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A Generalization of Torsion Graph for Modules

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Abstract

Let R be a commutative ring with identity and M an R-module. In this paper, we relate a graph to M, say $\Gamma(M)$, provided tsshat when $M = R$, $\Gamma(M)$ is exactly the classic zero-divisor graph.

Keywords

Commutative Ring, Graph, Anihilator

1. Introduction

The study of the relationship between graphs and R-modules is motivated by several factors. Firstly, graphs are powerful tools for representing various algebraic structures and their properties. By relating graphs to R-modules, we can leverage the visual and combinatorial nature of graphs to gain insights into the properties and behaviors of R-modules.

Secondly, this relationship opens up new avenues for applying graph-theoretic methods to problems in module theory and vice versa. For instance, graph invariants can provide new perspectives on module invariants, and techniques from module theory can be used to solve problems in graph theory. This interdisciplinary approach can lead to the discovery of novel results and the development of new methods in both fields.

Moreover, understanding the interplay between graphs and R-modules has potential applications in areas such as coding theory, where modules over rings are used to construct error-correcting codes, and in the study of networks, where graphs represent connections between different entities. Theoretical insights gained from this relationship can also contribute to advancements in homological algebra, representation theory, and other areas of mathematics.

In summary, the relationship between graphs and R-modules is important because it provides a rich framework for exploring and solving problems in both graph theory and module theory, with potential applications in various mathematical and practical domains.

In this section, we provide basic notions from graph theory and module theory needed throughout this paper. An undirected graph G is said to be connected if there is a path between any two distinct vertices. Let x and y be distinct vertices in G, the distance between x and y, denoted by $d(x,y)$, is defined as the length of a shortest path connecting x and y ($d(x,x)=0$ and $d(x,y)=\infty$ if no such path exists). The diameter of G is
$\operatorname{diam}(G)=\sup\{d(x,y) \mid x \text{ and } y \text{ are vertices of } G\}$.

A cycle of length n in G is a path of the form $x_1 - x_2 - x_3 - \cdots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. The girth of G, denoted by $\operatorname{gr}(G)$, is defined as the length of the shortest cycle in G, provided G contains a cycle; and $\operatorname{gr}(G)=\infty$ when no such cycle exists. A graph is said to be complete if any two distinct vertices are adjacent. A complete graph with n vertices is denoted by K_n.

Let M be an R-module and $S \subseteq M$. The annihilator of S is the set $\{r \in R \mid rS = 0\}$ and it is denoted by $\operatorname{Ann}(S)$. An element $x \in M$ is called a torsion element if $\operatorname{Ann}(x) \neq 0$. In case that $\operatorname{Ann}(x)=0$, then x is called torsion-free. An R-module M with $\operatorname{Ann}(M)=0$ is known as a faithful R-module.

In this paper, R denotes a commutative ring with 1, M a unitary R-module, and $t(M)$ be the set of torsion elements of M. It is worth noting that total graphs of commutative rings were introduced and investigated in [1]. Those graphs consider all elements of a commutative ring R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$ [1]. For a recent book on graphs from rings, see [2].

The classic zero-divisor graph has been extended to modules over commutative rings [3]. Two elements $m, n \in M$ are adjacent if and only if $(mR:M)(nR:M)M = 0$ [3]. This extension is considered as a straightforward generalization of the classic zero-divisor graph. In [4], the authors have linked two different graphs to an R-module M in regard to its first dual, $M^* = \operatorname{Hom}(M,R)$. Although these two graphs are not naturally generalizations of the classic zero-divisor graph, there are some rooted interconnections between them and the classic one. In this paper, we introduce a further new generalization of the classic zero-divisor graph, which is more natural and elegant than the preceding generalizations. In general, proofs of results are clearer than those proofs given for the corresponding results on the classic zero-divisor graph. Let $I(R)$ denote the set of all ideals in R. We introduce a mapping $*: M \times M \to I(R)$ that will be utilized for the generalization of the classic zero-divisor graph.

Definition 1.1. Let M be an R-module. For every two non-zero elements $x, y \in M$, we say that $x * y = y * x = \operatorname{Ann}(\operatorname{Ann}(x))\operatorname{Ann}(\operatorname{Ann}(y))$.

For an R-module M, let $t(M) = \{x \in M \mid xr = 0 \text{ for some non-zero } r \in R\}$ and let $t(M)^* = t(M) \setminus \{0\}$. Now we are ready to give our generalization of the classic zero-divisor graph. We link an undirected graph $\Gamma(M)$ to M with the set of vertices $t(M)^*$ such that for distinct elements $x, y \in t(M)^*$, the vertices x and y are adjacent provided that $x * y = 0$.

As we notice in the aftermath, the graph $\Gamma(M)$ is a perfect generalization of the classic zero-divisor graph.

2. Primary and Main Results on Γ(*M*)

A simple graph $G(V,E)$ is a pair of two sets: one is a finite nonempty set $V(G)$ of objects called vertices, and the other is a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called edges. A graph G is said to be connected if there exists a path between any pair of distinct vertices of G. The distance between x and y, denoted by $d(x,y)$, is defined as the length of the shortest path connecting x and y ($d(x,x)=0$ and $d(x,y)=\infty$ if no such path exists). The diameter of G, denoted by $\operatorname{diam}(G)$, is $\sup\{d(v,u) \mid v,u \in V(G)\}$. A graph G is called complete if there is an edge between each two distinct vertices. A complete graph with n vertices is denoted by K_n.

First of all, we will show that our definition of $\Gamma(M)$ is a generalization of the definition of $\Gamma(R)$ in [5].

Theorem 2.1. *Let R be a ring and* $x, y \in R$. *Then* $x * y = 0$ *if and only if* $xy = 0$.

Proof. For the forward direction, assume $\operatorname{Ann}(\operatorname{Ann}(x))\operatorname{Ann}(\operatorname{Ann}(y)) = 0$. It is known that $x \in \operatorname{Ann}(\operatorname{Ann}(x))$ and $y \in \operatorname{Ann}(\operatorname{Ann}(y))$, so we have $xy = 0$.

Conversely, assume $xy = 0$, but $\operatorname{Ann}(\operatorname{Ann}(x))\operatorname{Ann}(\operatorname{Ann}(y)) \neq 0$. Then we have $a \in \operatorname{Ann}(\operatorname{Ann}(x))$ and $b \in \operatorname{Ann}(\operatorname{Ann}(y))$ with $ab \neq 0$. Now $x \in \operatorname{Ann}(y)$ and $y \in \operatorname{Ann}(x)$, and we get $a\operatorname{Ann}(x) = b\operatorname{Ann}(y) = 0$. This implies that $ay = bx = 0$. However, $ab \neq 0$ assures that $b \notin \operatorname{Ann}(x)$ and so $bx \neq 0$, which is a contradiction.

Remark 1. For a ring R, if we take R as an R-module, then our graph is the same as the graph in [5] by Theorem 2.1.

Here the torsion graph $\Gamma(M)$ is a simple graph whose vertices are the non-zero torsion elements of M, and two distinct elements x, y are adjacent if and only if $x * y = 0$.

It is worth mentioning that Definition 1.1 is distinct from the notion of the torsion graph in [6].

Indeed, the torsion graph $G(M)$ is a simple graph whose vertices are the non-zero torsion elements of M, and two distinct elements x, y are adjacent if and only if $[x:M][y:M]M = 0$.

$(x,y) \in \Gamma(M)$ means x, y are adjacent in $\Gamma(M)$, and $(x,y) \in G(M)$ means x, y are adjacent in $G(M)$.

We illustrate the above observation with the next examples.

Example 1. Take $R = \mathbb{Z}$ and $M = \mathbb{Z}_4$. Now $(1,2) \in \Gamma(\mathbb{Z}_4)$. But $[1,\mathbb{Z}_4] = \mathbb{Z}$ and $[2,\mathbb{Z}_4] = 2\mathbb{Z}$, so $[1,\mathbb{Z}_4][2,\mathbb{Z}_4]\mathbb{Z}_4 = 2\mathbb{Z}_4 \neq 0$ which means that $(1,2) \notin G(M)$.

Example 2. Let K be a field. Take $R=\frac{K[X,Y]}{(X^3,Y^3)}$ and $M=\frac{K[X,Y]}{(X^3,Y^3,XY)}$. Now let $x=\bar{X}$ and $y=\bar{Y}$. Now we have $\mathrm{Ann}(x)=\langle X^2,Y\rangle$, then $\mathrm{Ann}(\mathrm{Ann}(x))=\langle X\rangle$. Also, $\mathrm{Ann}(y)=\langle Y^2,X\rangle$, then $\mathrm{Ann}(\mathrm{Ann}(y))=\langle Y\rangle$. This implies that $x*y\neq 0$. Also, $(x,y)\notin\Gamma(M)$. On the other hand, we have $[x,M]=\langle X\rangle$ and $[y,M]=M$, hence $[x,M][y,M]M=0$ which gives that $(x,y)\in G(M)$.

Theorem 2.2. *Let R be a ring and M be an R-module. If* $x\in t(M)^*$ *and* $\mathrm{Ann}(x)$ *is faithful, then* $x\sim y$ *for all* $y\in t(M)^*$.

Proof. Let $0\neq x\in t(M)^*$, then $\mathrm{Ann}(x)\neq 0$ and $\mathrm{Ann}(\mathrm{Ann}(x))=0$, so that for any $y\in t(M)^*$ we have $x\sim y$.

Remark 2. Let R be an integral domain and M be an R-module. If $x\in t(M)^*$, then $x:y$ for all $y\in t(M)^*$.

We recall Ganesan's Theorem: The ring $Z(R)$ is finite if and only if either R is finite or an integral domain. However, this does not hold for modules. That is, it is not the case that if M is finite, then either M is finite or torsion-free. The next example confirms this.

Example 3. Take $R=\mathbb{Z}$ and $M=\mathbb{Z}\oplus\mathbb{Z}_2$. This gives $t(M)=\{(0,1),(0,0)\}$ and M is neither finite nor torsion-free.

Lemma 2.3. *Let M be an R-module. If* $x*y=0$, *then* $x*ry=0$ *for all* $r\in R$.

Proof. Let $x\in M$ and $(x,y)\in\Gamma(M)$. We get that $x*y=0$. For any $r\in R$ we have $\mathrm{Ann}(x)\subseteq\mathrm{Ann}(rx)$ which implies that $\mathrm{Ann}(\mathrm{Ann}(x))\supseteq\mathrm{Ann}(\mathrm{Ann}(rx))$ and hence $y=0$.

Theorem 2.2 in [7] states that $\Gamma(R)$ is finite if and only if either R is finite or an integral domain. However, this is not the case for modules. This means there is an R-module with infinite $t(M)$ but $\Gamma(M)$ is empty (finite). The following example demonstrates this.

Example 4. Let K be a field and $R=\frac{K[X_1,X_2,\cdots]}{(X_1X_2,X_3,\cdots)}$. Put $M=\frac{R}{(X_1)}$. We have $\mathrm{Ann}(\mathrm{Ann}(x_{i_1},\cdots,X_{i_n}))=(X_2,X_3,\cdots)$. Now we get that $T(M)$ is empty and we have infinite torsion elements.

Definition 2.1. Let G be a graph. The girth of G denoted by $g(G)$ is the length of the shortest cycle in G. If G contains no cycles, then $g(G)=\infty$.

Note that if G contains a cycle, then $g(G)\leq 2\cdot\mathrm{diam}(G)+1$ by [[8], Proposition 13.2].

In [7], it was shown that the diameter of $\Gamma(R)$ does not exceed 3. In the same manner, we show the diameter of $\Gamma(M)$ also does not exceed 3 under the connectedness assumption.

Theorem 2.4. *Let R be a ring, M be an R-module, and* $\Gamma(M)$ *be not empty and connected. Then the diameter of* $\Gamma(M)\leq 3$.

Proof. Let $x,y\in V(M)$ be distinct. If $x*y=0$, then $d(x,y)=1$. So suppose that $x*y$ is nonzero. If $x*x=y*y=0$, then there exist $r\in\mathrm{Ann}(\mathrm{Ann}(x))$

and $s \in \text{Ann}(\text{Ann}(y))$ such that $rs \neq 0$. Then $s \notin \text{Ann}(x)$ because $r\text{Ann}(x) = 0$. So we have two nonzero elements y and sx. Now $\text{Ann}(s) \subseteq \text{Ann}(sx)$, which implies that $\text{Ann}(\text{Ann}(s)) \supseteq \text{Ann}(\text{Ann}(sx))$. We know that $s\text{Ann}(y) = 0$, which gives that $\text{Ann}(y) \subseteq \text{Ann}(s)$, so $\text{Ann}(\text{Ann}(s)) \subseteq \text{Ann}(\text{Ann}(y))$. Given that $y * y = 0$, this implies that $y \sim sx$. And because $\text{Ann}(\text{Ann}(sx)) \subseteq \text{Ann}(\text{Ann}(x))$ and $x * x = 0$, we have $sx \sim x$. Immediately, we have $y \sim sx \sim x$, which is a path of length 2, thus $d(x, y) = 2$.

If $x * x = 0$ and $y * y \neq 0$, we have two cases. One is when $\text{Ann}(\text{Ann}(y)) \subseteq \text{Ann}(x)$, meaning there exists $a \in \text{Ann}(\text{Ann}(y))$ with $ax \neq 0$. Since $y \in V(M)$, there exists $z \in M$ such that $z * y = 0$. So we have $a\text{Ann}(z) = 0$ because $z * y = 0$. This implies that $\text{Ann}(\text{Ann}(z)) \subseteq \text{Ann}(a) \subseteq \text{Ann}(ax)$. Also, we have $a\text{Ann}(y) = 0$, which gives that $\text{Ann}(y) \subseteq \text{Ann}(a) \subseteq \text{Ann}(ax)$. Hence, $\text{Ann}(\text{Ann}(ax)) \subseteq \text{Ann}(\text{Ann}(y))$, and that means $\text{Ann}(\text{Ann}(ax))\text{Ann}(\text{Ann}(z)) = 0$. Therefore, we have the path $x \sim ax \sim z \sim y$, so $d(x, y) \leq 3$.

The second case is when $\text{Ann}(\text{Ann}(y)) \subseteq \text{Ann}(x)$. Then we have for all $a \in \text{Ann}(\text{Ann}(y))$, $a \in \text{Ann}(x)$. But we have $\text{Ann}(x)\text{Ann}(\text{Ann}(x)) = 0$, which implies that $\text{Ann}(\text{Ann}(y))\text{Ann}(\text{Ann}(x)) = 0$. This contradicts $x * y \neq 0$. A similar argument holds if $y * y = 0$ and $x * x \neq 0$.

Thus, we may assume that $x * y$, $x * x$, and $y * y$ are all nonzero. Hence, there are $z, w \in V(M)$ with $z * x = w * y = 0$. If $\text{Ann}(x) \subseteq \text{Ann}(y)$, then $\text{Ann}(\text{Ann}(y)) \subseteq \text{Ann}(\text{Ann}(x))$. As $x * z = 0$, then $y * z = 0$. So we have $x \sim z \sim y$, and $d(x, y) = 2$. If $\text{Ann}(x) \not\subseteq \text{Ann}(y)$, then there exists $r \in \text{Ann}(x)$ with $ry \neq 0$. Thus, $\text{Ann}(\text{Ann}(r)) \subseteq \text{Ann}(\text{Ann}(x))$. But $\text{Ann}(\text{Ann}(x))\text{Ann}(\text{Ann}(x)) = 0$ implies that $\text{Ann}(\text{Ann}(r))\text{Ann}(\text{Ann}(x)) = 0$. We know that $\text{Ann}(\text{Ann}(ry)) \subseteq \text{Ann}(\text{Ann}(r))$, this leads to $\text{Ann}(\text{Ann}(ry))\text{Ann}(\text{Ann}(x)) = 0$. Therefore, we have $x \sim ry \sim w \sim y$, and $d(x, y) \leq 3$.

Theorem 2.5. *Let R be a ring, M be an R-module, and* $\Gamma(M)$ *be connected. If* $\Gamma(M)$ *contains a cycle, then* $\text{gr}(\Gamma(M)) \leq 4$.

Proof. Let x be a vertex in a cycle in $\Gamma(M)$ and $x * y = 0$, then there is $z \in M$ such that $x * z = 0$. Now if $x * y = 0$, then we have the following cycle $x \sim z \sim y \sim x$ with girth 3 ≤ 4. On the other hand, if $z * y \neq 0$, then by Theorem 2.4 there is a path $z \sim w_1 \sim w_2 \sim y$, so we have the following cycle $x \sim z \sim w_1 \sim w_2 \sim y \sim x$ with girth 4 ≤ 4.

Theorem 2.6. *Let R be a ring. Then* $Z(R)^2 = 0$ *if and only if* $\Gamma(M)$ *is complete for any non-torsion-free R-module M.*

Proof. Assume $Z(R)^2 = 0$ and let M be a non-torsion-free R-module. Let $x, y \in t(M)^*$. Then $\text{Ann}(\text{Ann}(x)) \subseteq Z(R)$ and $\text{Ann}(\text{Ann}(y)) \subseteq Z(R)$, hence $x * y = 0$, which gives that $\Gamma(M)$ is complete.

Conversely, we can assume $\Gamma(R)$ is complete. Now let $a, b \in Z(R)$, we have $a * b = 0$, then by Theorem 2.1 $ab = 0$, so $Z(R)^2 = 0$.

It should be noted that Theorem 2.6 assures that $\Gamma(\mathbb{Z}_n) \neq \Gamma((\mathbb{Z}_n)_{\mathbb{Z}})$. However, Proposition 1.2 in [9] implies that $\Gamma(\mathbb{Z}_n) = \Gamma((\mathbb{Z}_n)_{\mathbb{Z}})$. This illustrates that our definition of graphs for modules is distinct from the definition proposed in [9].

Theorem 2.7. *Let M be a simple R-module with a nonempty* $\Gamma(M)$. *Then* $\Gamma(M)$ *is complete if and only if* $(\mathrm{Ann}(\mathrm{Ann}(M)))^2 = 0$.

Proof. Because M is simple, then $\mathrm{Ann}(M) = \mathrm{Ann}(x)$ for every $x \in M$. Thus, $\mathrm{Ann}(M) = P$, where P is a maximal ideal of R, and because $P \subseteq \mathrm{Ann}(\mathrm{Ann}(P))$ and P is maximal, we are done. The converse is clear.

Lemma 2.8. *Let R be a ring and M be an R-module. Let* $x, y \in t(M)^*$. *If there exists* $r \in \mathrm{Ann}(x)$ *with* $ry \neq 0$, *then* $x * ry = 0$. *Moreover, if* $x * x = 0$ *and* $rx \neq 0$, *then* $x * rx = 0$.

Proof. Assume $x, y \in t(M)^*$ and there exists $r \in \mathrm{Ann}(x)$ with $ry \neq 0$. Now we have $\mathrm{Ann}(\mathrm{Ann}(x)) \subseteq \mathrm{Ann}(r) \subseteq \mathrm{Ann}(ry)$ and consequently we have $\mathrm{Ann}(\mathrm{Ann}(x)) \supseteq \mathrm{Ann}(\mathrm{Ann}(r)) \supseteq \mathrm{Ann}(\mathrm{Ann}(ry))$. But it is known that $\mathrm{Ann}(\mathrm{Ann}(x)) \cap \mathrm{Ann}(x) = 0$, hence $x * ry = 0$. To prove the second statement, as above we can see $\mathrm{Ann}(\mathrm{Ann}(rx)) \subseteq \mathrm{Ann}(\mathrm{Ann}(x))$, so $x * rx = 0$.

Theorem 2.9. *For every faithful R-module M, if* $\Gamma(M)$ *is not empty, then* $\Gamma(M)$ *is connected.*

Proof. Let $x, y \in t(M)^*$. Since $\Gamma(M)$ is not empty, there exist $z, w \in t(M)^*$ with $z * w = 0$. We have seven cases:

Case 1: If $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(z)$, $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(w)$, $\mathrm{Ann}(y) \subseteq \mathrm{Ann}(z)$, and $\mathrm{Ann}(y) \subseteq \mathrm{Ann}(w)$. Since M is faithful, there exist $m \in M$ with $rm \neq 0$ for some $r \in \mathrm{Ann}(x)$. By Lemma 2.8, we have the path $x \sim rm \sim z$. In the same way, we can find $s \in \mathrm{Ann}(y)$ and $n \in M$ with $sn \neq 0$, hence we have the path $y \sim sn \sim w$. The above argument gives the path $x \sim rm \sim z \sim w \sim sn \sim y$.

Case 2: If $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(z)$, $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(w)$, $\mathrm{Ann}(y) \subseteq \mathrm{Ann}(z)$, and $\mathrm{Ann}(y) \not\subseteq \mathrm{Ann}(w)$. As the above case, we can find the path $x \sim rm \sim w$ for some $r \in \mathrm{Ann}(x)$ and $m \in M$, and $x \sim an \sim z$ for some $a \in \mathrm{Ann}(x)$ and $m \in M$. Now, if $\mathrm{Ann}(w) \subseteq \mathrm{Ann}(y)$, then $y \sim z = 0$, hence we have the following path $x \sim rm \sim w \sim y$. If $\mathrm{Ann}(w) \not\subseteq \mathrm{Ann}(y)$ and $\mathrm{Ann}(y) \not\subseteq \mathrm{Ann}(w)$, we can find $t \in \mathrm{Ann}(y)$ with $tw \neq 0$ and $s \in \mathrm{Ann}(w)$ with $sy \neq 0$. Consequently, we have the path $y \sim tw \sim z \sim an \sim x$. In the same way, we can find a path from x to y if we have $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(z)$, $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(w)$, $\mathrm{Ann}(y) \not\subseteq \mathrm{Ann}(z)$, and $\mathrm{Ann}(y) \subseteq \mathrm{Ann}(w)$.

Case 3: If $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(z)$, $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(w)$, $\mathrm{Ann}(y) \not\subseteq \mathrm{Ann}(z)$, and $\mathrm{Ann}(y) \not\subseteq \mathrm{Ann}(w)$. As in case 1, we can find the path $x \sim rm \sim w$ for some $r \in \mathrm{Ann}(x)$ and $m \in M$ and the path $x \sim an \sim z$ for some $a \in \mathrm{Ann}(x)$ and $n \in M$. Of course, if $\mathrm{Ann}(z) \subseteq \mathrm{Ann}(y)$, then we have $w \sim y = 0$, consequently, we have the following path $x \sim rm \sim w \sim y$, and if $\mathrm{Ann}(w) \subseteq \mathrm{Ann}(y)$, then we have $z \sim y = 0$, consequently, we have the follsowing path $x \sim an \sim z \sim y$. Thus, we can find $c \in \mathrm{Ann}(y)$ with $cz \neq 0$ to get the path $y \sim cz \sim w \sim z \sim an \sim x$.

Case 4: Either $\text{Ann}(x) \not\subseteq \text{Ann}(z)$, $\text{Ann}(x) \subseteq \text{Ann}(w)$, $\text{Ann}(y) \subseteq \text{Ann}(z)$, and $\text{Ann}(y) \subseteq \text{Ann}(w)$, or $\text{Ann}(x) \subseteq \text{Ann}(z)$, $\text{Ann}(x) \not\subseteq \text{Ann}(w)$, $\text{Ann}(y) \subseteq \text{Ann}(z)$, and $\text{Ann}(y) \subseteq \text{Ann}(w)$. This is the same as case 2.

Case 5: If $\text{Ann}(x) \not\subseteq \text{Ann}(z)$, $\text{Ann}(x) \not\subseteq \text{Ann}(w)$, $\text{Ann}(y) \subseteq \text{Ann}(z)$, and $\text{Ann}(y) \subseteq \text{Ann}(w)$. This is as case 3.

Case 6: If $\text{Ann}(x) \not\subseteq \text{Ann}(z)$, $\text{Ann}(x) \subseteq \text{Ann}(w)$, $\text{Ann}(y) \not\subseteq \text{Ann}(z)$, and $\text{Ann}(y) \subseteq \text{Ann}(w)$. From case 1, we can find $x \sim rm \sim w$ and $z \sim sn \sim y$, consequently, we have the path $x \sim rm \sim w \sim z \sim sn \sim y$. This is the same if we assume $\text{Ann}(x) \subseteq \text{Ann}(z)$, $\text{Ann}(x) \not\subseteq \text{Ann}(w)$, $\text{Ann}(y) \subseteq \text{Ann}(z)$, and $\text{Ann}(y) \not\subseteq \text{Ann}(w)$.

Case 7: If we have $\text{Ann}(x) \not\subseteq \text{Ann}(z)$, $\text{Ann}(x) \not\subseteq \text{Ann}(w)$, $\text{Ann}(y) \not\subseteq \text{Ann}(z)$, and $\text{Ann}(y) \not\subseteq \text{Ann}(w)$. The above arguments give the path $x \sim rz \sim w \sim tz \sim y$ for some $r \in \text{Ann}(x)$ and $t \in \text{Ann}(y)$.

The faithfulness condition in Theorem 2.9 cannot be removed as illustrated by the next example.

Example 5. Let K be a field and $R = \frac{K[X_1, X_2, \cdots]}{(X_1X_2, X_3, \cdots, X_2X_3)}$. Put $M = \frac{R[Y]}{(X_1)}$.

Then we have the following.

- $\text{Ann}(\text{Ann}(X_2)) = (X_2)$
- $\text{Ann}(\text{Ann}(X_3)) = (X_3)$
- $\text{Ann}(\text{Ann}(X_i)) = (X_2, X_3, \cdots)$
- $\text{Ann}(\text{Ann}(Y)) = (X_2, X_3, \cdots)$
- $\text{Ann}(\text{Ann}(X_2X_{i_1} \cdots X_{i_n})) = (X_2)$ for all $X_{i_j} > 3$
- $\text{Ann}(\text{Ann}(X_3X_{i_1} \cdots X_{i_n})) = (X_2)$ for all $X_{i_j} > 3$
- $\text{Ann}(\text{Ann}(YX_{i_1} \cdots X_{i_n})) = (X_2, X_3, \cdots)$ for all $X_{i_j} > 3$
- $\text{Ann}(\text{Ann}(X_{i_1} \cdots X_{i_n})) = (X_2, X_3, \cdots)$ for all $X_{i_j} > 3$

So, we see $\Gamma(M)$ is not empty because we have the path $X_2 \sim X_3$, but not connected.

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Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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