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Bifurcations, Analytical and Non-Analytical Traveling Wave Solutions of (2 + 1)-Dimensional Nonlinear Dispersive Boussinesq Equation

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Abstract

For the (2 + 1)-dimensional nonlinear dispersive Boussinesq equation, by using the bifurcation theory of planar dynamical systems to study its corresponding traveling wave system, the bifurcations and phase portraits of the regular system are obtained. Under different parametric conditions, various sufficient conditions to guarantee the existence of analytical and non-analytical solutions of the singular system are given by using singular traveling wave theory. For certain special cases, some explicit and exact parametric representations of traveling wave solutions are derived such as analytical periodic waves and non-analytical periodic cusp waves. Further, two-dimensional wave plots of analytical periodic solutions and non-analytical periodic cusp wave solutions are drawn to visualize the dynamics of the equation.

Keywords

(2 + 1)-Dimensional Nonlinear Dispersive Boussinesq Equation, Bifurcations, Phase Portrait, Analytical Periodic Wave Solution, Periodic Cusp Wave Solution

1. Introduction

Many nonlinear phenomena in biology, fluid dynamics, chemistry, plasma physics, etc, can be described by nonlinear partial differential equations. Looking for explicit and exact solutions to nonlinear partial differential equations is a very meaningful and important task. These explicit and exact solutions may well describe various nonlinear phenomena in biology, fluid dynamics, chemistry, plasma physics, etc. In the past decades, many methods for obtaining explicit and exact solutions have been established such as improved modified extended tanh-function method [1] [2], Lie Group Method [3] [4], hyperbolic function method [5] [6], Jacobi elliptic function method [7] [8], sine-cosine method [9] [10], extended Kudryashov's method [11], improved sub-equation method [12] [13], enhanced (G′/G)-expansion method [14], extended (or improved) Hirota bilinear method [15] [16], Darboux transformation method [17] [18], Bäcklund transformation method [19], direct integration method [20], the ansatz method [21], method of separation of variables [22] and so on. However, the solution procedures become very complex and most of the methods fail or can obtain few special solutions as the degree of the nonlinearity is high. Therefore it is very important to do qualitative analysis of nonlinear partial differential equations. It can help one understand the dynamical properties of solutions even without obtaining any exact solution.

The Boussinesq equation, which describes the propagation of water waves, is one of the most classical integrable nonlinear partial differential equations and has attracted the attention of many researchers. Yan [23] introduced a family of Boussinesq equations with fully nonlinear dispersion

$$B(m,n):\ u_{tt}-\left(u^{n}\right)_{xx}-\left(u^{m}\right)_{xxxx}=0, \tag{1}$$

and presented some compacton solutions for $m=n$. Zhu *et al.* [24]-[26] researched the following three types of Boussinesq-like equations

$$u_{tt}+\left(u^{n}\right)_{xx}-\left(u^{m}\right)_{xxxx}=0, \tag{2}$$

$$u_{tt}-\left(u^{n}\right)_{xx}+\left(u^{m}\right)_{xxxx}=0, \tag{3}$$

and

$$u_{tt}+\left(u^{2n}\right)_{xx}+\left(u^{2n}\right)_{xxxx}=0, \tag{4}$$

and obtained some soliton solutions by using the extended decomposition method. Zhang *et al.* [27] investigated a more generalized form of the Boussinesq equations

$$u_{tt}+a\left(u^{n}\right)_{xx}+b\left(u^{m}\right)_{xxxx}=0, \tag{5}$$

and obtained several compacton and solitary solutions by using the integral approach. In 2022, Sun *et al.* [28] studied the Boussinesq equation in the following form

$$u_{tt}+2\left(u^{2}\right)_{xx}+\frac{1}{3}u_{xxxx}=0 \tag{6}$$

and derived the degenerate breather solutions through the generalized Darboux transformation. Yan [29] introduced a more generalized form of the (2 + 1)-dimensional Boussinesq equation with fully nonlinear dispersion

$$B(m,n,k,p):\ \left(u^{m}\right)_{tt}+\alpha\left(u^{n}\right)_{xx}+\beta\left(u^{k}\right)_{xxxx}+\gamma\left(u^{p}\right)_{yyyy}=0, \tag{7}$$

where $m,n,k,p\in Z^{+}$ and α,β,γ are real parameters. Equations (1)-(6) are only the special cases of Equation (7). By taking other different values of m,n,k,p and α,β,γ, Equation (7) contains a lot of well-known equations as special examples. When $m=n=1$, $\gamma=0$, equation (7) becomes the Boussinesq equation for $k=2$ and the modified Boussinesq equation for $k=3$. Yan [29] and Wazwaz [30] respectively constructed some solitary pattern solutions for $m=n=k=p$. Tian and Yu [31] obtained several compacton solutions of (7) for $k=p$. Up to now, various generalizations of the classical Boussinesq equations have been proposed and studied to probe the dynamical properties of water motions. Various exact solutions of the generalized Boussinesq equations have been constructed. Since looking for general solutions is much too difficult and only a few exact solutions can be solved, it becomes very important to do a qualitative analysis of the solutions. Here we focus on the traveling wave solutions to Equation (7) and discuss the dynamical behavior of the traveling wave solutions via bifurcation theory of planar dynamical systems.

We introduce the traveling wave transformation

$$u(x,y,t)=u(\xi),\ \xi=\omega_1 x+\omega_2 y-\omega_3 t, \tag{8}$$

where ω_1 and ω_2 are real constants while ω_3 is the wave speed. Then Equation (7) is reduced to the following nonlinear ordinary equation

$$\omega_3^2\left(u^m\right)_{\xi\xi}+\alpha\omega_1^2\left(u^n\right)_{\xi\xi}+\beta\omega_1^4\left(u^k\right)_{\xi\xi\xi\xi}+\gamma\omega_2^4\left(u^p\right)_{\xi\xi\xi\xi}=0. \tag{9}$$

Integrating (9) twice with respect to ξ and setting two integration constants as zero, we have

$$\omega_3^2 u^m+\alpha\omega_1^2 u^n+\beta\omega_1^4\left(u^k\right)_{\xi\xi}+\gamma\omega_2^4\left(u^p\right)_{\xi\xi}=0. \tag{10}$$

Without loss the generality, we suppose that $m\ge n$, $k\ge p$. We only consider (10) for the case $p>n$. It can be similarly treated for another case $p\le n$. Let $u(\xi)=(\phi(\xi))^{\frac{1}{n}}$, then we have

$$\phi_{\xi\xi}=-\frac{\left(\mu k(k-n)\phi^{\frac{k}{n}-2}+p(p-n)\phi^{\frac{p}{n}-2}\right)\phi_\xi^2+n^2\left(\nu\phi^{\frac{m}{n}}+\omega\phi\right)}{n\left(\mu k\phi^{\frac{k-n}{n}}+p\phi^{\frac{p-n}{n}}\right)}, \tag{11}$$

where $\mu=\frac{\beta\omega_1^4}{\gamma\omega_2^4},\nu=\frac{\omega_3^2}{\gamma\omega_2^4},\omega=\frac{\alpha\omega_1^2}{\gamma\omega_2^4}$. Let $\frac{\mathrm{d}\phi}{\mathrm{d}\xi}=z$. Then (11) is equivalent to the planar system

$$\begin{cases}\dfrac{\mathrm{d}\phi}{\mathrm{d}\xi}=z,\\[2ex] \dfrac{\mathrm{d}z}{\mathrm{d}\xi}=-\dfrac{\left(\mu k(k-n)\phi^{\frac{k}{n}-2}+p(p-n)\phi^{\frac{p}{n}-2}\right)z^2+n^2\left(\nu\phi^{\frac{m}{n}}+\omega\phi\right)}{n\left(\mu k\phi^{\frac{k-n}{n}}+p\phi^{\frac{p-n}{n}}\right)},\end{cases} \tag{12}$$

which has the first integral

$$H(\phi,z)=\left(\mu k\phi^{\frac{k-n}{n}}+p\phi^{\frac{p-n}{n}}\right)^2 z^2 +2n^2\left(\frac{\mu\nu k\phi^{\frac{m+k}{n}}}{m+k}+\frac{\mu\omega k\phi^{\frac{n+k}{n}}}{n+k}+\frac{p\nu\phi^{\frac{m+p}{n}}}{m+p}+\frac{p\omega\phi^{\frac{n+p}{n}}}{n+p}\right)=h, \tag{13}$$

where h is an arbitrary constant.

System (12) is a 7-parameter planar dynamical system depending on the parameter group (m,n,k,p,μ,ν,ω). Suppose that $\phi(\xi)$ is a continuous solution of (11) for $\xi\in(-\infty,\infty)$ and $\lim_{\xi\to\infty}\phi(\xi)=a$, $\lim_{\xi\to-\infty}\phi(\xi)=b$. Recall that 1) $\phi(\xi)$ is called a soliton solution if $a=b$; 2) $\phi(\xi)$ is called a kink or anti-kink soliton solution if $a\neq b$. Usually, a soliton solution of (11) corresponds to a homoclinic orbit of (12); a kink (or anti-kink) soliton solution of (11) corresponds to a heteroclinic orbit (or a so-called connecting orbit) of (12). Similarly, a periodic solution of (11) derives from a periodic orbit of (12). Thus, to investigate all possible bifurcations of soliton and periodic solutions of (11), we need to discuss all periodic annuli, homoclinic and heteroclinic orbits of (12). The bifurcation theory of dynamical systems [32] [33] plays a crucial role in the work.

We notice that there are three possible singularities in the right hand of the second Equation in (12) for $\phi=0$ and $\phi=\pm\phi_s$, where $\phi_s=\left(-\frac{p}{\mu k}\right)^{\frac{n}{k-p}}$. That is to say $\phi_{\xi\xi}$ has no definition on the above straight lines in the (ϕ,z)-phase plane, which implies that the analytical system (7) probably has singular traveling wave solutions. In this paper, to remove the singularities, we first introduce a scale transformation to transform the singular system into a regular system. We then study the bifurcation sets and the phase portraits of the regular system by using the bifurcation theory of dynamical systems. We next analyse the impact of singularity on the analyticity of solutions of (12) through the singular traveling wave theory developed by Li and Liu [34]. Finally, we obtain some sufficient conditions for the existence of analytical and non-analytical traveling wave solutions.

The rest of the paper is organized as follows. In Section 2, we analyse the bifurcation sets and phase portraits of the regular system (14). In Sections 3 and 4, we discuss the impact of singularity on the analyticity of solutions of (12) based on the phase portraits of the regular system (14) obtained in Section 2 and give some sufficient conditions guaranteeing the existence of analytical and non-analytical traveling wave solutions. In Section 5, we obtain some explicit and exact traveling wave solutions of (7) for certain special cases. The simulations and discussions are presented in Section 6. Finally, some conclusions are given in Section 7.

2. Bifurcations and Phase Portraits of Regular System (14)

In this section, we discuss the bifurcation parametric sets, bifurcation curves and phase portraits of (12).

Making the scale transformation $\mathrm{d}\xi = n\left(\mu k\phi^{\frac{k-n}{n}} + p\phi^{\frac{p-n}{n}}\right)\mathrm{d}\zeta$ in order to remove the singularity, then the singular system (12) is reduced to the following regular system

$$\begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\zeta} = n\left(\mu k\phi^{\frac{k-n}{n}} + p\phi^{\frac{p-n}{n}}\right)z, \\ \dfrac{\mathrm{d}z}{\mathrm{d}\zeta} = -\left(\mu k(k-n)\phi^{\frac{k}{n}-2} + p(p-n)\phi^{\frac{p}{n}-2}\right)z^2 - n^2\left(\nu\phi^{\frac{m}{n}} + \omega\phi\right), \end{cases} \tag{14}$$

which has the same topological phase portraits as (12) except on the singular lines $\phi = 0$ and $\phi = \pm\phi_s$. Both Equations (12) and (14) have the same first integral (13). The singular lines $\phi = 0$ and $\phi = \pm\phi_s$ are invariant straight-line solutions of (14).

For a given constant *h*, Equation (13) determines a set of invariant curves of (14), which contains different branches of curves. As the integration constant *h* varies, Equation (13) gives different families of orbits of (14) which correspond to different dynamical behavior.

To study the bifurcations and phase portraits of (14), we shall first investigate its all equilibrium points.

Equation (14) has two equilibrium points at $O(0,0)$ and $A^+(\phi_1,0)$ for $m-n$ is odd and has three equilibria at $O(0,0)$, A^+ and $A^-(-\phi_1,0)$ for $m-n$ is even, where $\phi_1 = \left(-\dfrac{\omega}{\nu}\right)^{\frac{n}{m-n}}$.

Equation (14) has two equilibrium points at $S_0^{\pm}\left(0, \pm\sqrt{-\dfrac{\omega}{6}}\right)$ on the straight line $\phi = 0$ when $p = 3n$ and $\omega < 0$.

Denote that

$$Y = \frac{n^2}{p(k-p)}\left(\nu\phi_s^{\frac{m}{n}-1} + \omega\right)\phi_s^{\frac{3n-p}{n}},$$

and

$$Z = \frac{n^2}{p(k-p)}\left(\nu(-\phi_s)^{\frac{m}{n}-1} + \omega\right)(-\phi_s)^{\frac{3n-p}{n}}.$$

Equation (14) has two equilibria at $S_1^{\pm}\left(\phi_s, \pm\sqrt{Y}\right)$ on the straight line $\phi = \phi_s$ for $Y > 0$ and has two equilibria at $S_2^{\pm}\left(-\phi_s, \pm\sqrt{Z}\right)$ on the straight line $\phi = -\phi_s$ for $Z > 0$.

Let $M(\phi_e, z_e)$ be the coefficient matrix of the linearized system of (14) at the equilibrium point (ϕ_e, z_e) and let $J(\phi_e, z_e) = \det(M(\phi_e, z_e))$. By the theory of planar dynamical systems, we know that the equilibrium (ϕ_e, z_e) is a saddle (or center) if $J < 0$ (or >0); it is a cusp if $J = 0$ and the Poincaré index of (ϕ_e, z_e) is zero. Thus, $S_i^{\pm}(i = 0,1,2)$ all are saddles.

Denote that $h_0^{\pm} = H\left(0, \pm\sqrt{-\frac{\omega}{6}}\right)$, $h_1^{\pm} = H\left(\pm\phi_1, 0\right)$, $h_2^{\pm} = H\left(\phi_s, \pm\sqrt{Y}\right)$ and $h_3^{\pm} = H\left(-\phi_s, \pm\sqrt{Z}\right)$, where $H(\phi, z)$ is defined by (13). It is obvious that $h_0^{\pm} = H(0,0) = 0$.

Here we are considering a physical model where only bounded solutions are meaningful. Thus we only pay attention to the bounded solutions of (14). In other words, we only consider the case when (14) has at least one center. Hence, we next assume that $\alpha < 0$, $\beta < 0$, $\gamma > 0$ and $\omega_1\omega_2\omega_3 \neq 0$. The other cases can be treated by the similar method. Suppose that $l_1, l_2 \in Z^+$. By using the above facts to do qualitative analysis, we obtain the following results.

Case I. $m - n = 2l_1 - 1$, $k = p$.

In this case, there is a unique singular straight line $\phi = 0$. For a fixed $\alpha < 0$, there exists a unique bifurcation curve $L_0 : \gamma = -\frac{\beta\omega_1^4}{\omega_2^4}$ which divides the second quadrant of the (β, γ)-parametric plane into two subregions

$$B_1 = \left\{(\beta,\gamma) \mid \beta < 0, \gamma > -\frac{\beta\omega_1^4}{\omega_2^4}\right\};\quad C_1 = \left\{(\beta,\gamma) \mid \beta < 0, 0 < \gamma < -\frac{\beta\omega_1^4}{\omega_2^4}\right\}.$$

Except on the singular line $\phi = 0$, Equation (14) has a unique equilibrium point A^+ which is a center for $(\beta,\gamma) \in B_1$ and a saddle for $(\beta,\gamma) \in C_1$. We only consider this case when $(\beta,\gamma) \in B_1$ since we only study bounded solutions of (14).

1) When $p < 3n$ and $(\beta,\gamma) \in B_1$, Equation (14) also exists another equilibrium point $O(0,0)$ besides A^+. O is a saddle point (or a cusp) when p is odd (or even). Equation (14) has a family of periodic orbits defined by $H(\phi, z) = h$ for $h \in \left(h_1^+, 0\right)$. These periodic orbits encircle the center A^+ and form a periodic annulus. Moreover, Equation (14) has a homoclinic orbit defined by $H(\phi, z) = 0$. The homoclinic orbit is homoclinic to the saddle O and is a boundary curve of the periodic annulus.

2) When $p = 3n$ and $(\beta,\gamma) \in B_1$, Equation (14) has two other equilibrium points $S_0^{\pm}$ on the singular line $\phi = 0$ besides A^+. S_0^+ and S_0^- both are saddle points. Equation (14) has a family of periodic orbits defined by $H(\phi, z) = h$ for $h \in \left(h_1^+, 0\right)$. These periodic orbits encircle the center A^+ and form a periodic annulus. Moreover, Equation (14) has two heteroclinic orbits $\overline{S_0^+S_0^-}$ and $S_0^+S_0^-$ defined by $H(\phi, z) = 0$. These two heteroclinic orbits form a two-point heteroclinic cycle $S_0^+S_0^-S_0^+$ which is a boundary curve of the periodic annulus.

3) When $p > 3n$ and $(\beta,\gamma) \in B_1$, Equation (14) has no other equilibrium point besides A^+ and has a family of large-scale periodic orbits defined by $H(\phi, z) = h$ for $h \in \left(h_1^+, 0\right)$. These periodic orbits encircle the center A^+ and form an unbounded periodic annulus.

Based on the above analysis and with the help of the mathematical software *MAPLE*, we can obtain the phase portraits of (14) shown in **Figure 1**.

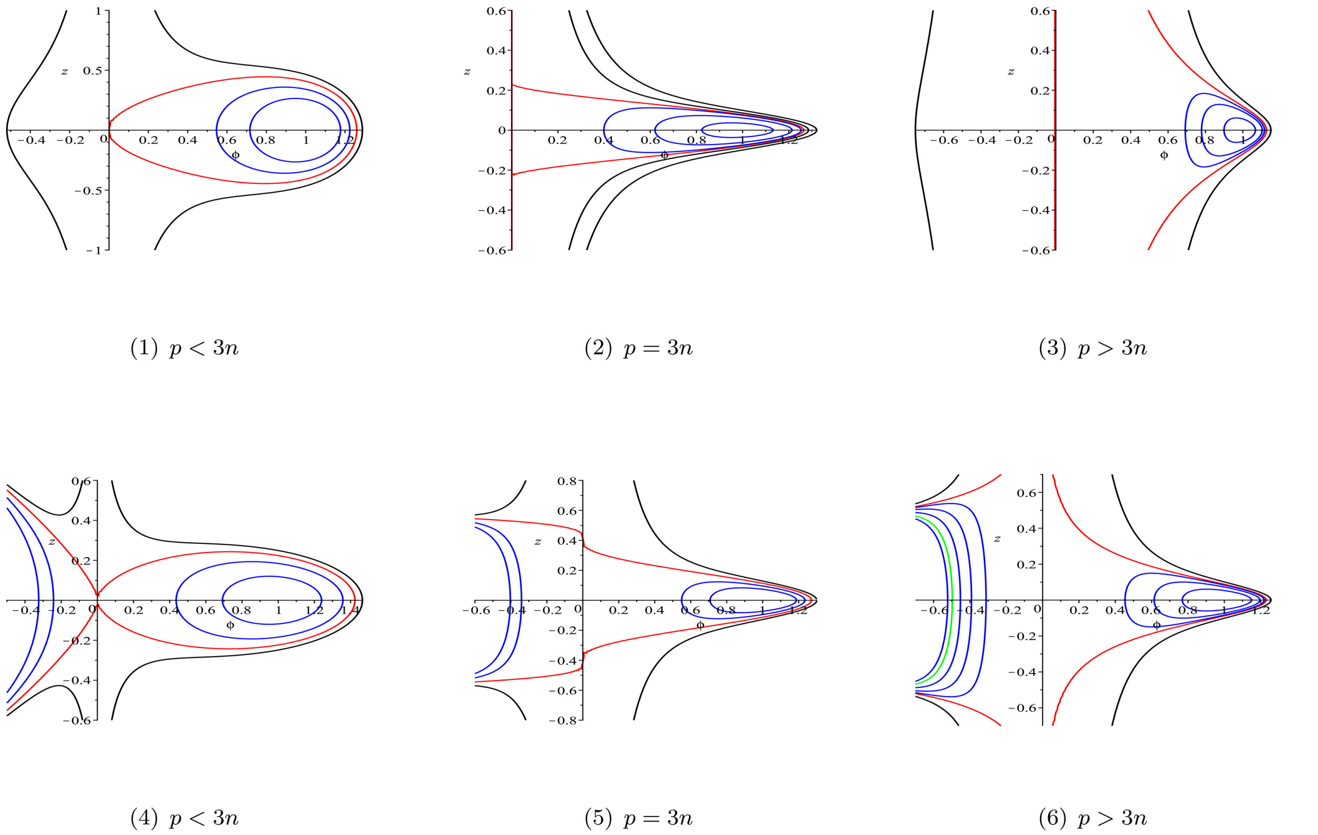


Figure 1. The phase portraits of (14) in Case I for $(\beta,\gamma)\in B_1$ (1)-(3) p is even; (4)-(6) p is odd.

Case II. $m-n=2l_1-1$, $k-p=2l_2-1$.

In this case, there exist two singular straight lines $\phi=0$ and $\phi=\phi_s$. For a fixed $\alpha<0$, the second quadrant of the (β,γ)-parametric plane is divided into three subregions

$$B_2=\{(\beta,\gamma)\,|\,\beta<0,\gamma>-\beta\Delta_2\};$$
$$C_2=\{(\beta,\gamma)\,|\,\beta<0,-\beta\Delta_1<\gamma<-\beta\Delta_2\};$$
$$D_2=\{(\beta,\gamma)\,|\,\beta<0,0<\gamma<-\beta\Delta_1\}$$

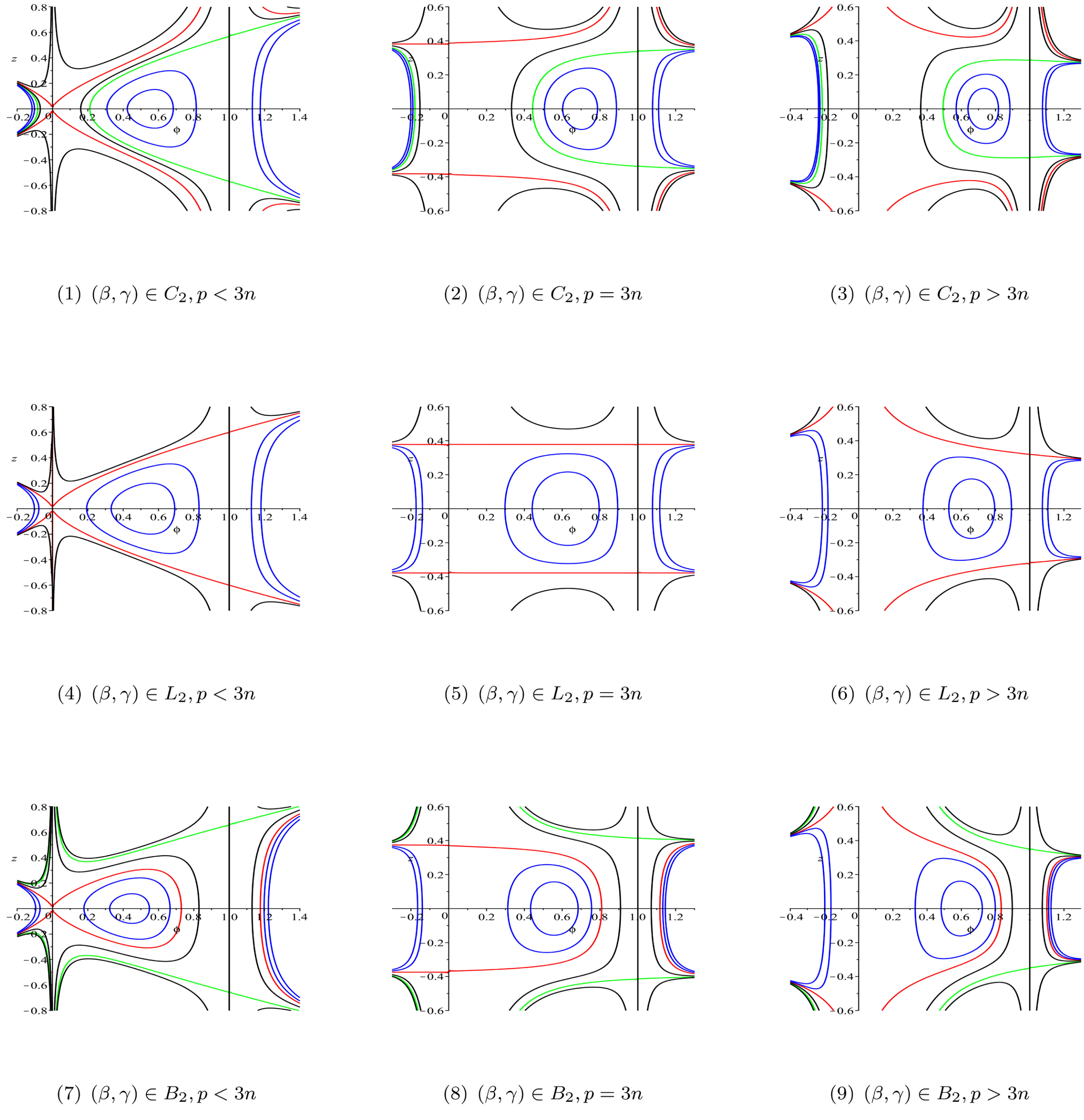
by the bifurcation curves $L_1:\gamma+\Delta_1\beta=0$ and $L_2:\gamma+\Delta_2\beta=0$, where

$$\Delta_1=\frac{k\omega_1^4}{p\omega_2^4}\left(-\frac{\alpha\omega_1^2}{\omega_3^2}\right)^{\frac{k-p}{m-n}},\quad \Delta_2=\frac{k\omega_1^4}{p\omega_2^4}\left(-\frac{(m+k)(m+p)\alpha\omega_1^2}{(n+k)(n+p)\omega_3^2}\right)^{\frac{k-p}{m-n}}.$$

On the ϕ-axis in the (ϕ,z)-phase plane, the system (14) has two equilibrium points O and A^+. A^+ is a saddle for $(\beta,\gamma)\in D_2\cup L_1$ and a center for $(\beta,\gamma)\in B_2\cup C_2\cup L_2$. We do not consider the case $(\beta,\gamma)\in D_2\cup L_1$ since we only pay attention to the bounded solutions of (14). The phase portraits of (14) are displayed in Figure 2.

Case III. $m-n=2l_1-1$, $k-p=2l_2$.

In this case, there exist three singular straight lines $\phi=0$ and $\phi=\pm\phi_s$. For a fixed $\alpha<0$, there are the same bifurcation curves and bifurcation sets as in Case II. Except on the singular line $\phi=0$, Equation (14) has a unique equilibrium A^{+} which is a saddle for $(\beta,\gamma)\in D_2\cup L_1$ and a center for $(\beta,\gamma)\in B_2\cup C_2\cup L_2$. The pase portraits of (14) are shown in **Figure 3**.



(1) $(\beta,\gamma)\in C_2, p<3n$

(2) $(\beta,\gamma)\in C_2, p=3n$

(3) $(\beta,\gamma)\in C_2, p>3n$

(4) $(\beta,\gamma)\in L_2, p<3n$

(5) $(\beta,\gamma)\in L_2, p=3n$

(6) $(\beta,\gamma)\in L_2, p>3n$

(7) $(\beta,\gamma)\in B_2, p<3n$

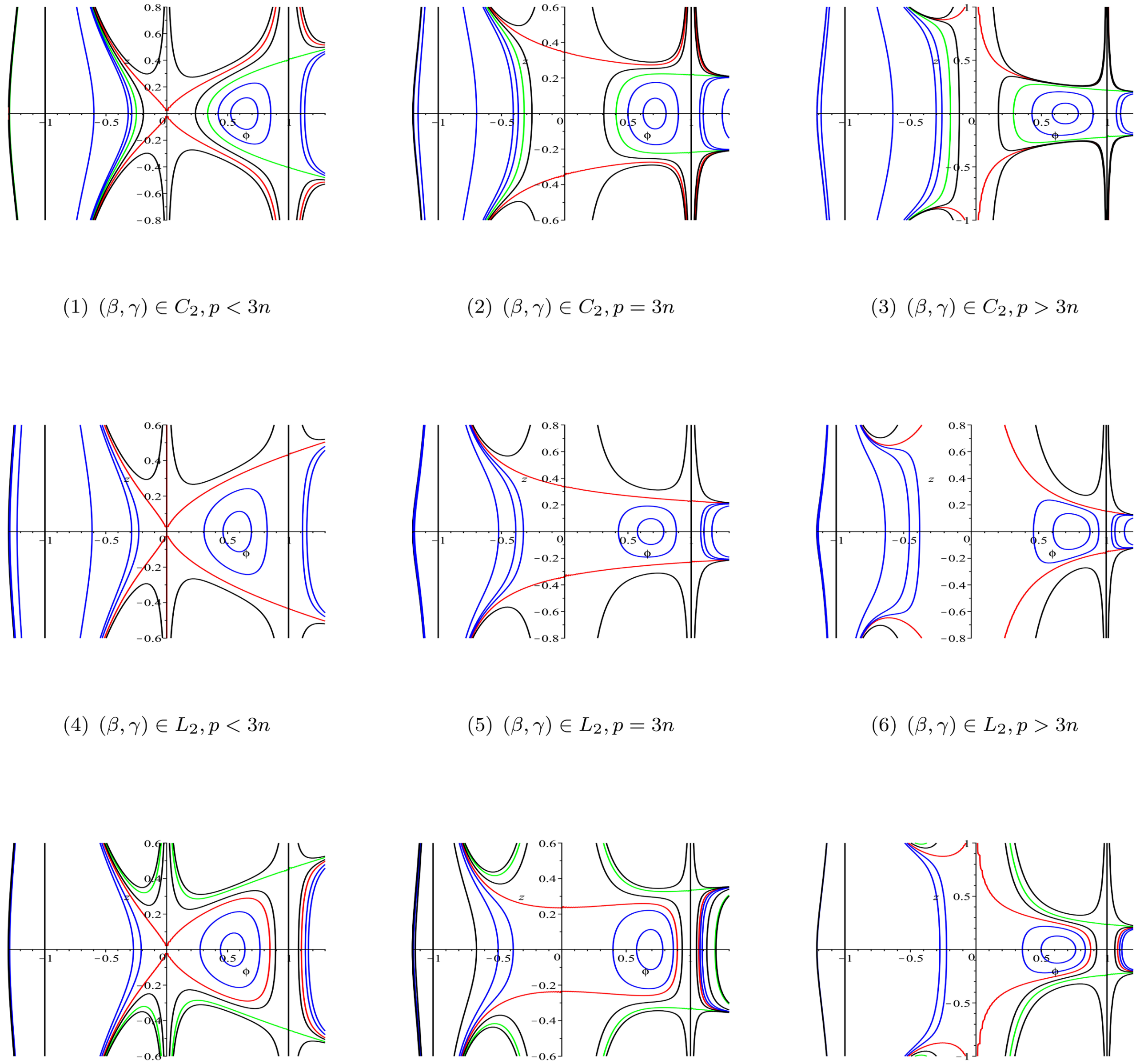
(8) $(\beta,\gamma)\in B_2, p=3n$

(9) $(\beta,\gamma)\in B_2, p>3n$

Figure 2. The phase portraits of (14) in Case II.

Case IV. $m-n=2l_1$, $k=p$.

In this case, there exists a unique singular line $\phi=0$. For a fixed $\alpha<0$, there are the same bifurcation sets as in Case I. We only consider $(\beta,\gamma)\in B_1$. For $(\beta,\gamma)\in C_1$, it can be similarly discussed. Except on the singular line, system (14) has two equilibrium points A^{+} and A^{-}. A^{+} is a center and A^{-} is a saddle when p is even; A^{+} and A^{-} both are centers when p is odd. The phase portraits of (14) are displayed in Figure 4.



(1) $(\beta, \gamma) \in C_2, p < 3n$ (2) $(\beta, \gamma) \in C_2, p = 3n$ (3) $(\beta, \gamma) \in C_2, p > 3n$

(4) $(\beta, \gamma) \in L_2, p < 3n$ (5) $(\beta, \gamma) \in L_2, p = 3n$ (6) $(\beta, \gamma) \in L_2, p > 3n$

(7) $(\beta, \gamma) \in B_2, p < 3n$ (8) $(\beta, \gamma) \in B_2, p = 3n$ (9) $(\beta, \gamma) \in B_2, p > 3n$

Figure 3. The phase portraits of (14) in Case III.

Case V. $m-n=2l_1$, $k-p=2l_2-1$.

In this case, there exist two singular lines $\phi=0$ and $\phi=\phi_s$. For a fixed $\alpha<0$, there are the same bifurcation sets as in Case II. Except on the singular lines, system (14) has two equilibrium points A^+ and A^-. A^+ is a saddle for $(\beta,\gamma)\in D_2\cup L_1$ and a center for $(\beta,\gamma)\in B_2\cup L_2\cup C_2$. A^- is a center when p is odd and a saddle when p is even. The phase portraits of (14) are shown in Figure 5 and Figure 6, respectively.

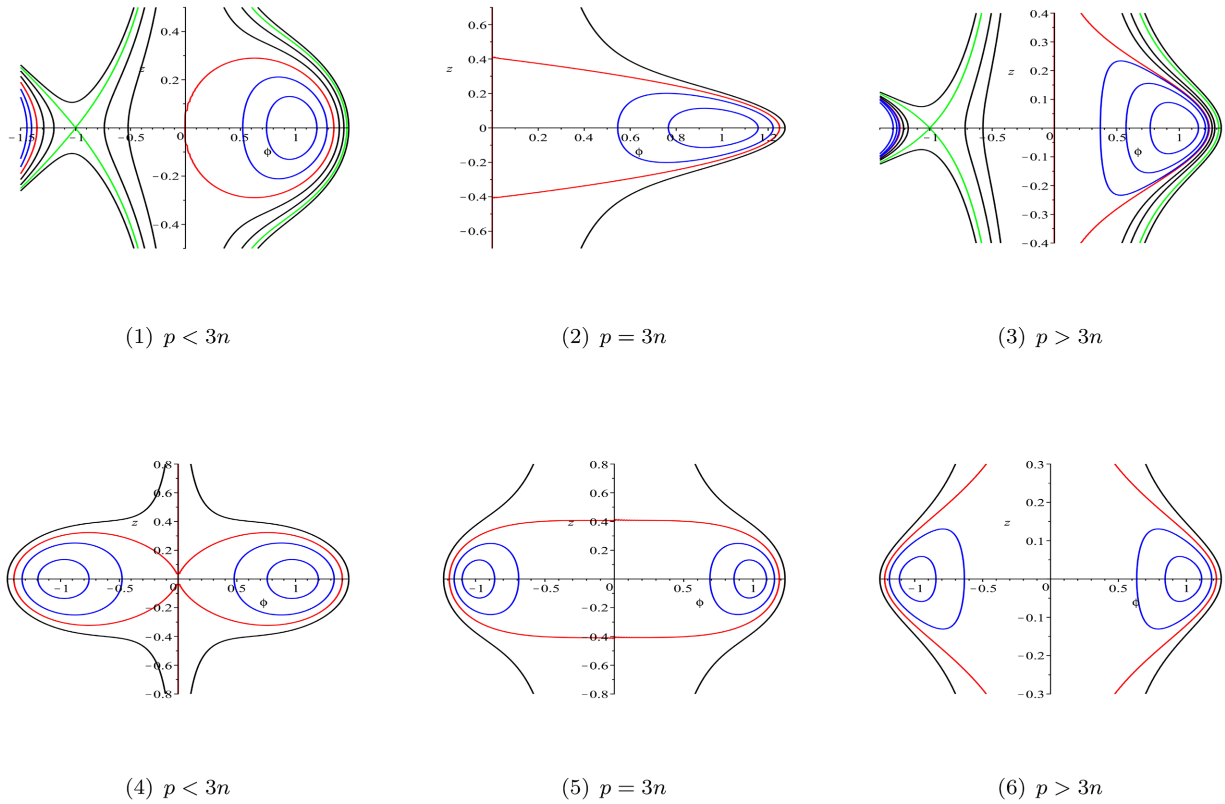


Figure 4. The phase portraits of (14) in Case IV for $(\beta, \gamma) \in B_1$: (1)-(3) p is even; (4)-(6) p is odd.

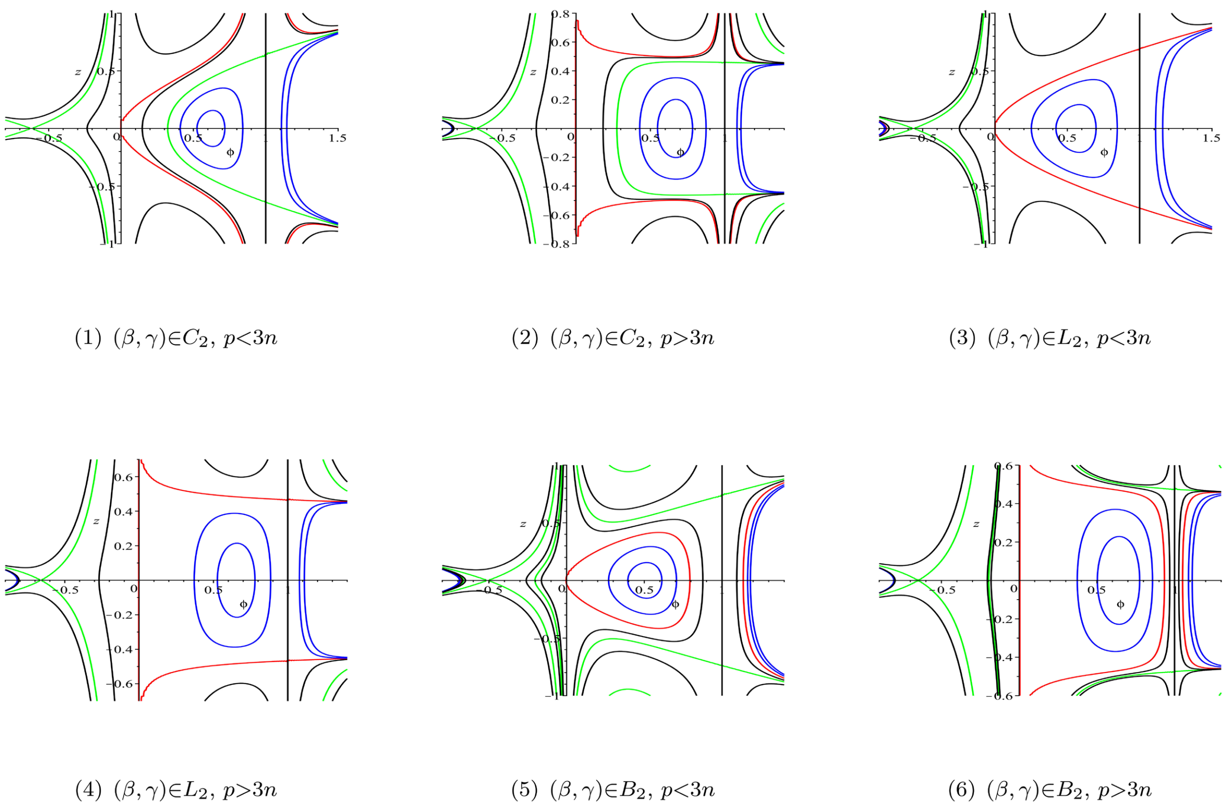
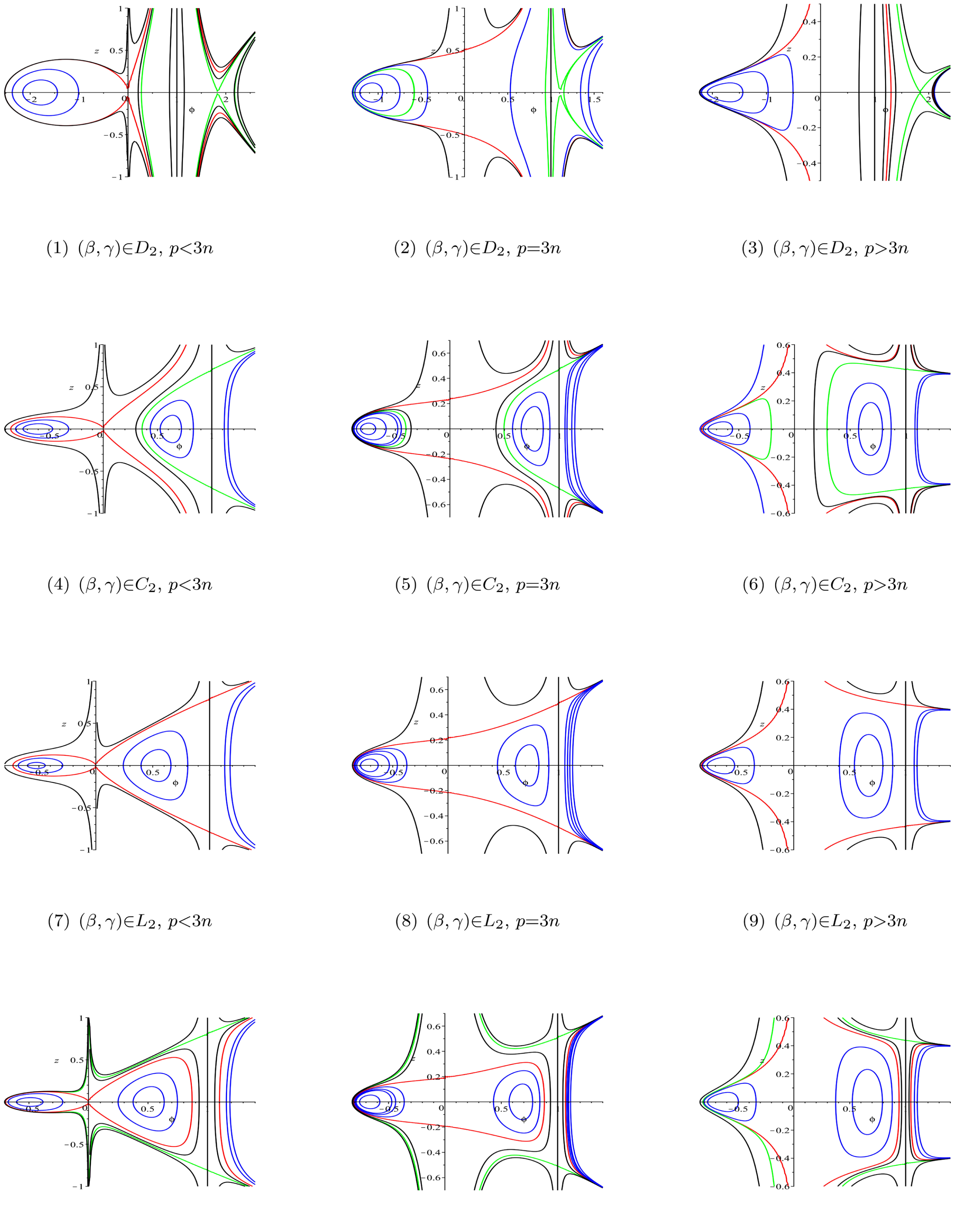


Figure 5. The phase portraits of (14) in Case V and p is even.



(1) $(\beta,\gamma)\in D_2,\ p<3n$ (2) $(\beta,\gamma)\in D_2,\ p=3n$ (3) $(\beta,\gamma)\in D_2,\ p>3n$

(4) $(\beta,\gamma)\in C_2,\ p<3n$ (5) $(\beta,\gamma)\in C_2,\ p=3n$ (6) $(\beta,\gamma)\in C_2,\ p>3n$

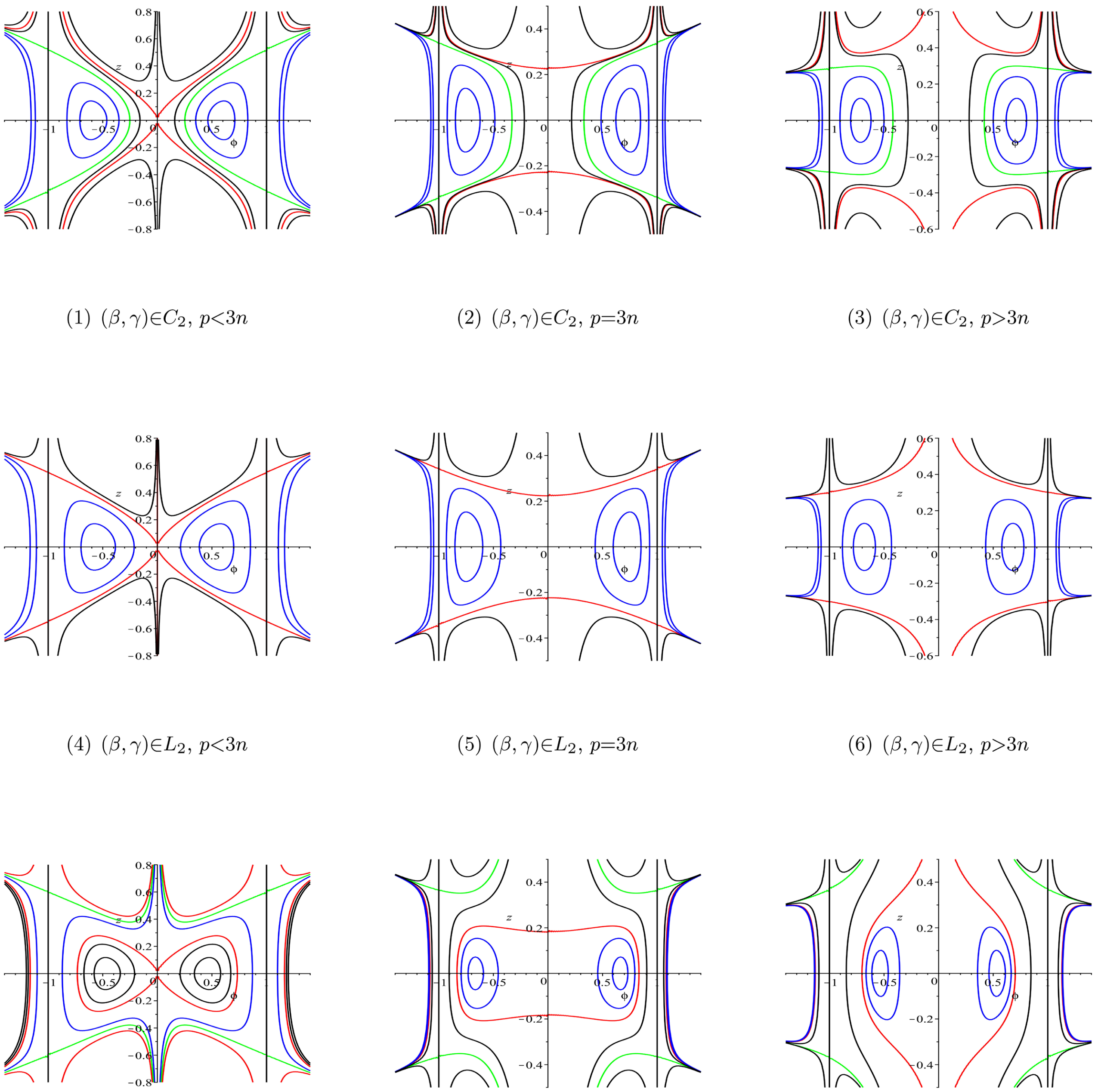
(7) $(\beta,\gamma)\in L_2,\ p<3n$ (8) $(\beta,\gamma)\in L_2,\ p=3n$ (9) $(\beta,\gamma)\in L_2,\ p>3n$

(10) $(\beta,\gamma)\in B_2,\ p<3n$ (11) $(\beta,\gamma)\in B_2,\ p=3n$ (12) $(\beta,\gamma)\in B_2,\ p>3n$

Figure 6. The phase portraits of (14) in Case V and p is odd.

Case VI. $m-n=2l_1$, $k-p=2l_2$.

In this case, there exist three singular lines $\phi=0$ and $\phi=\pm\phi_s$. For a fixed $\alpha<0$, there are the same bifurcation sets as in Case II. Except on the singular lines, (14) has two equilibrium points A^+ and A^-. When p is odd, both of them are centers for $(\beta,\gamma)\in C_2\cup L_2\cup B_2$ and saddles for $(\beta,\gamma)\in L_1\cup D_2$. One is a center and another is a saddle when p is even. The phase portraits of (14) are displayed in Figure 7 and Figure 8, respectively.

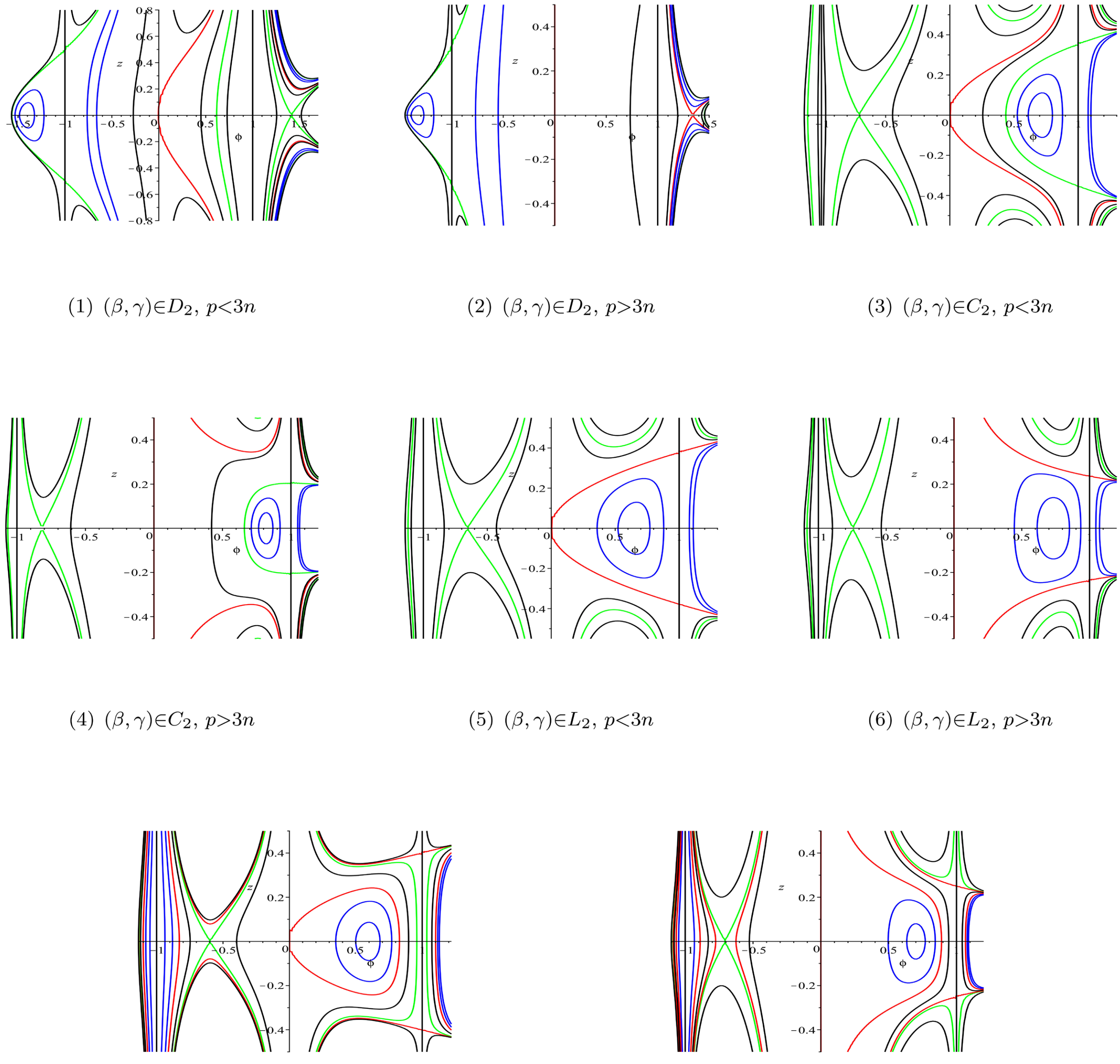


(1) $(\beta,\gamma)\in C_2$, $p<3n$ (2) $(\beta,\gamma)\in C_2$, $p=3n$ (3) $(\beta,\gamma)\in C_2$, $p>3n$

(4) $(\beta,\gamma)\in L_2$, $p<3n$ (5) $(\beta,\gamma)\in L_2$, $p=3n$ (6) $(\beta,\gamma)\in L_2$, $p>3n$

(7) $(\beta,\gamma)\in B_2$, $p<3n$ (8) $(\beta,\gamma)\in B_2$, $p=3n$ (9) $(\beta,\gamma)\in B_2$, $p>3n$

Figure 7. The phase portraits of (14) in Case V and p is odd.



(1) $(\beta,\gamma)\in D_2,\ p<3n$ (2) $(\beta,\gamma)\in D_2,\ p>3n$ (3) $(\beta,\gamma)\in C_2,\ p<3n$

(4) $(\beta,\gamma)\in C_2,\ p>3n$ (5) $(\beta,\gamma)\in L_2,\ p<3n$ (6) $(\beta,\gamma)\in L_2,\ p>3n$

(7) $(\beta,\gamma)\in B_2,\ p<3n$ (8) $(\beta,\gamma)\in B_2,\ p>3n$

Figure 8. The phase portraits of (14) in Case V and p is even.

3. Existence of Analytical Traveling Wave Solutions

We discuss the existence of analytical traveling wave solutions of (7). As mentioned above, the singular traveling wave system (12) has the same orbits as the regular system (14) except on the singular straight lines $\phi=0$ or $\phi=\pm\phi_s$. The transformation of variables $\mathrm{d}\xi = n\left[\mu k\phi^{\frac{k-n}{n}} + p\phi^{\frac{p-n}{n}}\right]\mathrm{d}\zeta$ only derives the difference between the parametric representations of the orbits of (12) and those of (14) when $\phi\neq 0$ and $\phi\neq\pm\phi_s$. Each solution of (14) is analytical because (14) is an analytical system. Thus the traveling wave solutions of (12), which correspond to those orbits far from the three singular lines in the (ϕ, z)-phase plane, are analytical.

When an orbit of (14) is close to, even crosses the singular line $\phi = 0$, does it correspond to an analytical solution of (12)? For example, for the case $(\beta, \gamma) \in B_1$ and $n < p < 3n$, there is an orbit of (14) which intersects with the singular line $\phi = 0$ at the origin O (see **Figure 1(1)** and **Figure 1(4)**, etc.). We here consider the case in **Figure 1(1)**. The same conclusions can be drawn for other similar cases by the same method. It follows from **Figure 1(1)** that Equation (14) has an analytical homoclinic orbit Γ_1^0 defined by $H(\phi, z) = 0$ and a family of analytical periodic orbits Γ_1^h defined by $H(\phi, z) = h$ for $h \in (h_1^+, 0)$. The orbit Γ_1^0 intersects the ϕ axis at $(\phi_1^0, 0)$ ($\phi_1^0 > 0$) besides the origin. An analytical homoclinic orbit of the regular system (14) is not always an analytical homoclinic orbit of the singular system (12) because of the difference between the time scales ξ and ζ. We have the following conclusion on Γ_1^0.

Lemma 3.1. Suppose that $m - n = 2l_1 - 1$, $k = p < 3n$, $(\beta, \gamma) \in B_1$. The orbit Γ_1^0 is an analytical periodic orbit of the singular system (12) although it is an analytical homoclinic orbit of the regular system (14).

Proof. Assume that $(\phi_1^0(\xi), z_1^0(\xi))$ is the solution of (12) defined by the orbit Γ_1^0 with the initial value condition $(\phi_1^0(0), z_1^0(0)) = (\phi_1^0, 0)$. From $H(\phi, z) = 0$, one obtains

$$z_1^0(\xi) = \pm\sqrt{\frac{-2n^2\omega}{p(n+p)(\mu+1)}}\,\phi^{\frac{3n-p}{2n}}\sqrt{1+\frac{(n+p)\nu}{(m+p)\omega}\phi^{\frac{m-n}{n}}}, \tag{15}$$

where $\phi = \phi_1^0(\xi) \in [0, \phi_1^0]$. Taking ξ as a time variable and letting T_1 be the time that the solution $(\phi_1^0(\xi), z_1^0(\xi))$ moves from $(\phi_1^0, 0)$ to the origin along the orbit Γ_1^0. It follows from the first Equation in (12) that $\mathrm{d}\xi = \frac{\mathrm{d}\phi}{z}$. Then one can derive from (15) that

$$T_1 = \left|\int_0^{T_1}\mathrm{d}\xi\right| = \left|\lim_{\phi\to 0^+}\int_\phi^{\phi_1^0}\frac{\mathrm{d}\phi}{z_1^0(\xi)}\right| = \int_0^{\phi_1^0}\Omega(\phi)\mathrm{d}\phi, \tag{16}$$

where

$$\Omega(\phi) = \frac{\sqrt{\frac{p(n+p)(\mu+1)}{-2n^2\omega}}}{\phi^{\frac{3n-p}{2n}}\sqrt{1+\frac{(n+p)\nu}{(m+p)\omega}\phi^{\frac{m-n}{n}}}}.$$

The integral in (16) is an improper integral since its integrand function $\Omega(\phi)$ is unbounded in the right neighborhood of $\phi = 0$. On the other hand, we have

$$\lambda = \lim_{\phi\to 0^+}\phi^{\frac{3n-p}{2n}}\Omega(\phi) = \sqrt{\frac{p(n+p)(\mu+1)}{-2n^2\omega}}.$$

The improper integral in (16) is convergent since $\lambda > 0$ and $0 < \frac{3n-p}{2n} < 1$ from $n < p < 3n$. Hence T_1 is a finite constant. That is to say, the origin O can be reached within a limited time, which means that O is a regular point rather than a singular point of (12) although it is a saddle of (14). Therefore the orbit Γ_1^0 is an analytical periodic orbit rather than a homoclinic orbit of (12). □

From Lemma lemma 3.1, an analytical homoclinic orbit of the regular system perhaps corresponds to an analytical periodic orbit of the singular system. According to the above analysis and the phase portraits obtained in Section 0, we draw the following conclusions on the existence of analytical traveling waves of (7).

Theorem 3.2 (Analytical Periodic Travelling Waves)

1) Equation (7) has at least one family of analytical periodic travelling wave solutions corresponding to $H(\phi, z) = h$, $h \in [0, h_1^+)$ or $h \in (h_1^+, 0]$, if any one of the following conditions holds.

a) $m - n = 2l_1 - 1$, $3n > k = p > n$ and $(\beta, \gamma) \in B_1$ (see Figure 1(1) and Figure 1(4));

b) $m - n = 2l_1 - 1$, $3n > p > n$, $k > p$ and $(\beta, \gamma) \in B_2$ (see Figure 2(7) and Figure 3(7));

c) $m - n = 2l_1$, $3n > k = p > n$ and $(\beta, \gamma) \in B_1$ (see Figure 4(1) and Figure 4(4));

d) $m - n = 2l_1$, $3n > p > n$, $k > p$ and $(\beta, \gamma) \in B_2$ (see Figure 5(5), Figure 6(10), Figure 7(7) and Figure 8(7)).

2) Equation (7) has at least one family of analytical periodic travelling wave solutions corresponding to $H(\phi, z) = h$, $h \in [0, h_1^-)$ or $h \in (h_1^-, 0]$, if any one of the following conditions holds.

a) $m - n = 2l_1$, $3n > k = p > n$, p is odd and $(\beta, \gamma) \in B_1$ (see Figure 4(4));

b) $m - n = 2l_1$, $k - p = 2l_2 - 1$, $3n > p > n$, p is odd and $(\beta, \gamma) \in D_2 \cup C_2 \cup L_2 \cup B_2$ (see Figure 6(1), Figure 6(4), Figure 6(7) and Figure 6(10));

c) $m - n = 2l_1$, $k - p = 2l_2$, $3n > p > n$, p is odd and $(\beta, \gamma) \in B_2$ (see Figure 7(7)).

4. Existence of Non-Analytical Travelling Wave Solutions

In Section 3, we have come to the conclusion that the analytical orbits of (14), which are far from the singular straight lines $\phi = 0$ and $\phi = \pm\phi_s$ or intersect with the singular straight lines at the origin, are still analytical orbits of (12). In this section, we discuss the dynamical behavior of the orbits that are infinitely close to the singular line or intersect the singular lines at other points except the origin.

We study these cases in which there exists uncountable infinity many periodic orbits being infinitely close to the singular lines $\phi = 0$ or $\phi = \pm\phi_s$. We confine our attention to the case in Figure 1(2) for example. The same results can be drawn for other similar cases by the same method. We see from Figure 1(2) that

Equation (14) has a family of periodic orbits on the right-half (ϕ, z)-phase plane defined by $H(\phi, z) = h$ for $h \in (h_1^+, 0)$. These orbits are infinitely close to the singular line $\phi = 0$ as $h \to 0^-$, which leads to the waveforms of periodic cusp waves. We point out the following lemma on periodic cusp waves, whose proof is similar to Theorems 3.1 and 3.2 in [34].

Lemma 4.1. The boundary curves of a periodic annulus are the limit curves of closed orbits inside the annulus. If these boundary curves contain a segment of the singular straight line $\phi = 0$ of (12), $z = \phi_\xi$ rapidly jumps in a very short time interval ξ along this segment and near this segment.

Let $(\phi(\xi), y(\xi))$ be the solution of Equation (12) corresponding to the closed orbit near the singular line $\phi = 0$. From Lemma lemma 4.1, $z(\xi) = \phi'(\xi)$ quickly changes its symbol from negative to positive near $\phi = 0$ and the waveform of $\phi(\xi)$ forms a periodic cusp wave. Based on the phase portraits in Section 0, we have the following conclusions.

Theorem 4.2 (Non-analytical Periodic Cusp Waves)

1) Corresponding to $H(\phi, z) = h$, $h \in [0, h_1^+)$ (or $h \in (h_1^+, 0]$) defined by (13), Equation (7) has at least one family of non-analytical periodic waves; when h varies from h_1^+ to 0, these periodic waves will gradually lose their analyticity, and evolve from analytical periodic waves to non-analytical periodic cusp waves and finally approach to a periodic cusp wave, if any one of the following conditions holds.

a) $m > n$, $k = p = 3n$, $(\beta, \gamma) \in B_1$ (see **Figure 1(2)**, **Figure 1(5)**, **Figure 4(2)**, **Figure 4(5)**);

b) $m > n$, $k > p > n$, $p \le 3n$, $(\beta, \gamma) \in L_2$ (see **Figure 2(4)**, **Figure 2(5)**, **Figure 3(4)**, **Figure 3(5)**, **Figure 5(3)**, **Figure 6(7)**, **Figure 6(8)**, **Figure 7(4)**, **Figure 7(5)**);

c) $m > n$, $k > p = 3n$, $(\beta, \gamma) \in B_2$ (see **Figure 2(8)**, **Figure 3(8)**, **Figure 6(11)**, **Figure 7(8)**).

2) Corresponding to $H(\phi, z) = h$, $h \in (h_1^+, h_2^+]$ or $h \in [h_2^+, h_1^+)$, Equation (7) has at least one family of non-analytical periodic waves; when h varies from h_1^+ to h_2^+, these periodic waves will gradually lose their analyticity, and evolve from analytical periodic waves to non-analytical periodic cusp waves and finally approach to a periodic cusp wave, if $m > n$, $k > p > n$ and $(\beta, \gamma) \in C_2$ (see **Figures 2(1)-(3)**, **Figures 3(1)-(3)**, **Figure 5(1)**, **Figure 5(2)**, **Figures 6(4)-(6)**, **Figures 7(1)-(3)**, **Figure 8(3)**, **Figure 8(4)**);

Remark 4.3. Corresponding to $H(\phi, z) = h$, $h \in (h_1^-, 0]$ (or $[0, h_1^-)$ or $(h_1^-, h_2^-]$ or $[h_2^-, h_1^-)$), we also have the similar results as the Theorem 0.4. We omit these statements to save the space.

We consider the last cases in which the regular system (14) has uncountable infinity many unbounded open orbits Γ^h which are infinitely close to the singular lines $\phi = 0$ or $\phi = \pm\phi_s$, with $|z| = \left|\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right| \to +\infty$ as $\phi \to 0$ (or $\pm\phi_s$) for $(\phi, z) \in \Gamma^h$. The case in **Figure 1(1)** is taken as an example. Similar conclusions can be derived for those similar cases by using the similar method. We see from Figure 1(1) that Equation (14) has uncountable infinity many open orbits Γ^h defined by $H(\phi,z)=h$ for $h\in(0,+\infty)$. These open orbits lie in the right-half (ϕ,z)-phase plane and intersect the ϕ-axis at $(\phi^h,0)$. Assume that $(\phi^h(\xi),z^h(\xi))$ is the special solution of (12) under the initial condition $(\phi^h(0),z^h(0))=(\phi^h,0)$. It follows from (13) that

$$z^h(\xi)=\pm\left[\mu k\left(\phi^h(\xi)\right)^{\frac{k-n}{n}}+p\left(\phi^h(\xi)\right)^2\right]^{-1}\sqrt{g\left(\phi^h(\xi)\right)}, \tag{17}$$

where

$$g(\phi)=h-2n^2\left(\frac{\mu\nu k\phi^{\frac{m+k}{n}}}{m+k}+\frac{\mu\omega k\phi^{\frac{n+k}{n}}}{n+k}+\frac{p\nu\phi^{\frac{m+p}{n}}}{m+p}+\frac{p\omega\phi^{\frac{n+p}{n}}}{n+p}\right).$$

Let T^h be the time that the phase point $(\phi^h(\xi),z^h(\xi))$ spend moving from the initial position $(\phi^h,0)$ to the singular line $\phi=0$ along the open orbit Γ^h. From (17) and the first Equation in (12), we have

$$T^h=\left|\lim_{\phi^h(\xi)\to0^+}\int_{\phi^h(0)}^{\phi^h(\xi)}\frac{\mathrm{d}\phi^h(\xi)}{z^h(\xi)}\right|=\int_0^{\phi^h}\frac{\mu k\phi^{\frac{k-n}{n}}+p\phi^2}{\sqrt{g(\phi)}}\mathrm{d}\phi. \tag{18}$$

The improper integral in (18) is convergent since $g(\phi)\in C\left[0,\phi^h\right)$ and the singular point $\phi=\phi^h$ is a simple real root of $g(\phi)$. Thus T^h is a finite positive number for every fixed $h\in(0,+\infty)$ and the solution $\phi^h(\xi)$ only exists in the finite time interval $\left[-T^h,T^h\right]$ with $\phi^h(\pm T^h)=0$ but the derivative $\frac{\mathrm{d}\phi^h}{\mathrm{d}\xi}(\pm T^h)=z^h(\pm T^h)=\pm\infty$. Thus the open orbits Γ^h of the regular system (14) correspond to the breaking wave solutions (*i.e.*, the so-called blow-up solutions) of the singular traveling wave system (12).

Theorem 4.4 (Non-analytical Breaking Waves) Corresponding to $H(\phi,z)=h$, $h\in(0,+\infty)$ or $h\in(0,h_1^-)$ or $h\in(0,h_2^+)$ or $h\in(0,h_3^+)$, Equation (7) has at least one family of breaking waves, if any one of the following conditions holds.

1) $m>n$, $k=p$ and $(\beta,\gamma)\in B_1$ (see Figure 1 and Figure 4);

2) $m-n=2l_1-1$, $k-p=2l_2-1$ and $(\beta,\gamma)\in B_2\cup C_2$ (see Figures 2(1)-(3) and Figures 2(7)-(9));

3) $m-n=2l_1-1$, $k-p=2l_2$ and $(\beta,\gamma)\in B_2\cup L_2\cup C_2$ (see Figure 3);

4) $m-n=2l_1$, $k-p=2l_2-1$ and $(\beta,\gamma)\in B_2\cup L_2\cup C_2$ (see Figure 5 and Figure 6);

5) $m-n=2l_1$, $k-p=2l_2$, p is odd and $(\beta,\gamma)\in B_2$ (see Figures 7(7)-(9));

6) $m-n=2l_1$, $k-p=2l_2$, p is even and $(\beta,\gamma)\in B_2\cup L_2\cup C_2\cup D_2$ (see Figure 8).

5. Explicit Exact Analytical and Non-Analytical Traveling Wave Solutions

In this section, we construct some explicit exact parametric representations of analytical and non-analytical traveling wave solutions of (7) for certain special conditions.

1) $m-n=2l_1-1$, $k=p>n$, $(\beta,\gamma)\in B_1$.

a) When $m=k=p=2n$, and n is odd, for $h=0$, from the first integral (13), one obtains

$$z=\pm\sqrt{\frac{\nu}{4(\mu+1)}}\sqrt{\phi\left(-\frac{4\omega}{3\nu}-\phi\right)}. \tag{19}$$

Substituting (19) into the first Equation in (12) leads to

$$\int_0^{\phi}\pm\frac{\mathrm{d}\phi}{\sqrt{\phi\left(-\frac{4\omega}{3\nu}-\phi\right)}}=\pm\sqrt{\frac{\nu}{4(\mu+1)}}\int_0^{\xi}\mathrm{d}\xi, \tag{20}$$

which gives an analytical periodic solution of (12)

$$\phi_1(\xi)=-\frac{4\omega}{3\nu}\sin^2\left(\sqrt{\frac{\nu}{16(\mu+1)}}\xi\right). \tag{21}$$

So $B(2n,n,2n,2n)$ has an analytical periodic wave solution (see Figure 1(1))

$$u_1(x,y,t)=\left(\phi_1(\xi)\right)^{\frac{1}{n}}=\left(-\frac{4\omega}{3\nu}\sin^2\left(\sqrt{\frac{\nu}{16(\mu+1)}}\xi\right)\right)^{\frac{1}{n}}. \tag{22}$$

b) When $m=2n$, $k=p=3n$, and n is odd, the regular system (14) has a two-point heteroclinic cycle defined by $H(\phi,z)=0$ in (13), which corresponds to a non-analytical periodic cusp wave solution of (7) (see Figure 1(5))

$$u_2(x,y,t)=\left(-\frac{5\omega}{4\nu}-\frac{\nu\xi^2}{30(\mu+1)}\right)^{\frac{1}{n}},\ \xi\in\left[-\frac{T_1}{2},\frac{T_1}{2}\right] \tag{23}$$

with the period $T_1=\frac{5}{\nu}\sqrt{-6\omega(\mu+1)}$.

c) When $m=4n$, $k=p=3n$, and n is odd, for $h=0$, $B(4n,n,3n,3n)$ has a non-analytical periodic cusp wave solution (see Figure 1(5))

$$u_3(x,y,t)=\left(1-\frac{\sqrt{3}\left(1-\mathrm{cn}(\sigma_1\xi,k_1)\right)}{1+\mathrm{cn}(\sigma_1\xi,k_1)}\right)^{\frac{1}{n}}\rho_1^{\frac{1}{n}},\ \xi\in\left[-\frac{T_2}{2},\frac{T_2}{2}\right] \tag{24}$$

with the period $T_2=\frac{2}{\sigma_1}\mathrm{cn}^{-1}\left(2-\sqrt{3},k_1\right)$, where $\sigma_1=\sqrt{\frac{2\sqrt{3}\rho_1\nu}{21(\mu+1)}}$, $\rho_1=\sqrt[3]{-\frac{7\omega}{4\nu}}$, $k_1=\frac{1}{\sqrt{6}-\sqrt{2}}$.

2) $m-n=2l_1-1$, $k-p=2l_2-1$, $(\beta,\gamma)\in L_2$.

a) When $n\in Z^+$, for $h=0$, $B(6n+3,4n+2,10n+5,8n+4)$ has a non-analytical periodic cusp wave solution (see Figure 2(4))

$$u_4(x,y,t)=\left(-\frac{\omega}{12}\xi^2\right)^{\frac{1}{4n+2}},\ \xi\in\left[-\frac{T_3}{2},\frac{T_3}{2}\right] \tag{25}$$

with the period $T_3=\frac{16}{3\nu}\sqrt{-3\omega}$.

b) When n is odd, for $h=0$, $B(2n,n,4n,3n)$ has a sawtooth-shaped periodic cusp wave solution (see **Figure 2(5)**)

$$u_5(x,y,t)=\left(\sqrt{-\frac{\omega}{6}}|\xi|\right)^{\frac{1}{n}},\ \xi\in\left[-\frac{T_4}{2},\frac{T_4}{2}\right] \tag{26}$$

with the period $T_4=\frac{3}{\nu}\sqrt{-6\omega}$.

c) When n is odd, for $h=0$, $B(2n,n,3n,2n)$ has a periodic cusp wave solution (see **Figure 2(4)**):

$$u_6(x,y,t)=\left(-\frac{\omega}{12}\xi^2\right)^{\frac{1}{n}},\ \xi\in\left[-\frac{T_5}{2},\frac{T_5}{2}\right] \tag{27}$$

with the period $T_5=\frac{4}{\nu}\sqrt{5\nu}$.

3) $m-n=2l_1$, $k=p>n$, $(\beta,\gamma)\in B_1$.

a) When $n\in Z^+$, for $h=0$, $B(3n,n,2n,2n)$ has an analytical periodic wave solution (see **Figure 4(1)**)

$$u_7(x,y,t)=\left(\frac{\rho_2\,\mathrm{sn}^2\left(\sigma_2\xi,\frac{1}{\sqrt{2}}\right)}{2-\mathrm{sn}^2\left(\sigma_2\xi,\frac{1}{\sqrt{2}}\right)}\right)^{\frac{1}{n}}, \tag{28}$$

where $\rho_2=\sqrt{-\frac{5\omega}{3\nu}}$, $\sigma_2=\sqrt{\frac{\rho_2\nu}{10(\mu+1)}}$.

b) When n is even, for $h=0$, $B(4n,n,3n,3n)$ has a non-analytical periodic wave solution with the same representation as (24) (see **Figure 4(2)**).

c) When n is even, for $h=0$, $B(2n,n,3n,3n)$ has a non-analytically periodic cusp wave solution with the same representation as (23) (see **Figure 4(2)**).

d) When n is odd, for $h=0$, $B(3n,n,3n,3n)$ has two non-analytical periodic cusp wave solutions (see **Figure 4(5)**)

$$u_8^{\pm}(x,y,t)=\pm\left(\sqrt{-\frac{3\omega}{2\nu}}\cos\left(\sqrt{\frac{\nu}{9(\mu+1)}}\xi\right)\right)^{\frac{1}{n}},\ \xi\in\left[-\frac{T_6}{2},\frac{T_6}{2}\right], \tag{29}$$

with the same period $T_6=3\pi\sqrt{\frac{\mu+1}{\nu}}$.

e) When n is odd, for $h=0$, $B(5n,n,3n,3n)$ has two periodic cusp wave solutions (see **Figure 4(5)**):

$$u_9^{\pm}(x,y,t)=\pm\left(\rho_3\,\mathrm{cn}\left(\rho_3\sigma_3\xi,\frac{1}{\sqrt{2}}\right)\right)^{\frac{1}{n}},\ \xi\in\left[-\frac{T_7}{2},\frac{T_7}{2}\right], \tag{30}$$

with the same period $T_7 = \frac{2}{\rho_3\sigma_3} K\left(\frac{1}{\sqrt{2}}\right)$, where $\rho_3 = \sqrt[4]{\frac{2\omega}{-\nu}}$, $\sigma_3 = \sqrt{\frac{\nu}{6(\mu+1)}}$.

4) $m-n=2l_1$, $k-p=2l_2-1$, $(\beta,\gamma)\in L_2$.

a) When n is odd, for $h=0$, $B(3n,n,3n,2n)$ has a non-analytical periodic cusp wave solution (see Figure 5(3)):

$$u_{10}(x,y,t)=\left[\sqrt{-\frac{2\omega}{5\nu}}\left(\cosh\left(\sqrt[4]{\frac{5\omega\nu}{-72}}\xi\right)-1\right)\right]^{\frac{1}{n}},\ \xi\in\left[-\frac{T_8}{2},\frac{T_8}{2}\right], \tag{31}$$

with the period $T_8 = 2\sqrt[4]{\frac{-72}{5\omega\nu}}\cosh^{-1}\left(\frac{7}{2}\right)$.

b) When $n\in Z^+$, for $h=0$, $B(8n+4,4n+2,10n+5,8n+4)$ has a non-analytical periodic cusp wave solution (see Figure 5(3))

$$u_{11}(x,y,t)=\left[\frac{\sqrt{-21\omega\nu}}{252}\xi^2+\frac{\sqrt{-3\omega}}{6}|\xi|\right]^{\frac{1}{2n+1}},\ \xi\in\left[-\frac{T_9}{2},\frac{T_9}{2}\right], \tag{32}$$

with the period $T_9 = \frac{6\left(\sqrt{15}-\sqrt{7}\right)}{\sqrt{\nu}}$.

c) When n is odd, for $h=0$, $B(3n,n,4n,3n)$ has two periodic cusp wave solutions (see Figure 6(8))

$$u_{12}(x,y,t)=\left[\frac{\sqrt{-210\omega\nu}}{420}\xi^2+\frac{\sqrt{-6\omega}}{6}|\xi|\right]^{\frac{1}{n}},\ \xi\in\left[-\frac{T_{10}}{2},\frac{T_{10}}{2}\right], \tag{33}$$

with the period $T_{10} = \frac{2\left(\sqrt{77}-\sqrt{35}\right)}{\sqrt{\nu}}$ and

$$u_{13}(x,y,t)=\left[\frac{\sqrt{-210\omega\nu}}{420}\xi^2-\frac{\sqrt{-6\omega}}{6}|\xi|\right]^{\frac{1}{n}},\ \xi\in\left[-\frac{T_{11}}{2},\frac{T_{11}}{2}\right], \tag{34}$$

with the period $T_{11} = 2\sqrt{\frac{35}{\nu}}$.

5) $m-n=2l_1$, $k-p=2l_2$, $(\beta,\gamma)\in L_2$.

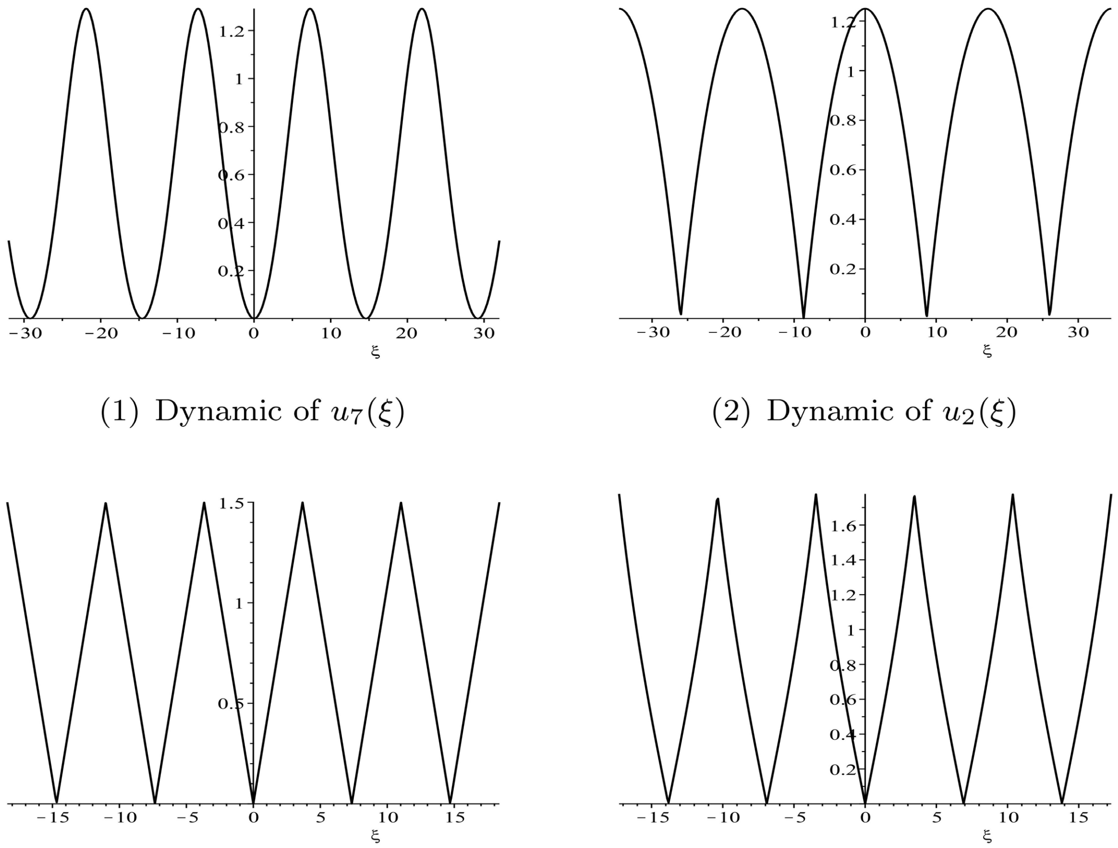
a) When n is odd, for $h=0$, $B(5n,n,5n,3n)$ has two sawtooth-shaped periodic cusp wave solutions (see Figure 7(5))

$$u_{14}^{\pm}(x,y,t)=\pm\left[\sqrt[4]{\frac{15\omega}{-8\nu}}\sinh\left(\sqrt[4]{-\frac{2\omega\nu}{135}}|\xi|\right)\right]^{\frac{1}{n}},\ \xi\in\left[-\frac{T_{12}}{2},\frac{T_{12}}{2}\right], \tag{35}$$

with the same period $T_{12} = 2\sqrt[4]{\frac{-135}{2\omega\nu}}\sinh^{-1}\left(\frac{2}{\sqrt[4]{3}}\right)$.

6. Simulations and Discussions

In this part of the manuscript, we present the simulations of some typical new solutions of Equation (7) which can help us better understand the previous conclusions in Sections 8 and 8 (see Figure 9).



(1) Dynamic of $u_7(\xi)$ (2) Dynamic of $u_2(\xi)$

(3) Dynamic of $u_5(\xi)$ (4) Dynamic of $u_{14}^{+}(\xi)$

Figure 9. Dynamics of the exact solutions for $\mu=1$, $\nu=1$, $\omega=-1$, $n=1$.

Sub-**Figure 9(1)** shows the behavior of the solution $u_7(x,y,t)$ with parameters $\mu=1$, $\nu=1$, $\omega=-1$, $n=1$. Here we observe the analytical periodic wave solution. The orbit corresponding to this periodic solution is a periodic orbit of the singular system (12) but is a homoclinic orbit of the regular system (14). A homoclinic orbit of a regular system usually gives rise to an analytical solitary wave solution. A solitary wave and a periodic wave have completely different dynamic properties. It is the singularity that causes an analytical homoclinic orbit of the regular system to become an analytical periodic orbit of the corresponding singular system. The wave plot of **Figure 9(1)** verifies the correctness of Theorem theorem 3.2.

Sub-**Figure 9(2)** shows the behavior of the solution $u_2(x,y,t)$ with parameters $\mu=1$, $\nu=1$, $\omega=-1$, $n=1$, which presents the non-analytical periodic cusp wave solution. The orbit corresponding to this periodic cusp wave is a two-point heteroclinic cycle of the regular system (14) but gives rise to a non-analytical periodic cusp wave solution for the singular system (12).

Sub-**Figure 9(3)** and **Figure 9(4)** show the behaviors of solutions $u_5(x,y,t)$ and $u_{14}^{+}(x,y,t)$ respectively with parameters $\mu=1$, $\nu=1$, $\omega=-1$, $n=1$, which present two non-analytical sawtooth-shaped periodic cusp wave solutions.

The orbits corresponding to the two periodic cusp wave are two four-point heteroclinic cycles of the regular system (14). Each of the two heteroclinic cycles sequentially connects to four equilibrium points: S_0^+, S_0^-, S_1^- and S_1^+ but gives rise to a saw-shaped periodic cusp wave solution for the singular system (12).

7. Conclusions

In this paper, the (2 + 1)-dimensional nonlinear dispersive Boussinesq equation has been investigated. It is very difficult to directly study the dynamic properties of the corresponding traveling wave equation since it is a singular system with three possible singularities. So a scale transformation is introduced to remove the singularities and transform the singular system into a regular system. Then the bifurcations and phase portraits of the regular system are discussed by using the bifurcation theory of dynamical system. By using the phase portraits of the regular system and utilizing the singular traveling wave theory to analyze the impact of singularity on the analyticity of the solutions of the original equation (7), some sufficient conditions for the existence of analytical and non-analytical solutions are obtained. Many explicit and exact traveling wave solutions are given which can verify the correctness of the conclusions in Sections 8 and 8.

The method used in this paper is very effective and can also be used to study other nonlinear equations with singularities.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix. Main Variables Used in This Paper

Some main variables used in this paper are listed in Table A1.

Table A1. Main variables used in this paper.

$\mu = \frac{\beta\omega_1^4}{\gamma\omega_2^4}$	$\nu = \frac{\omega_3^2}{\gamma\omega_2^4}$	$\omega = \frac{\alpha\omega_1^2}{\gamma\omega_2^4}$
$\phi_s = \left(-\frac{p}{\mu k}\right)^{\frac{n}{k-p}}$	$\phi_1 = \left(-\frac{\omega}{\nu}\right)^{\frac{n}{m-n}}$	$h_0^{\pm} = H\left(0, \pm\sqrt{-\frac{\omega}{6}}\right)$
$h_1^{\pm} = H\left(\pm\phi_1, 0\right)$	$h_2^{\pm} = H\left(\pm\phi_s, \pm\sqrt{Y}\right)$	$h_3^{\pm} = H\left(-\phi_s, \pm\sqrt{Z}\right)$