

Dispersion Relations in Diffraction in Time

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How to cite this paper: Godoy, S. and Villa, K. (2024) Dispersion Relations in Diffraction in Time. *Applied Mathematics*, 15, 464-468.

<https://doi.org/10.4236/am.2024.157028>

Received: June 5, 2024

Accepted: July 23, 2024

Published: July 26, 2024

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Abstract

In agreement with Titchmarsh's theorem, we prove that dispersion relations are just the Fourier-transform of the identity, $g(x') = \pm \text{Sgn}(x')g(x')$, which defines the property of being a truncated functions at the origin. On the other hand, we prove that the wave-function of a generalized diffraction in time problem is just the Fourier-transform of a truncated function. Consequently, the existence of dispersion relations for the diffraction in time wave-function follows. We derive these explicit dispersion relations.

Keywords

Diffraction in Time, Dispersion Relations, Hilbert Transforms

1. Introduction

Similarities between optics and quantum mechanics have long been recognized. One example of this symmetry was obtained by Moshinsky [1] who addressed the following quantum shutter problem. Consider a mono-energetic beam, $E = (\hbar k)^2/2m$, of free particles, moving parallel to the x -axis. For negative times, the beam is interrupted at $x = 0$ by a perfectly absorbing shutter perpendicular to the beam. Suddenly, at time $t = 0$, the shutter is opened, allowing for $t > 0$ the free time-evolution of the beam of particles. What is the time-dependent density, $|\psi(x, t)|^2$, observed at an arbitrary distance x from the shutter?

The shutter problem implies solving the time-dependent Schrödinger equation with an initial condition given by a truncated function at $x = 0$,

$$\psi(x, 0) = \begin{cases} e^{ikx} & x < 0 \\ 0 & x > 0 \end{cases} = e^{ikx}\theta(-x), \quad (1)$$

where $\theta(-x)$ denotes the step down function at $x = 0$. For $x > 0$, Moshinsky proved that the free propagation of the initial truncated beam, has a time-de-

pendent probability density given by:

$$|\psi(x, t)|^2 = \frac{1}{2} \left[C(\xi) + \frac{1}{2} \right]^2 + \frac{1}{2} \left[S(\xi) + \frac{1}{2} \right]^2, \quad (2)$$

here $C(\xi) + iS(\xi) \equiv \int_0^\xi \exp(i\pi u^2/2) du$ denotes the complex Fresnel function and the argument ξ is given by

$$\xi(x, t; k) \equiv \sqrt{\frac{m}{\pi \hbar t}} \left(\frac{\hbar k}{m} t - x \right). \quad (3)$$

The quantum probability density given by Equation (2) looks in time similar to the optical expression in space for the light intensity in the Fresnel diffraction by a straight-edge [2]. For a fixed position, $X \equiv kx = 1$, the plot of the probability density as a function of time, $T \equiv Et/\hbar$, is shown in **Figure 1**.

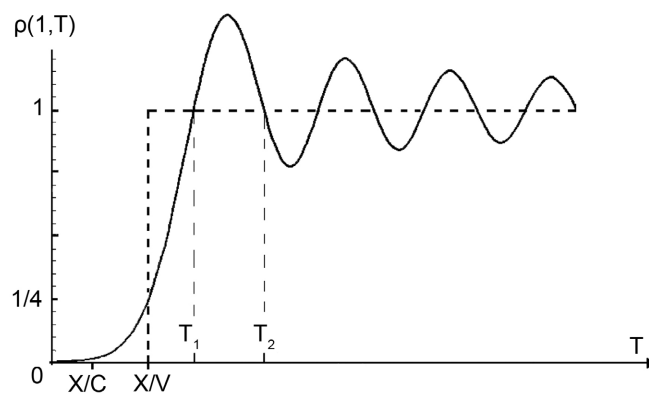


Figure 1. 1 Schrödinger diffraction in time.

Diffraction in time is a pure quantum phenomenon, and similar oscillations should arise after a sudden closing and opening gates in nanoscopic circuits [3]. For a review on the subject see [4]. With adequate potentials added to the model, it has been used to study transient dynamics of tunneling matter waves [5], and the transient response to abrupt changes of the interaction potential in semiconductor structures and quantum dots [6]. There is, in summary, a strong motivation for a thorough understanding of transient time oscillation in beams of matter.

In nanoscopic circuits, the words “closing” and “opening” gates describe truncated wave functions in x -domain, $\psi(x) = g(x)\theta(\pm x)$, which reminds us of a similar concept, the one known as “causal” functions in time-domain: $f(t) = h(t)\theta(t)$. We know that according to the Titchmarsh’s theorem [7], the Fourier transform: $F(\omega) \equiv \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt$, of the causal function, is such that $F(\omega)$ satisfies dispersion relations. These are a pair of reciprocal Hilbert-transform relations [7].

$$\operatorname{Re} F(\omega) = \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{\operatorname{Im} F(\omega')}{\omega - \omega'} d\omega', \quad \operatorname{Im} F(\omega) = -\frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{\operatorname{Re} F(\omega')}{\omega - \omega'} d\omega', \quad (4)$$

here, P, denotes the Cauchy’s principal value of the integral.

The similarity between causal functions and truncated functions, strongly suggests that dispersion relations must also exist in the wave-function of the diffraction in time problem. As far as we know, this topic has never been reported before. The purpose of this paper to show that this is indeed true and in this paper we derive the explicit dispersion relations satisfied by a generalized diffraction in time problem.

2. Diffraction in Time as a Fourier-Transform of a Truncated Function

The generalized diffraction in time problem is the solution of an arbitrary time-dependent Schrödinger equation with the initial condition given by

$$\psi(x, 0) = e^{ikx} \theta(-x). \quad (5)$$

Denoting by, $G(x, t | x', 0)$, the Green's function (time propagator) of the time-dependent Schrödinger equation, the time evolution $\psi(x, t)$ of the initial condition is given by

$$\psi(x, t; k) = \int_{-\infty}^{\infty} G(x, t | x', 0) e^{ikx'} \theta(-x') dx'. \quad (6)$$

By simplicity in the notation, since (x, t) will be kept constants in the present development, from now on we will omit the (x, t) variables and rewrite Equation (6) as simple as

$$\psi(k) = \int_{-\infty}^{\infty} G(x') \theta(-x') e^{ikx'} dx'. \quad (7)$$

Denoting by $g(x') \equiv G(x') \theta(-x')$ (a step down truncated function at $x' = 0$), and using the symbol, $\mathcal{F}[\](k)$, to denote the Fourier-transform operator, we can write

$$\psi(k) = \int_{-\infty}^{\infty} g(x') e^{ikx'} dx' \equiv \mathcal{F}[g(x')](k) \quad (8)$$

Here we have the important result. Any diffraction in time wave-function, $\psi(x, t; k) \equiv \psi(k)$, is the Fourier-transform of a truncated function $g(x')$ at $x' = 0$. The purpose of this paper is to show that for any diffraction in time problem, the corresponding wave-function, $\psi(k)$, satisfies dispersion relations (in the k domain).

3. Dispersion Relations for the Diffraction in Time Wave-Function

Any step down truncated function at $x' = 0$, $g(x') \equiv G(x') \theta(-x')$, satisfies the identity:

$$g(x') = -\text{Sgn}(x') g(x'). \quad (9)$$

This statement is clear, since for $x' < 0$, $\text{Sgn}(x') = -1$, then Equation (9), becomes $g(x') = g(x')$, which means $g(x') = G(x') \neq 0$ for $x' < 0$. On the other hand, for $x' > 0$, $\text{Sgn}(x') = +1$, then Equation (9) becomes $g(x') = -g(x')$ which means $g(x') = 0$ for $x' > 0$. This is the definition of a truncated func

tion at $x' = 0$.

We claim that Equation (9), which is the mathematical statement of being a truncated function in the x' -domain, is nothing but the Fourier-transform pair of the dispersion relations in k -domain. We show this next.

Taking the Fourier transform of Equation (9) we have:

$$\mathcal{F}[g(x')](k) = -\mathcal{F}[Sgn(x')g(x')](k). \quad (10)$$

Next, using the Fourier-transform identities:

$$\mathcal{F}[A(x)B(x)](k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[A(x)](k-k') \mathcal{F}[B(x)](k') dk' \quad (11)$$

$$\mathcal{F}[Sgn(x)](k) = 2i \frac{P}{k} \quad P \equiv \text{Cauchy's principal part}, \quad (12)$$

then, Equation (10) becomes a Hilbert-transform identity.

$$\mathcal{F}[g(x')](k) = \frac{-i}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{F}[g(x')](k')}{k-k'} dk'. \quad (13)$$

Equation (13) is a consequence of $g(x')$ being a (step-down) truncated function at $x' = 0$. On the other hand, from Equation (8), we know that the diffraction in time wave-function $\psi(k)$ satisfies: $\psi(k) = \mathcal{F}[g(x')](k)$, therefore, we can rewrite Equation (13) as a Hilbert-transform identity for $\psi(x, t; k)$

$$\psi(x, t; k) = \frac{-i}{\pi} P \int_{-\infty}^{\infty} \frac{\psi(x, t; k')}{k-k'} dk'. \quad (14)$$

In terms of real and imaginary components, this last equation can be rewritten as:

$$\text{Re}[\psi(k)] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Im}[\psi(k')]}{k-k'} dk'; \quad \text{Im}[\psi(k)] = \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Re}[\psi(k')]}{k-k'} dk'. \quad (15)$$

For a generalized diffraction in time wave-function, these are the explicit dispersion relations.

Dispersion relations, what for? Diffraction in time is a time-dependent scattering process. Assuming that for all $(x < 0, t)$ we know the wave-function for $k > 0$ (incident wave-function), then from Equation (14) we can find the corresponding $(x < 0, t)$ wave-function for $k < 0$ (reflected wave-function). In other words, dispersion relations give a connection between the time-dependent incident and reflected wave-functions.

4. Conclusion

We have proved that dispersion relations given by Equation (14) is just the Fourier-transform of the identity, $g(x') = -Sgn(x')g(x')$, which is the statement of $g(x')$ being a truncated function. Since diffraction in time wave-function $\psi(x, t; k)$ is the Fourier-transform of a truncated one, the existence of dispersion relations in diffraction in time follows.

Acknowledgements

Sincere thanks to the members of JMP for their professional performance, and

special thanks to managing editor: *Nancy Ho*, for a rare attitude of high quality.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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