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Graph-Induced by Modules via Tensor Product

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Abstract

This paper investigates the connections between ring theory, module theory, and graph theory through the graph $G(R)$ of a ring R. We establish that vertices of $G(R)$ correspond to modules, with edges defined by the vanishing of their tensor product. Key results include the graph's connectivity, a diameter of at most 3, and a girth of at most 7 when cycles are present. We show that the set of modules $S(R)$ is empty if and only if R is a field, and that for semisimple rings, the diameter is at most 2. The paper also discusses module isomorphisms over subrings and localization, as well as the inclusion of $G(T)$ within $G(R)$ for a quotient ring T, highlighting that the reverse inclusion is not guaranteed. Finally, we provide an example illustrating that a non-finitely generated module M does not imply $M \otimes M = 0$. These findings deepen our understanding of the interplay among rings, modules, and graphs.

Keywords

Graph Theory, Commutative Ring, Tensor Product, Connected, Diameter, Semisimple Ring

1. Introduction

Graphs serve as powerful mathematical tools for modeling complex systems and relationships in various fields, ranging from computer science to biology and social networks. On the other hand, modules over rings constitute fundamental algebraic structures with diverse applications in algebra, geometry, and representation theory. The intersection of these two realms gives rise to an intriguing area of study known as graph-induced by modules, which captures algebraic properties induced by the underlying structure of a graph.

Let R be a commutative ring with unity, and let $S(R)$ denote its set of all R-modules M that there exists an R-mdule N with $M \otimes N = 0$. And M is called tensor zero-divisor. We associate a simple graph $G(R)$ to R, where the vertices are the nonzero tensor zero-divisors $S(R)$, and for distinct vertices, they are adjacent if and only if $M \otimes N = 0$. Consequently, $G(R)$ is the empty graph if and only if R is a field.

In this article, we delve into the realm of graph-induced by modules through the lens of tensor product theory. The tensor product, a cornerstone operation in algebra, offers a versatile framework for combining algebraic structures in a systematic manner. By leveraging the rich interplay between graphs and algebraic structures, we aim to explore the construction, properties, and applications of graph-induced modules.

The notion of graph-induced modules arises from the desire to understand and exploit the algebraic properties encoded within the topology of a graph. Each vertex and edge of a graph carries intrinsic algebraic information, which can be harnessed to define modules over appropriate rings. These modules capture not only the combinatorial aspects of the graph but also the underlying algebraic symmetries and dependencies.

One of the key insights driving our investigation is the recognition of the tensor product as a unifying tool for constructing graph-induced modules. By carefully manipulating tensor products associated with vertices and edges of a graph, we can systematically build modules that encapsulate the algebraic essence of the underlying graph structure. This approach not only provides a coherent framework for constructing graph-induced by modules but also unveils deep connections between graph theory and algebra.

Throughout the article, when we refer to a "ring," we mean a commutative ring, and when we use the term "equality," it signifies isomorphism. This clarification helps to streamline the language and ensures consistency in our notation.

We aim to provide a comprehensive overview of the theory of graph-induced by modules, emphasizing the role of tensor product techniques in their construction and analysis. We will explore various examples.

2. Important Result

Great, starting with clear definitions, theorems, and notations is an excellent way to establish a solid foundation for the subsequent material.

Definition 2.1 [1] *A graph G consists of a set of vertices (or nodes) denoted by* $V(G)$, *and a set of edges (or links) that connect pairs of vertices. The set of vertices of G is denoted by* $V(G)$. *Each vertex typically represents an entity or a point of interest in a problem context. Two vertices x and y are considered adjacent (connected) in G, denoted by* $x-y$, *if there exists an edge that directly connects them.*

Definition 2.2 *Let R be a ring. The set* $S(R)$ *to consist of all non-trivial R-modules M for which there exists another non-trivial R-module N such that the* *tensor product of M and N is zero. Here M is a representative element from each isomorphism class of R-modules.*

Definition 2.3 *We define the graph of a ring R, denoted by* $G(R)$, *as a graph where the vertices are elements of* $S(R)$, *and two vertices M and N are adjacent if and only if their tensor product* $M \otimes N$ *is zero.*

Prior to delving into theorem 2.1, we require the subsequent definitions from [2].

1) The diameter of a graph is the maximum distance between any pair of vertices in the graph. In other words, it is the length of the longest shortest path between any two vertices. The diameter of a graph is commonly denoted by the symbol $\operatorname{diam}(G)$.

2) The girth of a graph is the length of the shortest cycle in the graph. A cycle is a closed path where the starting and ending vertices are the same, and it doesn't repeat any other vertices or edges. The girth of a graph is often denoted by the symbol $g(G)$.

Theorem 2.1 *For a ring R. Then we have the following on the graph* $G(R)$.

1) *The graph* $G(R)$ *of a ring R is connected*;

2) *the diameter of* $G(R)$ *is at most* 3;

3) *If* $G(R)$ *contains a cycle, then the girth* $g(G(R))$ *is at most* 7.

Proof. Naturally We can combine cases 1 and 2 into a single proof by using the definition of connected components. This is the method. Let $x, y \in S$ be different. We examine the following cases:

Case 1. $d(x,y)=1$ if $x \otimes y = 0$ it follows that $\operatorname{diam}(G(R)) \leq 3$.

Case 2. If $x \otimes x = y \otimes y = 0$ and $x \otimes y \neq 0$, then we have the following path $x - x \otimes y - y$ that means $\operatorname{diam}(G(R)) \leq 3$.

Case 3. If $x \otimes x = 0, y \otimes y \neq 0$ and $x \otimes y \neq 0$, so that we have an element $z \in S - \{x, y\}$ with $y \otimes z = 0$. In the case of $x \otimes z = 0$ we get the following path $x - z - y$ with length 2 that means $\operatorname{diam}(G(R)) \leq 3$. Otherwise *i.e.* $x \otimes z \neq 0$ we get then we have the fallowing path $x - x \otimes z - y$ and its length is 2 that means $\operatorname{diam}(G(R)) \leq 3$.

Case 4. If $x \otimes x \neq 0, y \otimes y = 0$ and $x \otimes y \neq 0$. The proof of this case is similarly to proof of case 3.

Case 5. If $x \otimes x \neq 0, x \otimes y \neq 0$ and $y \otimes y \neq 0$. In this case there is $z, w \in S - \{x, y\}$ with $x \otimes z = 0$ and $y \otimes w = 0$. If $z = w$ we have the following path $x - z - y$ of length 2 and hence $\operatorname{diam}(G(R)) \leq 3$. In the case of $z \neq w$ and $z \otimes w = 0$ we have the following path $x - z - w - y$ with length 3 that means $\operatorname{diam}(G(R)) \leq 3$, but if $z \otimes w \neq 0$ then we will get the following path $x - z \otimes w - y$ with length 2 that means $\operatorname{diam}(G(S)) \leq 3$.

To prove 3. By Proposition 1.3.2 in [2].

Lemma 2.2 *Let R be a ring and F be a free R-module, for any R-module M then* $F \otimes M \neq 0$.

Proof. By Theorem 1.6.6 in [3] $F \cong \oplus R$. So that by Theorem 2.2.6 we have $F \otimes M \cong \oplus R \otimes M \cong \oplus(R \otimes M)$. And by Proposition 3.14 in [4] we get $F \otimes M \cong \oplus M$ which is not zero.

Theorem 2.3 *Let R be a ring then* $S(R)$ *is empty if and only if R is a field.*

Proof. Assume $S(R)$ is empty we want to prove that *R* is afield. By contradiction, suppose *R* is not a field then there is an ideal $0 \neq I \neq R$ of *R*. Now Let *E* be an invective *R*-module. It is known by Theorem 2.4.5 in [3] that *E* is divisible. But by Exercise 2 p.31 in [4] we have $R/I \otimes E \cong E/IE$. Because *E* is divisible we have $IE = E$ and $I \neq R$ we have $R/I \neq 0$ $R/I \otimes E = 0$ which means that $S(R)$ is not empty which is a contradiction so we have that *R* is a field.

Conversely If *R* is a field then by Example 1.6.3 in [3] any *R*-module *M* is free. Let M, N are *R*-modules, then M, N are free. These arguments and Lemma 2.2 give $S(R)$ is empty.

Theorem 2.4 *If R is a semisimple ring that isn't a field, then the diameter of the graph of* $G(R)$, *is at most* 2.

Proof. Suppose *M* and *N* belong to $S(R)$. If $M \otimes N = 0$, there's nothing more to demonstrate. However, if $M \otimes N \neq 0$, and $0 \neq I$ represents a proper ideal of *R*, then due to the injective and divisible nature of *M* and *N* by Theorem 3.5.16 and Theorem 2.4.5 in [3] , respectively, we have $IN = N$ and $IM = M$. Thus, we can establish by proof of Theorem 2.3 $R/I \otimes M = 0$ and $R/I \otimes N = 0$, forming a path of length 2: $N - R/I - M$.

3. The Localization

Definition 3.1 [3] *Consider a ring R, and let A represent the set of all regular elements* (*those that are not zero divisors*) *in R. The localization of R by A, denoted as* $A^{-1}R$, *is termed the quotient ring of R and symbolized as* $T(R)$.

Theorem 3.1 *For a commutative ring R, and B an* $A^{-1}R$ *-module, there exists an isomorphism between B and* $A^{-1}B$ as $A^{-1}R$ *-modules.*

Proof. Now, both *B* and $A^{-1}B$ are $A^{-1}R$ -modules. Define $f: B \to A^{-1}B$ by $f(b) = \frac{1}{1}b$. We aim to prove that *f* is an $A^{-1}R$ -isomorphism.

For any $x, y \in B$, we observe:

$$f(x+y) = \frac{1}{1}(x+y) = \frac{1}{1}x + \frac{1}{1}y = f(x) + f(y).$$

Moreover, for $f\left(\frac{r}{s}x\right)$, where $\frac{r}{s} \in A^{-1}R$ and $x \in B$, we find:

$$f\left(\frac{r}{s}x\right) = \frac{1}{1}\left(\frac{rx}{s}\right) = \frac{r}{s}x = \frac{r}{s}\frac{1}{1}x = \frac{r}{s}f(x).$$

To prove *f* is a monomorphism, assume $f(x) = 0$ for some $x \in B$. Then $\frac{1}{1}x = 0$, implying there exists $t \in A$ such that $tx = 0$, and thus $x = 0$ in *B*.

Now, to prove *f* is an epimorphism, let $x \in B$, noting that $x = \frac{1}{1}x$. Thus, $f(x) = x$. These arguments establish *B* as isomorphic to $A^{-1}B$ as $A^{-1}R$-modules.

Theorem 3.2 *Consider a subring R of a ring T. Let M and N be T-modules. If M and N are isomorphic as T-modules, then they are also isomorphic as R-modules.*

Proof. Suppose $M \cong_T N$. Naturally, both *M* and *N* are *R*-modules, as *R* is a subring of *T*. Since they are isomorphic as *T*-modules, there exists an isomorphism map *f* from *M* onto *N*. This map *f* is both onto and one-to-one. To show that *f* is an *R*-homomorphism, consider $a, b \in M$. We have $f(a+b) = f(a) + f(b)$ because *f* is a *T*-homomorphism. Now, let $r \in R \subseteq T$. Then $f(ra) = rf(a)$ because $r \in T$, demonstrating that *f* is an *R*-homomorphism. Consequently, *f* being an *R*-isomorphism implies that $M \cong_R N$.

Theorem 3.3 *Consider a ring R and let T be the quotient ring of R. Then,* $G(T)$ *is a subset of* $G(R)$.

Proof. Suppose *M* and *N* are *T*-modules with $M \otimes_T M = 0$. We aim to demonstrate $M \otimes_R N = 0$.

By Theorem 3.1, we know that $M \otimes_R N \cong A^{-1}(M \otimes_R N)$. Additionally, according to Proposition 3.7 in [4], we have $A^{-1}(M \otimes_R N) \cong A^{-1}M \otimes_T A^{-1}N$.

Utilizing Theorem 3.1 and Theorem 3.2, we establish $M \cong_T A^{-1}M$ and $N \cong_T A^{-1}N$. Consequently, $M \otimes_R N \cong_R M \otimes_T N = 0$.

Remark 3.4 *While R being a subring of L implies* $G(L)$ *is a subset of* $G(R)$, *this inclusion is not always guaranteed. The subsequent examples illustrate this point.*

Example 3.5 *Consider a field K and define* $R = K \times K$. *While K is a subset of R, according to Theorem* 2.3, $G(R)$ *is non-empty, while* $G(K)$ *is empty. Thus, we observe* $G(R) \not\subseteq G(K)$.

Lemma 3.6 *Suppose R is a local ring with a maximal ideal M. For any finitely generated R-module A, if* $A \otimes A = 0$, *then* $A = 0$.

Proof. This directly follows from Exercise 3 p.31 in [4].

Lemma 3.7 *If M is a nonzero finitely generated R-module, then* $M \otimes M \neq 0$.

Proof. Suppose, by contradiction, that *M* is a nonzero finitely generated *R*-module with $M \otimes M = 0$. Then, $(M \otimes M)_P = 0$ for all maximal ideals *P*. However, by Proposition 3.7 in [4], we know that $(M \otimes M)_P \cong M_P \otimes M_P = 0$. By Lemma 3.6, this implies $M_P = 0$ for all maximal ideals *P*. Consequently, by Proposition 3.8 in [4], we conclude that $M = 0$, which contradicts the assumption that *M* is nonzero. Thus, $M \otimes M \neq 0$.

Remark 3.8 *If M is not a finitely generated R-module, it's not generally true that* $M \otimes_R M \neq 0$. *The following example illustrates this.*

Example 3.9 *Let R be a local domain that is not a field, and let Q be its quotient field. Consider* $M = Q / R$. *Then,* $M \otimes_R M = 0$.

4. Complemented Graphs

Sure, let's dive into some definitions.

Definition 4.1 [5] *In a graph G For vertices x and y of G, then* $x \le y$ *if x and y are not adjacent and each vertex of G adjacent to y is also adjacent to x.*

Certainly! Let's explore equivalent conditions for the relation $x \le y$ in a graph $G(R)$ for a given ring *R*.

Before we delve into the equivalent conditions, let's first introduce the following definitions.

Definition 4.2 *In the context of a ring R and a set C comprising R-modules, if M is an element of C, then the annihilator of M, denoted* $Anc(M)$, *is the subset of C consisting of all modules N such that the tensor product* $M \otimes N$ *equals zero.*

Definition 4.3 *In the context of a ring R, R is termed a C-reduced ring if, for any nonzero R-module M, the tensor product* $M \otimes M$ *is nonzero.*

Lemma 4.1 *Considering a ring R and elements* $M, N \in S$ *then* $M \le N$ *if and only if* $Anc(N) - M \subseteq Anc(M) - N$ *and* $M \otimes N \ne 0$.

Proof. Assuming $M \le N$, then $M \otimes N \ne 0$, implying $M \notin Anc(N)$ and $N \notin Anc(M)$. Now, if $X \in Anc(N)$, which implies $X \otimes M = 0$. Consequently, we infer that *X* is adjacent to *N*, implying *X* is also adjacent to *M*. This deduction leads to $X \otimes M = 0$, implying $X \in Anc(M) = Anc(M) - N$. And it's trivially evident that $M \otimes N \ne 0$.

Conversely, let's define $A = Anc(N) - M$ and $B = Anc(M) - N$. If $A \subseteq B$, then since $M \otimes N \ne 0$ by assumption, if *X* is adjacent to *N*, then $X \in A \subseteq B$, which implies that *X* is adjacent to *M*.

Based on Lemma 4.1, we can derive the following observations.

Remark 4.2

1) If $M \otimes M \ne 0$, then $M \le N$ if and only if $Anc(N) \subseteq Anc(M)$.

2) If *R* is *C*-reduced, then $M \le N$ if and only if $Anc(N) \subseteq Anc(M)$.

Definition 4.4 [6] *Consider a graph G and two elements x and y belonging to G. We define a relation ~ between x and y as follows:* $x \sim y$ *if and only if* $x \le y$ $y \le x$.

The relation ~ is an equivalent relation according to [6].

Corollary 4.3 *If R is a C-reduced ring and for any* $M, N \in G(R)$, $M \sim N$ *if and only if* $Anc(M) = Anc(N)$.

Proof. By the definition itself and Remark 4.2.

There exists a *C*-reduced ring, as demonstrated in the following example.

Example 4.4 *A ring is C-reduced if it is semisimple.*

Proof. Assuming *R* is a semisimple ring, Theorem 3.5.19 in [3] state that $R \cong K_1 \times \cdots \times K_n$ where each K_i is a field for $i = 1$ to *n*. Let *P* be a maximal ideal of *R*. Since $R \cong K_1 \times \cdots \times K_n$, R_P is a field. Now assume $0 \ne M$ is an *R*-module. According to Proposition 3.8 in [4] there exists maximal ideal *P* of *R* such that $M_P \ne 0$. By Lemma 2.2 and Example 1.6.3 in [3] we have $M_P \otimes_{R_P} M_P \ne 0$.

However, Proposition 3.7 in [4] implies that $M_P \otimes_{R_P} M_P \cong (M \otimes_R M)_P$ and consequently by Proposition 3.8 in [4] we have $M \otimes_R M \neq 0$ indicating that *R* is *C*-reduced.

Definition 4.5 [5] *In a graph G If* $x, y \in V(G)$ *and* $x \neq y$ *we say that x and y are orthogonal denoted as* $x \perp y$ *if and only if x and y are adjacent and there is no* $z \in V(G)$ *which is adjacent to both x and y.*

For a commutative ring *R*, we will present an alternative definition of orthogonality in the graph $G(R)$ through the following lemma.

Lemma 4.5 *Let R be a ring,* $M, N \in S(R)$. *Let* $M \otimes M \neq 0$ *or* $N \otimes N \neq 0$. *Then* $M \perp N$ *if and only if* $Anc(M) \cap Anc(N) = 0$ *and* $M \otimes N = 0$.

Proof. Assume $M, N \in S(R)$ and $M \perp N$. Then by the definition we have $M \otimes N = 0$. Now Let $X \in Anc(M) \cap Anc(N)$, then $M \otimes X = N \otimes X = 0$ that means *X* is adjacent to both *M* and *N* which is a contradiction. That gives $X = 0$ Conversely, suppose that $Anc(M) \cap Anc(N) = 0$ and $M \otimes N = 0$. Then *N* is adjacent to *M*. Now if *X* is adjacent to *M* and *N* then $M \otimes X = N \otimes X = 0$ that means $X \in Anc(M) \cap Anc(N) = 0$, thus $X = 0, \notin S(R)$.

Remark 4.6 *If R is C-reduced then if* $M, N \in S(R)$, *then* $M \perp N$ *if and only if* $Anc(M) \cap Anc(N) = 0$.

Proof. Directly from Lemma 4.5.

Definition 4.6 [6] *The graph G is called is called complemented if for each vertex a of G, there is a vertex b of G (called complement of a) such that* $a \perp b$ *and that G is uniquely complemented if G is complemented and whenever* $a \perp b$ *and* $a \perp c$, *then* $b \sim c$.

Lemma 4.7 *Let R be a ring and* $M; N \in S(R)$. *Then the following statements are equivalent.*

(1) $M \perp N; M \otimes M \neq 0$, and $N \otimes N \neq 0$.

(2) $M \otimes N = 0$ and $M \oplus N$ is a regular element of *C*. Here *M* is a regular element means there is no nonzero *N* in *C* with $M \otimes N = 0$.

Proof. "(1) $\Rightarrow$ (2)". $M \otimes N = 0$ directly by Lemma 4.5. Now assume that $(M \oplus N) \otimes X = 0$ where $X \in S(R)$. We have from Theorem 2.2.6 in [3] $(M \oplus N) \otimes X = M \otimes X \oplus N \otimes X = 0$ which implies that $M \otimes X = N \otimes X = 0$ which gives that $X \in Anc(M) \cap Anc(N) = 0$ hence $x = 0$ which is a contradiction because $X \notin S(R)$.

(2) $\Rightarrow$ (1). Since $M \oplus N$ is regular, $(M \oplus N) \otimes M \neq 0$. But by assumption we have $M \otimes N = 0$ this implies that $M \otimes M \neq 0$. In the same way we can show that $N \otimes N \neq 0$. Let $X \in Anc(M) \cap Anc(N)$. Thus, $(M \oplus N) \otimes X = 0$ is implied. However, since $M \oplus N$ is regular, $X = 0$ must exist. $Anc(M) \cap Anc(N) = 0$ as a result.

Lemma 4.8 *Let R be a C-reduced ring and* $M; N; L \in S(R)$. *If* $M \perp N$ *and* $M \perp L$, *then* $N \sim L$. *Thus* $G(R)$ *is uniquely complemented if and only if* $G(R)$ *is complemented.*

Proof. By Remark 4.6 we have $Anc(M) \cap Anc(N) = Anc(M) \cap Anc(L) = 0$, so that $N \otimes L \neq 0$. So N and L are not adjacent. Now suppose that $X \otimes N = 0$ for some $X \in S(R)$. Then $(X \otimes N) \otimes M = X \otimes (M \otimes N) = 0$ and $(X \otimes L) \otimes N = (X \otimes N) \otimes L = 0$. Thus $X \otimes L \in Anc(M) \cap Anc(L) = 0$ implies That $X \otimes L = 0$. Hence $L \leq N$. Similarly, $N \leq L$, and thus $N \sim L$. The last statement is easy.

Open Problem

I think the converse of Theorem 2.4 is true. That means if $\text{diam}(G(R)) \leq 2$ then R is semisimole ring.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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