16	9	7	13	4	2	0
2	6	4	15	0	8	12	5	3	11	9
3	5	14	1	9	11	6	4	2	0	7
4	13	2	0	10	7	5	3	1	8	6
5	3	1	19	8	6	4	2	9	7	12
6	2	18	9	7	5	3	0	8	11	4
7	17	0	8	6	4	1	9	10	5	3
8	1	9	7	5	2	0	19	6	4	16
9	0	8	6	3	1	18	7	5	15	2

Figure 7. $\Re_1$.

	10	11	12	13	14	15	16	17	18	19
0	19	18	16	15	3	13	12	7	11	10
1	17	15	14	4	12	11	8	10	19	18
2	14	13	5	11	10	9	19	18	17	16
3	12	6	10	19	0	18	17	16	15	13
4	7	19	18	1	17	16	15	14	12	11
5	18	17	2	16	15	14	13	11	10	8
6	16	3	15	14	13	12	10	19	9	17
7	4	14	13	12	11	19	18	0	16	15
8	13	12	11	10	18	17	1	15	14	5
9	11	10	19	17	16	2	14	13	6	12

Figure 8. $\Re_2$.

Constructed orthogonal Latin rectangles (**Figure 9**)

	10	11	12	13	14	15	16	17	18	19
0	9 19	7 18	4 16	2 15	17 3	8 13	6 12	14 7	3 11	1 10
1	8 17	5 15	3 14	16 4	9 12	7 11	13 8	4 10	2 19	0 18
2	6 14	4 13	15 5	0 11	8 10	12 9	5 19	3 18	11 17	9 16
3	5 12	14 6	1 10	9 19	11 0	6 18	4 17	2 16	0 15	7 13
4	13 7	2 19	0 18	10 1	7 17	5 16	3 15	1 14	8 12	6 11
5	3 18	1 17	19 2	8 16	6 15	4 14	2 13	9 11	7 10	12 8
6	2 16	18 3	9 15	7 14	5 13	3 12	0 10	8 19	11 9	4 17
7	17 4	0 14	8 13	6 12	4 11	1 19	9 18	10 0	5 16	3 15
8	1 13	9 12	7 11	5 10	2 18	0 17	19 1	6 15	4 14	16 5
9	0 11	8 10	6 19	3 17	1 16	18 2	7 14	5 13	15 6	2 12

Figure 9. Orthogonal Latin rectangle.

Rounding off the Latin rectangle after repeating 3 elements. (**Figure 10**)

	10	11	12	13	14
0	9 19	7 18	4 16	2 15	17 3
1	8 17	5 15			9 16
2	6 14	4 13			
3	5 12				
4					
5					
6			7 13		

Figure 10. Rounding off the orthogonal Latin rectangle.

Orthogonal Latin rectangle forms an array of 12 distinct quaternions:

{0, 9, 10, 19}, {0, 7, 11, 18}, {0, 4, 12, 16}, {0, 3, 14, 17}, {1, 8, 10, 17}, {1, 5, 11, 15}, {2, 6, 10, 14}, {3, 5, 10, 12}, {2, 4, 11, 13}, {6, 7, 12, 13}, {0, 2, 16, 18}, {1, 9, 14, 16}.

These 12 different quaternions arrays do not have an inclusion relationship therefore the set formed constitutes an antichain:

{{0, 9, 10, 19}, {0, 7, 11, 18}, {0, 4, 12, 16}, {0, 3, 14, 17}, {1, 8, 10, 17}, {1, 5, 11, 15}, {2, 6, 10, 14}, {3, 5, 10, 12}, {2, 4, 11, 13}, {6, 7, 12, 13}, {0, 2, 16, 18}, {1, 9, 14, 16}}.

This results in a parameter of $(n = 20, k = 12, \delta_r = 3r + 3)$ and a combined neural code with a weight of 4. Its corresponding codeword is:

(10000000011000000001) (10000001000100000010) (10001000000010001000)
(10010000000000100100) (01000000101000000100) (01000100000100010000)
(00100010001000100000) (00010100001001000000) (00101000000101000000)
(00000011000011000000) (10100000000000001010) (01000000010000101000).

Consider the case $v = 4m + 1$ below.

Construction 3: One column on top of construction 12 constitutes $[2m] \times [2m+1]$ Latin rectangle, the first Latin rectangle adds a column $r_{i,2m+1} = \frac{p_+}{2} + i \in p_+$, and the second Latin rectangle adds a column

$r_{i,2m+1} = \frac{p_+}{2} - i \in p_-$.

Lemma 5: Construction 3 is an orthogonal Latin rectangle without fixed points.

Proof: the first Latin rectangle obviously occurs at most once in the same column of elements, $r_{0,0}, r_{0,1}, \cdots, r_{0,2m+1}$ the elements belonging to p_+ in this row are all odd, and the one element added is even. For row i, if i is even, $-i-1$ is odd, and $\frac{p_+}{2}+i$ is even; if i is odd, $-i-1$ is even, and $\frac{p_+}{2}+i$ is odd; therefore the same element cannot occur in the same row. The same proof for the second Latin rectangle, *i.e.*, that there are no fixed points for either Latin rectangle, and no fixed points for the orthogonal Latin rectangle.

Consider below $v = 4m+2, p_+ = p_- = 2m+1$.

Construct 4: $v = 4m+2, p_+ = p_- = 2m+1$, construct orthogonal Latin rectangle of $[2m+1] \times [2m+1]$.

The element of the set of positive sub-Latin rectangle of the first Latin moment $\Re_1$ is $\{i, -i-1\}$ at $i \in \left[\frac{P_+-1}{2}-1\right]$; the elements of the set of negative sub-Latin rectangle are $\left\{\frac{P_+-1}{2}, \frac{P_+-1}{2}-1\right\}$ at $i = \frac{P_+-1}{2}$.

The elements of the set of positive sub-Latin rectangle of the second Latin rectangle $\Re_2$ are $\left\{\frac{P_+-1}{2}, \frac{P_+-1}{2}-1\right\}$ for $i = \frac{P_+-1}{2}$; the elements of the set of negative sub-Latin rectangle are $\{i, -i-1\}$ for $i \in \left[\frac{P_+-1}{2}-1\right]$.

Lemma 6: Construction 4 is a Latin rectangle without fixed points, and orthogonal Latin rectangle does not contain fixed points.

Proof: it is sufficient to prove that one Latin rectangle has no fixed point, and the second Latin rectangle is proved using the same steps.

1) There is no such thing as an element appearing twice in the first column, and the subsequent columns are sequentially minus one from the first column, so there can be no such thing as an element appearing twice, *i.e.*, there are no identical elements in the same column.

2) Let $\{i, j\}$ and $\{i', j'\}$ be two different elements belonging to the set of positive Latin rectangle, and let $i'-i = j'-j$, By the construction 4 know

$$i'-i = j'-j = -i'-1-(-i-1) = -(i'-i),$$

that is, $i'-i=0$ or $2(i'-i) = p_+$ obviously cannot be satisfied, a contradiction; and belongs to the negative sub-Latin rectangle of the set of a column of a row of only one element therefore the peer will not be duplicated, that is, to satisfy the regularity.

3) Let $r_{i,j} = i \to r_{i+1,j-1} = i+1 \underset{\cdots}{\to} r_{i+j,0} = i+j$, Contradiction with Construct 4. For $r_{i,j} = j$ the case is the same as for $r_{i,j} = i$, it is proved that there is no fixed point.

The two Latin rectangles have no fixed points, *i.e.*, neither do their mutually orthogonal Latin rectangle.

Example 3: $v=10, p_+ = p_- = 5$, rounding off the Latin rectangle after repeating 3 elements. (**Figure 11**)

	5	6
0	9 4	8 2
1	7 3	

Figure 11. Rounding off the orthogonal Latin rectangle.

The orthogonal Latin rectangle forms 3 distinct quaternions:

{0, 4, 9, 5}, {0, 2, 6, 8}, {1, 3, 7, 5}.

These 3 different quaternions do not have an inclusion relationship therefore the set formed constitutes an antichain:

{{0, 4, 9, 5}, {0, 2, 6, 8}, {1, 3, 7, 5}}

This results in a parameter of $(n=10, k=3, \delta_r = 3r+3)$ and the combined neural code with weight 4 has the corresponding codeword:

(1001100001) (1010001010) (0101010100)

Example 4: $v=14, p_+ = p_- = 7$, the orthogonal Latin rectangle forms 3 distinct quaternion arrays (**Figure 12**):

	7	8	9
0	13 6	12 4	11 2
1	11 5	10 3	
2	9 4		

Figure 12. Rounding off the orthogonal Latin rectangle.

The orthogonal Latin rectangle forms 6 distinct quaternions:

{0, 7, 6, 13}, {0, 4, 8, 12}, {0, 2, 11, 9}, {1, 5, 7, 11}, {1, 3, 8, 10}, {2, 4, 7, 9}.

These six different quaternions do not have an inclusion relationship therefore the set formed constitutes an antichain:

{{0, 7, 6, 13}, {0, 4, 8, 12}, {0, 2, 11, 9}, {1, 5, 7, 11}, {1, 3, 8, 10}, {2, 4, 7, 9}}.

This results in a parameter of $(n=14, k=6, \delta_r = 3r+3)$ and the combined neural code with weight 4 has the corresponding codeword:

(10100000010100) (01000101000100) (01010000101000) (00101001010000) (10000011000001) (10001000100010).

Consider the case $v=4m+3$ below.

Construct 5: One column on top of construct 4 constitutes the Latin rectangle of $[2m+1]\times[2m+2]$,

the first Latin rectangle adds a column $r_{i,2m+2} = \frac{p_+ - 1}{2} + i \in p_-$,

the second Latin rectangle adds a column $r_{i,2m+2} = \frac{p_+ - 1}{2} + i \in p_+$.

Lemma 7: Construction 5 is an orthogonal Latin rectangle without fixed points.

Proof: two proofs of similarity of Latin rectangle. The following only proves the first one. Obviously the same column element occurs at most once, $r_{0,0}, r_{0,1}, \cdots, r_{0,2m+1}$ is only one element belonging to p_- in this row and it is odd, the added one is an element belonging to p_- and it is even. For row i, if i is even, $-i-1$ is odd and $\frac{p_+-1}{2}+i$ is even; if i is odd, $-i-1$ is even and $\frac{p_+-1}{2}+i$ is odd, so there can be no identical element in the row.

3.2. Construction Based on Disjoint Steiner Systems

F is a family of sets, If the set in F can be split into $F_1, F_2, \cdots, F_n$, satisfying that (V, F_i) is a (t', k, v) system $1 \le i \le n$, then (V, F) is t'-partitionable. (t, k, v) In layman's terms, this means that each set of size k in the family of sets formed by v elements has at most $t-1$ elements that intersect every two sets.

Let (V_1, F^1) be t_1-partitionable, with partition classes $F_1^1, F_2^1, \cdots, F_n^1$, then (V_2, F^2) is t_2-partitionable, with partition classes $F_1^2, F_2^2, \cdots, F_n^2$, assuming that V_1 and V_2 do not intersect, then a new system can be formed, blocks are

$$\left\{A \bigcup B : A \in F_i^1, D \in F_i^2, 1 \le i \le n\right\}$$

Lemma 8 [16] [19]: The above V_1 and V_2 form a new system with parameters

$$\left(t = \max\left(k_2 + t_1, k_1 + t_2\right), k_1 + k_2, v_1 + v_2\right). \quad (5)$$

Proof: the new system contains $v = v_1 + v_2$ points, each block has $k = k_1 + k_2$ points, and the block

$$\left\{A \bigcup B : A \in F_i^1, B \in F_i^2, 1 \le i \le n\right\}.$$

Let $A_1, A_2 \in F_i^1$, $B_1, B_2 \in F_i^2$ then

$$t = \max\left(\left(A_1 \cup B_1\right) \cap \left(A_2 \cup B_2\right)\right).$$

And because the value of $\max\left(\left(A_1 \cup B_1\right) \cap \left(A_2 \cup B_2\right)\right)$ is only in two cases: the first one where B_1 is the same as B_2,

$$A_1 \cap A_2 = t_1 - 1;$$

the second where A_1 is the same as A_2,

$$B_1 \cap B_2 = t_2 - 1, \text{ with } |B_i| = k_2, |A_i| = k_1,$$

Since,

$$\max\left(\left(A_1 \cup B_1\right) \cap \left(A_2 \cup B_2\right)\right) = \max\left(k_2 + t_1, k_1 + t_2\right) - 1,$$

So,

$$t = \max\left(\left(A_1 \cup B_1\right) \cap \left(A_2 \cup B_2\right)\right),$$

That is, the parameters of the system constituting the new

$$\left(t = \max\left(k_2 + t_1, k_1 + t_2\right), k_1 + k_2, v_1 + v_2\right).$$

Construct 6: Based on 1-partitionable and 0-partitionable constructs $(2,4,v)$.

The 1-factorization is $(2,4,2m)$ system 1-partitionable with $2m-1$ classes, the 1-factorization can be expressed as $(1,2,2m)$ and $2m-1$ points can form $(1,1,2m-1)$ with one point as a block of districts, which is 0-partitionable. By Lemma 6 a new system parameter can be formed as $(2,3,4m-1)$. Obviously the number of block of $(2,3,4m-1)$ is $m(2m-1)=2m^2-m$, and $2m^2-m$ points different from the previous ones are added to each block of $(2,3,4m-1)$ respectively to form a new system parameter as $(2,4,2m^2+3m-1)$.

$(2,4,2m^2+3m-1)$ forms $2m^2-m$ quaternion, each of which has at most one intersecting element and no containment relation, so the set family formed by these quaternion is an antichain, *i.e.*, it can be constituted as a 4-weighted combinatorial neural code with the parameters $\left(n=2m^2+3m-1, k=2m^2-m, \delta_r=3r+3\right)$.

Theorem 1: Construct 6 forms a combinatorial neural code with parameter

$$\left(n=2m^2+3m-1, k=2m^2-m, \delta_r=3r+3\right) \tag{6}$$

3.3. The Based Group Divisible Design Construct (2, 4, v), v Is a Multiple of 12

Group Divisible Design (GD design) [18]: with X as a given set of positive integers, K and M as a given set of positive integers, let $\boldsymbol{D}=(V,\boldsymbol{G},\boldsymbol{A})$ be a finite associative structure, where V is a v-element set, G constitutes a division of V, the elements of A are called the block, and the elements of G are called the group, if the following conditions are satisfied:

- For any $B \in \boldsymbol{A}$, there is $|B| \in K$;
- For any $G \in \boldsymbol{G}$, there is $|G| \in M$;
- For any $B \in \boldsymbol{A}$ and any $G \in \boldsymbol{G}$, then $|B \cap G| \le 1$;
- Any pair of elements of V belonging to different groups is contained in exactly λ block at the same time.

Then D is said to be a Group Divisible Design or GD design, denoted $GD(K,X,M;V)$.

Theorem 2: $GD(4,1,3;v)$, v is a multiple of 12,

$$\left|B(4,1;v+1)\right| = \frac{v}{3} + \frac{v(v-3)}{12} \tag{7}$$

Example 7: Let $v=\{1,2,\cdots,12\}$, the groups in $GD(4,1,3;v)$ are {1, 2, 3}, {4, 5, 6}, {7, 8, 9}, {10, 11, 12}; blocks: {1, 4, 7, 10}, {1, 5, 8, 11}, {1, 6, 9, 12}, {2, 4, 8, 12}, {2, 5, 9, 10}, {2, 6, 7, 11}, {3, 4, 9, 11}, {3, 5, 7, 12}, {3, 6, 8, 10}. Adding a point 13 to each of these groups results in 4 new blocks: {1, 2, 3, 13}, {4, 5, 6, 13}, {7, 8, 9, 13}, {10, 11, 12, 13}. Together with the above, the 9 blocks form a $B(4,1;13)$, $|B(4,1;13)|=13$. The above quaternion array forms a combinatorial neural code with parameters $(n=13, K=13, \delta_r=3r+3)$ with the following structure:

(1001001001000) (1000100100100) (1000010010010) (0101000100010)
(0100100011000) (0100011000100) (0010101000010) (0011000010100)
(1110000000001) (0010010101000) (0001110000001) (0000001110001)
(0000000001111)

3.4. Based on the Transversal Design Construction (t, k, v), $v = km$, m, k Is Even

Transversal design [18]: i) The set V has kv elements called points; ii) Partition of V into sets $\{G_i : i \in [k]\}$, each G_i contains v points called groups; iii) A set B of k-subsets is called a block if it satisfies the following two conditions: a. Each group and each block intersect by exactly 1 point; b. t-subsets of V appear in only one block or two and more points are in one group but not all points.

Theorem 3: $TD(t,k,v)$, $v = km$, m,k is even, then

$$\left|(k-1,k,v) = m^{k-1} + 2^{\frac{k}{2}} \times \binom{m}{2} \times \left(\frac{m}{2}\right)^{\frac{k-2}{2}}\right| \tag{8}$$

Proof: $v = km$ divide the v points into k groups, each group has m points, will be v points with two-dimensional label $[k]\times[m]$, based on the transversal design there are already m^{k-1} blocks, because it is determined that one is selected for each group, and the last is determined by determining the $k-1$ in front of it. Below add new block which is formed by

$$\{a_1, a_2, \cdots, a_{k-1}, a_k\},$$

where $\{a_1, a_2\} \in \Phi_i^{g_1}$, $\{a_3, a_4\} \in \Phi_i^{g_2}$, $\cdots$, $\{a_{k-1}, a_k\} \in \Phi_i^{g_{\frac{k}{2}}}$, and

$$g_1 \in [0,1], g_2 \in [2,3], \cdots, g_{\frac{k}{2}} \in [k-2, k-1], \Phi^{(g)} = \left\{\Phi_1^{(g)}, \Phi_2^{(g)}, \cdots, \Phi_{m-1}^{(g)}\right\}$$

is a 1-factor decomposition of m elements in each group. There are a total of k groups for $g \in [k-1]$ because g consists of $\left\{g_1, g_2, \cdots, g_{\frac{k}{2}}\right\}$, and each g_i has two choices, *i.e.*, there are $2^{\frac{k}{2}}$ choices for g. If $\left\{g_1, g_2, \cdots, g_{\frac{k}{2}}\right\}$ is fixed, $\{a_1, a_2\}$ are related so there are $\binom{m}{2}$ choices, and $\{a_3, a_4\}$ have $\frac{m}{2}$ choices, and similarly $\{a_{k-1}, a_k\}$ also have $\frac{m}{2}$ choices, then $\{a_1, a_2, \cdots, a_{k-1}, a_k\}$ has a total of $2^{\frac{k}{2}} \times \binom{m}{2} \times \left(\frac{m}{2}\right)^{\frac{k-2}{2}}$ choices. So

$$\left|(k-1,k,v)\right| = m^{k-1} + 2^{\frac{k}{2}} \times \binom{m}{2} \times \left(\frac{m}{2}\right)^{\frac{k-2}{2}}$$

$TD(t,k,v)$ constructs a family of sets of set size k and the sets in the family intersect at most $t-1$ identically, which do not have an inclusion relation, thus constituting a combinatorial neural code. $TD(k-1,k,v)$ constitutes a combinatorial neural code with parameters

$$\left(n=v, K=m^{k-1}+2^{\frac{k}{2}}\times\binom{m}{2}\times\left(\frac{m}{2}\right)^{\frac{k-2}{2}}, \delta_r=2r+2\right)$$

Since there are four choices for it follows that the total number of added blocks is 48.

Example 8: Suppose that is a transversal design whose with points of the form $Z_4\times\{0,1,2,3\}$, and blocks

$$\{\{(i_0,0),(i_1,1),(i_2,2),(i_3,3)\}: i_0,i_1,i_2,i_3\in Z_4, i_3=i_0+i_1+i_2\}$$

Since there are four choices for each of i_0,i_1,i_2 it follows that we have $4^3=64$ blocks.

The factorization of interest reads as

$$\begin{aligned}\Phi^g &= \{\Phi_1^{(g)},\Phi_2^{(g)},\Phi_3^{(g)}\}\\ &= \{\{\{(0,g),(1,g)\},\{(2,g),(3,g)\}\},\\ &\quad \{\{(0,g),(2,g)\},\{(1,g),(3,g)\}\},\\ &\quad \{\{(0,g),(3,g)\},\{(1,g),(2,g)\}\}\}\end{aligned}$$

Where $g\in\{0,1,2,3\}$, for a $g_1\in\{0,1\}$ and a $g_2\in\{2,3\}$, we obtain 12 blocks:

$$\begin{aligned}&\{\{(0,g_1),(1,g_1),(0,g_2),(1,g_2)\},\{(0,g_1),(1,g_1),(2,g_2),(3,g_2)\},\\ &\{(2,g_1),(3,g_1),(0,g_2),(1,g_2)\},\{(2,g_1),(3,g_1),(2,g_2),(3,g_2)\},\\ &\{(0,g_1),(2,g_1),(0,g_2),(2,g_2)\},\{(0,g_1),(2,g_1),(1,g_2),(3,g_2)\},\\ &\{(1,g_1),(3,g_1),(0,g_2),(2,g_2)\},\{(1,g_1),(3,g_1),(1,g_2),(3,g_2)\},\\ &\{(0,g_1),(3,g_1),(0,g_2),(3,g_2)\},\{(0,g_1),(3,g_1),(1,g_2),(2,g_2)\},\\ &\{(1,g_1),(2,g_1),(0,g_2),(3,g_2)\},\{(1,g_1),(2,g_1),(1,g_2),(2,g_2)\}\}\end{aligned}$$

Since there are four choices for g_1,g_2 it follows that the total number of added blocks is 48. There are 112 blocks in total, *i.e.*, they constitute a combinatorial neural code with parameters $(n=16, K=112, \delta_r=2r+2)$.

4. Conclusion

This paper focuses on constructing a class of combinatorial neural codes using orthogonal Latin rectangle, transversal designs, and group divisible designs. Section 1 introduces the research background and properties of combinatorial neural codes. Section 2 introduces antichain codes as a class of combinatorial neural codes, and the associated lemmas. Section 3 constructs combinatorial neural codes with parameters $(n,k,\delta_r=3r+3)$ using the method of orthogonal Latin rectangle and it has all 4-weight distributions; then a 4-weight code with parameters $(n=2m^2+3m-1, K=2m^2-m, \delta_r=3r+3)$ is constructed by using the disjointed Steiner system; and then a 4-weight code with parameters $\left(n, k=\frac{n}{3}+\frac{n(n-3)}{12}, \delta_r=3r+3\right)$ 4-weighted code; and finally the combined neural code with parameters (n, k, δ_r) is constructed based on the transversal design.

Acknowledgements

The authors gratefully acknowledge the discussion with Tang Chunming, Southwest Jiaotong University.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Atick, J.J. (1992) Could Information Theory Provide an Ecological Theory of Sensory Processing? *Network*: *Computation in Neural Systems*, **3**, 4-44. https://doi.org/10.3109/0954898X.2011.638888

[2] Borst, A. and Theunissen, F.E. (1999) Information Theory and Neural Coding. *Nature Neuroscience*, **2**, 947-957. https://doi.org/10.1038/14731

[3] O'Keefe, J. and Dostrovsky, J. (1971) The Hippocampus as a Spatial Map. Preliminary Evidence from Unit Activity in the Freely-Moving Rat. *Brain Research*, **34**, 171-175. https://doi.org/10.1016/0006-8993(71)90358-1

[4] Cotardo, G. and Ravagnani, A. (2022) Parameters of Codes for the Binary Asymmetric Channel. *IEEE Transactions on Information Theory*, **68**, 2941-2950. https://doi.org/10.1109/tit.2022.3147593

[5] Chen, A., Frick, F. and Shiu, A. (2019) Neural Codes, Decidability, and a New Local Obstruction to Convexity. *SIAM Journal on Applied Algebra and Geometry*, **3**, 44-66. https://doi.org/10.1137/18m1186563

[6] Gross, E., Obatake, N. and Youngs, N. (2018) Neural Ideals and Stimulus Space Visualization. *Advances in Applied Mathematics*, **95**, 65-95. https://doi.org/10.1016/j.aam.2017.10.002

[7] Curto, C., Itskov, V., Veliz-Cuba, A. and Youngs, N. (2013) The Neural Ring: An Algebraic Tool for Analyzing the Intrinsic Structure of Neural Codes. *Bulletin of Mathematical Biology*, **75**, 1571-1611. https://doi.org/10.1007/s11538-013-9860-3

[8] Curto, C., Itskov, V., Morrison, K., Roth, Z. and Walker, J.L. (2013) Combinatorial Neural Codes from a Mathematical Coding Theory Perspective. *Neural Computation*, **25**, 1891-1925. https://doi.org/10.1162/neco_a_00459

[9] Borst, A. and Theunissen, F.E. (1999) Information Theory and Neural Coding. *Nature Neuroscience*, **2**, 947-957. https://doi.org/10.1038/14731

[10] Curto, C., Itskov, V., Morrison, K., Roth, Z. and Walker, J.L. (2013) Combinatorial Neural Codes from a Mathematical Coding Theory Perspective. *Neural Computation*, **25**, 1891-1925. https://doi.org/10.1162/neco_a_00459

[11] Fu, F.-W., Ling, S. and Xing, C. (2003) New Lower Bounds and Constructions for Binary Codes Correcting Asymmetric Errors. *IEEE Transactions on Information Theory*, **49**, 3294-3299. https://doi.org/10.1109/tit.2003.820028

[12] Firer, M. and Walker, J.L. (2016) Matched Metrics and Channels. *IEEE Transactions on Information Theory*, **62**, 1150-1156. https://doi.org/10.1109/tit.2015.2512596

[13] Kløve, T. (1981) Errors Correcting Codes for Asymmetric Channel. Department of Informatics, University of Bergen.

[14] Qureshi, C.M. (2019) Matched Metrics to the Binary Asymmetric Channels. *IEEE Transactions on Information Theory*, **65**, 1106-1112. https://doi.org/10.1109/tit.2018.2885782

[15] Cotardo, G. and Ravagnani, A. (2020) Parameters of Combinatorial Neural Codes. arXiv: 2011.101v1.

[16] Gunby, B., He, X., Narayanan, B., *et al.* (2022) Antichain Codes. arXiv: 2212.08406.

[17] Sperner, E. (1928) Ein Satz über Untermengen einer endlichen Menge. *Mathematische Zeitschrift*, **27**, 544-548. https://doi.org/10.1007/bf01171114

[18] Zhang, X., Jing, X. and Feng, K. (2023) Optimal Combinatorial Neural Codes with Matched Metric δ_r: Characterization and Constructions. *IEEE Transactions on Information Theory*, **69**, 5440-5448. https://doi.org/10.1109/TIT.2023.3266010

[19] Shen, H. (2008) Combinatorial Design Theory. 2nd Edition, Shanghai Jiao Tong University Press.

[20] Gabrys, R., Dau, H., Colbourn, C.J. and Milenkovic, O. (2019) Set-Codes with Small Intersections and Small Discrepancies. 2019 *IEEE International Symposium on Information Theory* (*ISIT*), Paris, 7-12 July 2019, 2359-2363. https://doi.org/10.1109/isit.2019.8849651.