

Variational Iterative Method Applied to Variational Problems with Moving Boundaries

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ABSTRACT

In this paper, He's variational iterative method has been applied to give exact solution of the Euler Lagrange equation which arises from the variational problems with moving boundaries and isoperimetric problems. In this method, general Lagrange multipliers are introduced to construct correction functional for the variational problems. The initial approximations can be freely chosen with possible unknown constant, which can be determined by imposing the boundary conditions. Illustrative examples have been presented to demonstrate the efficiency and applicability of the variational iterative method.

Keywords: Variational Iterative Method; Variational Problems; Moving Boundaries; Isoperimetric Problems

1. Introduction

In modeling a large class of problems arising in science, engineering and economics, it is necessary to minimize amounts of a certain functional. Because of the important role of this subject, special attention has been given to these problems. Such problems are called variational problems, see [1,2].

The simplest form of a variational problem can be considered as

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x)) dx, \quad (1)$$

where v is the functional which its extremum must be found. Functional v can be considered by two kinds of boundary conditions. In the fixed boundary problems, the admissible function $y(x)$ must satisfy following boundary conditions

$$y(x_0) = y_0, y(x_1) = y_1 \quad (2)$$

In moving boundary problems, at least one of the boundary points of the admissible function is movable along a boundary curve. Further more many applications of the calculus of variations lead to problems in which not only boundary conditions, but also a quite different type of conditions known as constraints, are imposed on the admissible function. The necessary condition for the admissible solutions of such problems has to satisfy the Euler-Lagrange equation which is generally nonlinear.

In this work we consider He's variational iterative method as a well known method for finding both analytic

and approximate solutions of differential equations. Here, the problem is initially approximated with possible unknowns. Then a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory [3].

Variational iterative method is applied on various kinds of problems [4-31].

Author of [32] solved variational problems with moving boundaries with Adomian decomposition method. Variational iterative method was applied to solve variational problems with fixed boundaries (see [11,27,30]). In this work we obtain exact solution of variational problems with moving boundaries and isoperimetric problems by variational iterative method. In fact, variational iterative method is applied to solve the Euler-Lagrange equation with prescribed boundary conditions. To present a clear overview of the procedure several illustrative examples are included.

2. Variational Iterative Method

In variational iterative method which is stated by He [3], solutions of the problems are approximated by a set of functions that may include possible constants to be determined from the boundary conditions. In this method the problem is considered as

$$Ly + Ny = g(x), \quad (3)$$

where L is a linear operator, and N is a nonlinear operator. $g(x)$ is an inhomogeneous term. By using the variational iterative method, the following correct func-

tional is taken into account

$$y_{n+1} = y_n + \int_0^x \lambda (Ly_n(s) + N\tilde{y}_n(s) - g(s)) ds, \quad (4)$$

where λ is Lagrange multiplier [5], the subscript n denotes the n -th approximation, \tilde{y}_n is as a restricted variation *i.e.* $\delta\tilde{y}_n = 0$ [6-8]. Taking the variation from both sides of the correct functional with respect to y_n and imposing $\tilde{y}_{n+1} = 0$, the stationary conditions are obtained. By using the stationary conditions the optimal value of the λ can be identified.

The successive approximation y_k ($k \geq 1$) can be established by determining a general lagrangian multiplier λ and initial solution y_0 . Since this procedure avoids the discretization of the problem, it is possible to find the closed form solution without any round off error.

In the case of m equations, the equations are rewritten in the form of:

$$L_i(y_i) + N_i(y_1, \dots, y_m) = g_i(x), i = 1, \dots, m, \quad (5)$$

where L_i is a linear with respect to y_i , and N_i is nonlinear part of the i th equation. In this case the correct functionals are produced as

$$y_{i(n+1)} = y_{in} + \int_0^x \lambda_i (L_i(y_{in}(s)) + N(\tilde{y}_{1n}(s), \dots, \tilde{y}_{mn}(s)) - g(s)) ds, \quad (6)$$

and the optimal values of the $\lambda_i, i = 1, \dots, m$ are obtained by taking the variation from both sides of the correct functionals and finding stationary conditions using $\delta y_{i(n+1)} = 0, i = 1, \dots, m$.

3. Statement of the Problem

3.1. Moving Boundary Problems

The necessary condition for the solution of problem (1) is to satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0, \quad (7)$$

The general form of the variational problem (1) is

$$v[y_1, y_2, \dots, y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n) dx, \quad (8)$$

Here, the necessary condition for the extremum of the functional (8) is to satisfy the following system of second-order differential equations

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \frac{\partial F}{\partial y'_i} = 0, i = 1, 2, \dots, n \quad (9)$$

In fixed boundary problems, Euler-Lagrange equation must be considered by the boundary conditions, but for

the problems with variable boundaries, Euler-Lagrange equation must satisfy natural boundary conditions or transversality conditions which will be described in the following theorems.

For the problems with variable boundaries, we have two cases:

Type 1: As the first case, those problems are considered for which at least one of the boundary points move freely along a line parallel to the y -axis. Indeed at this point $y(x)$ is not specified. In this case all admissible functions have the same domain $[x_0, x_1]$ and satisfy the Euler-Lagrange equation in this interval. Furthermore such functions have to satisfy conditions called natural boundary conditions stated in the following theorem.

Theorem 3.1. Suppose the function $y = y(x)$ in $C^1[x_0, x_1]$, yields a relative minimum of the functional (1) that for which $y(x_0) = y_0, y(x_1) = y_1$ is arbitrary (free right endpoint) and $y(x_0), y(x_1)$ are arbitrary (free endpoints). Then $y_0(x)$ satisfies, the following natural boundary conditions, respectively:

$$\frac{\partial F}{\partial y'}(x_1, y_0(x_1), y'_0(x_1)) = 0 \quad (10)$$

or

$$\begin{aligned} & \frac{\partial F}{\partial y'}(x_0, y_0(x_0), y'_0(x_0)) \\ & = \frac{\partial F}{\partial y'}(x_1, y_0(x_1), y'_0(x_1)) = 0 \end{aligned} \quad (11)$$

Type 2: For the second case, the beginning and end points (or only one of them) can move freely on given curves $y = \varphi(x), y = \psi(x)$. In this case, a function $y(x)$ is required, which emanates at some $x = x_0$ from the curve $y = \varphi(x)$ and terminates for some $x = x_1$ on the curve $y = \psi(x)$ and minimizes the functional (1). In this problem, the points x_0, x_1 are not known, and they must satisfy the necessary conditions called transversality conditions, described in the following theorem.

Theorem 3.2. If the function $y = y_0(x) \in C^1[x_0, x_1]$, which emanates at some $x = x_0$ from the curve $y = \varphi(x) \in C^1(-\infty, +\infty)$ and terminates for some $x = x_1$ on the curve $y = \psi(x) \in C^1(-\infty, +\infty)$, yields a relative minimum for functional (1), where $F \in C^1(R)$, R being a domain in the (x, y, y') space that contains all lineal elements of $y = y_0(x)$, then it is necessary that $y = y_0(x)$ to satisfy the Euler-Lagrange equation in the interval $[x_0, x_1]$ and at the point of exit and the point of entrance, the following transversality conditions to be satisfied:

$$\begin{aligned} & \frac{\partial F}{\partial y'}(x_0, y_0(x_0), y'_0(x_0))(\psi'(x_0) - y'_0(x_0)) \\ & + F(x_0, y_0(x_0), y'_0(x_0)) = 0 \end{aligned} \quad (12)$$

$$\begin{aligned} & \frac{\partial F}{\partial y'}(x_1, y_0(x_1), y'_0(x_1))(\varphi'(x_1) - y'_0(x_1)) \\ & + F(x_1, y_0(x_1), y'_0(x_1)) = 0 \end{aligned} \tag{13}$$

In the case that one of the points is fixed, then the transversality condition has to be held at the other point. One can consider transversality conditions for the problems with more than one unknown functions. For example, in to minimize two dimensional case, a vector function $y(x) = (y_1(x), y_2(x))$ is looked for such that

$$v[y_1, y_2] = \int_{x_0}^{x_1} F(x, y_1, y_2, y'_1, y'_2) dx, \tag{14}$$

in which $y_1(x_0) = y_{1,x_0}, y_2(x_0) = y_{2,x_0}$ and the endpoint lies on a two-dimensional surface that is given by $x = u(y_1, y_2)$. Here the transversality conditions at $x = x_1$ are:

$$\left(\frac{\partial u}{\partial y_1} F + \left(1 - \frac{\partial u}{\partial y_1} y'_1 - \frac{\partial u}{\partial y_2} y'_2 \right) \frac{\partial u}{\partial y'_1} F \right) (x_1) = 0, \tag{15}$$

$$\left(\frac{\partial u}{\partial y_2} F + \left(1 - \frac{\partial u}{\partial y_1} y'_1 - \frac{\partial u}{\partial y_2} y'_2 \right) \frac{\partial u}{\partial y'_2} F \right) (x_1) = 0, \tag{16}$$

In which $(y_1^0(x), y_2^0(x))$ is an admissible vector function.

For further information on transversality conditions, specially for the proofs of Theorems 3.1 and 3.2 and conditions (15), (16), see [2].

Example 3.1. Consider the following functional:

$$J[y] = \int_0^T a(by(t) - y'(t) - c^*)^2 dt, \tag{17}$$

In which $a, b > 0, c^* > 0$ and $y(t)$ is the amount of a capital at time t (see [1]).

Here, the capital stock $y(0)$ at the initial time $t = 0$ of the planning period is assumed to be known: $y(0) = y_0$; on the other hand, the planner won't wish to explain how large the capital would be at time $t = T$. Therefore, there is a variational problem with free right endpoint. Here we let $a = b = c^* = 1, T = 1$, and $y_0 = 2$ which has the analytical solution $y(t) = 1 + e^t$. The corresponding Euler-Lagrange equation is:

$$y''(t) - y(t) + 1 = 0.$$

Now natural boundary condition at $t = 1$ is as following:

$$\frac{\partial f}{\partial y'}(1, y(1), y'(1)) = -2(y(1) - y'(1) - 1)_{t=1} = 0$$

Therefore, the following boundary conditions are:

$$y(0) = 2, y(1) - y'(1) - 1 = 0. \tag{18}$$

By using variational iterative method we consider the

following functional is considered:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(y''_n(z) - y_n(z) + 1) dz,$$

Taking the variation from both sides of the correct functional with respect to y_n given:

$$\begin{aligned} \delta y_{n+1}(t) &= \delta y_n(t) + \delta \int_0^t \lambda(y''_n(z) - y_n(z) + 1) dz \\ &= (\delta y_n(t) + \lambda(z) \delta y'_n(z))_{z=t} - (\lambda'(z) \delta y_n(z))_{z=t} \\ &+ \int_0^t ((\lambda'' - \lambda) \delta y_n(z)) dz = 0 \end{aligned}$$

For all variations δy_n and $\delta y'_n$. The following stationary conditions are obtained:

$$\begin{aligned} \lambda''(z) - \lambda(z) &= 0, \\ (\lambda(z))_{z=t} &= 0 \\ (1 - \lambda'(z))_{z=t} &= 0 \end{aligned}$$

So that $\lambda(z) = \frac{1}{2} e^{z-t} - \frac{1}{2} e^{t-z}$. Therefore iterative formula can be found as:

$$\begin{aligned} y_{n+1}(t) &= y_n(t) \\ &+ \int_0^t \left(\frac{1}{2} e^{z-t} - \frac{1}{2} e^{t-z} \right) (y''_n(z) - y_n(z) + 1) dz, \end{aligned}$$

If $y_0 = Ae^t + Be^{-t}$, then

$$\begin{aligned} y_1(t) &= Ae^t + Be^{-t} + \int_0^t \left(\frac{1}{2} e^{z-t} - \frac{1}{2} e^{t-z} \right) (+1) dz \\ &= Ae^t + Be^{-t} + \frac{1}{2} (e^{z-t} + e^{t-z})^t \\ &= \left(A - \frac{1}{2} \right) e^t + \left(B - \frac{1}{2} \right) e^{-t} + 1 \end{aligned}$$

By imposing (18) $A = \frac{3}{2}, B = \frac{1}{2}$ are resulted. Which yields the exact solutions of the problem (see **Figure 1**).

Example 3.2. We want to find the shortest distance from the point $A(1,1,1)$ to the sphere

$$x^2 + y^2 + z^2 = 1$$

This problem is reduced to optimize the following functional:

$$J[y, z] = \int_{x_1}^1 \sqrt{1 + y'^2(x) + z'^2(x)} dx \tag{19}$$

where the point $B(x_1, y_1, z_1)$ must lie on the sphere, with the exact solution $y_1 = x, z_1 = x$, see [33]. The corresponding Euler Lagrange equations for this problem

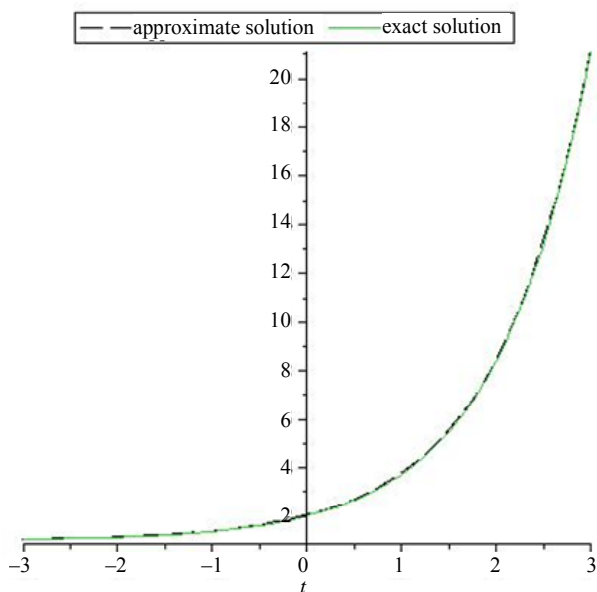


Figure 1. The graphs of approximated and exact solution for Example 3.1.

are:

$$\frac{d}{dx} \left(\frac{y'}{\sqrt{1+y'^2(x)+z'^2(x)}} \right) = 0,$$

$$\frac{d}{dx} \left(\frac{z'}{\sqrt{1+y'^2(x)+z'^2(x)}} \right) = 0.$$

So that

$$\frac{y'}{\sqrt{1+y'^2(x)+z'^2(x)}} = e, \quad \frac{z'}{\sqrt{1+y'^2(x)+z'^2(x)}} = f.$$

In above equations “e” and “f” are constant, so they can be rewritten as:

$$y' - e\sqrt{1+y'^2(x)+z'^2(x)} = 0,$$

$$z' - f\sqrt{1+y'^2(x)+z'^2(x)} = 0.$$

The transversality conditions are:

$$\left[\sqrt{1+y'^2+z'^2} - \frac{y'^2}{\sqrt{1+y'^2+z'^2}} + \left(\frac{-x}{\sqrt{1-x^2-y^2}} - z' \right) \frac{z'}{\sqrt{1+y'^2+z'^2}} \right]_{x=x_1} = 0 \tag{20}$$

$$\left[\frac{y'}{\sqrt{1+y'^2+z'^2}} + \frac{z'}{\sqrt{1+y'^2+z'^2}} \cdot \frac{-y}{\sqrt{1-x^2-y^2}} \right]_{x=x_1} = 0 \tag{21}$$

By using variational iteration method results:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda_1 \left(y'_n(s) - e\sqrt{1+\tilde{y}_n'^2(s)+\tilde{z}_n'^2(s)} \right) ds$$

and

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda_2 \left(z'_n(s) - f\sqrt{1+\tilde{y}_n'^2(s)+\tilde{z}_n'^2(s)} \right) ds$$

The variation from both sides of above equations for finding the optimal value of λ is:

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda_1 y'_n(s) ds$$

$$= \delta y_n(x) + [\lambda_1 \delta y_n(s)]_{s=x} - \int_0^x \lambda_1' \delta y_n(s) ds = 0$$

and

$$\delta z_{n+1}(x) = \delta z_n(x) + \delta \int_0^x \lambda_2 z'_n(s) ds$$

$$= \delta z_n(x) + [\lambda_2 \delta z_n(s)]_{s=x} - \int_0^x \lambda_2' \delta z_n(s) ds = 0$$

Therefore

$$[1 + \lambda_1(s)]_{s=x} = 0, \quad [\lambda_1'(s)]_{s=x} = 0.$$

and

$$[1 + \lambda_2(s)]_{s=x} = 0, \quad [\lambda_2'(s)]_{s=x} = 0$$

which yields:

$$\lambda_1(s) = -1, \lambda_2(s) = -1.$$

So that the following iterative formulas are obtained:

$$y_{n+1}(x) = y_n(x) + \int_0^x (-1) \left(y'_n(s) - e\sqrt{1+\tilde{y}_n'^2(s)+\tilde{z}_n'^2(s)} \right) ds$$

$$z_{n+1}(x) = z_n(x) + \int_0^x (-1) \left(z'_n(s) - f\sqrt{1+\tilde{y}_n'^2(s)+\tilde{z}_n'^2(s)} \right) ds$$

If $y_0(x) = ax + b, z_0(x) = cx + d$ then we have:

$$y_1 = ax + b - \int_0^x \left(a - e\sqrt{1+a^2+c^2} \right) ds$$

$$= e\sqrt{1+a^2+c^2}x + b$$

and

$$z_1 = cx + d - \int_0^x \left(c - f\sqrt{1+a^2+c^2} \right) ds$$

$$= f\sqrt{1+a^2+c^2}x + d$$

By choosing $(x_0, y_0, z_0) = (1, 1, 1)$,

$$y_1 = (-b + 1)x + b, z_1 = (-d + 1)x + d.$$

Imposing (20) and (21) lead to, $b = 0, d = 0, x_1 = \frac{\sqrt{3}}{3}$.
therefore:

$$y_1 = x, z_1 = x.$$

which is the exact solution.

3.2. Isoperimetric Problems

Assume that two functions $G(x, y, y')$ and $F(x, y, y')$ are given. Among all curves $y = y(x) \in C^1[x_0, x_1]$ along which the functional

$$K[y] = \int_{x_0}^{x_1} G(x, y, y') dx$$

assumes a given value l , determine the one for which the functional

$$J[y] = \int_{x_0}^{x_1} F(x, y, y') dx$$

Gives an external value. Suppose that F and G have continuous first and second partial derivatives for $x_0 \leq x \leq x_1$ and for arbitrary values of the variables y and y' .

Euler's theorem: If a curve $y = y(x)$ extremizes the functional $J[y] = \int_{x_0}^{x_1} F(x, y, y') dx$ under the conditions

$$K[y] = \int_{x_0}^{x_1} G(x, y, y') dx = l,$$

$$y(x_0) = y_0, y(x_1) = y_1$$

and if $y = y(x)$ is not an extremal of the functional K , there exists a constant λ such that the curve $y = y(x)$ is an extremal of the functional

$$L = \int_{x_0}^{x_1} [F(x, y, y') + \lambda G(x, y, y')] dx$$

The necessary condition for the solution of this problem is to satisfy the Euler-Lagrange equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx} \frac{\partial H}{\partial y'} = 0$$

with given boundary conditions in which $H = F + \lambda G$ for further information (see [2]).

Example 3.3. It is aimed to find the minimum of the functional

$$J[y] = \int_0^\pi y'^2(x) dx \tag{22}$$

Such that

$$\int_0^\pi y^2(x) dx = 1 \tag{23}$$

and

$$y(0) = 0, y(\pi) = 0 \tag{24}$$

With exact solution $y(x) = \pm \sqrt{\frac{2}{\pi}} \sin x$ [19]. According to the following auxiliary functional:

$$L = \int_0^1 (y'^2 + \lambda y^2) dx$$

and the corresponding Euler-Lagrange equation:

$$2\lambda y - \frac{d}{dx}(2y') = 0$$

so

$$y'' - \lambda y = 0$$

By applying He's variational iterative method results

$$y_{n+1}(x) = y_n(x) + \int_0^x (\gamma(y_n''(s) - \lambda y_n(s)) ds$$

To find the optimal value of λ following equation is required:

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) + [\gamma(\delta y_n'(s))]_{s=x} \\ -[\gamma'(\delta y_n(s))]_{s=x} &+ \int_0^x \delta y_n''(s) - \gamma \lambda \delta y_n(s) ds = 0 \end{aligned}$$

Therefore, the stationary conditions are obtained in the following form:

$$\begin{aligned} [1 - \gamma']_{s=x} &= 0, \\ [\gamma(s)]_{s=x} &= 0, \\ [\gamma'' - \gamma \lambda]_{s=x} &= 0. \end{aligned}$$

which yields

$$\gamma = s - x$$

and the desired sequence is

$$y_{n+1}(x) = y_n(x) + \int_0^x (s - x)(y_n''(s) - \lambda y_n(s)) ds$$

By choosing $y_0 = a \sin(cx) + b \cos(cx)$

$$y_1(x) = a \sin(cx) + b \cos(cx)$$

$$\begin{aligned} &+ \int_0^x (s - x)((-ac^2 - \lambda a) \sin(cs) + (-bc^2 - \lambda b) \cos(cs)) ds \\ &= -\frac{\lambda a \sin(cx)}{c^2} - \frac{\lambda b \cos(cx)}{c^2} + b + \frac{\lambda b}{c^2} + acx + \frac{\lambda ax}{c} \end{aligned}$$

Imposing (24) on this function given

$$b = 0, y_1(x) = -\frac{\lambda a \sin(cx)}{c^2} + acx + \frac{\lambda ax}{c} \tag{25}$$

If $\lambda = 0$ then from (24) $ac = 0$, but from (23)

$$ac = \sqrt{\frac{3}{\pi^3}}, \text{ which is a contradiction.}$$

Now imposing (24), we have: $\lambda = -\frac{c^3\pi}{-\sin(c\pi) + c\pi}$

so $\lambda < 0$. and it is known that in this case imposing (24) on the Euler Lagrange equation yields

$$c = \sqrt{-\lambda}, -\lambda = k^2 (k = 1, 2, \dots)$$

Hence:

$$y(x) = a \sin kx$$

and from (23) $a = \pm\sqrt{\frac{2}{\pi}}$. But y must be extremal when $0 \leq x \leq \pi$, therefore:

$$y(x) = \pm\sqrt{\frac{2}{\pi}} \sin x$$

As it is observed that this solution is equal to exact solution (see **Figure 2**).

Example 3.4. The objective is to find an extremum of the functional

$$J[y(x), z(x)] = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z) dx \quad (26)$$

Such that

$$\int_0^1 (y'^2 - xy' - z'^2) dx = 2 \quad (27)$$

and

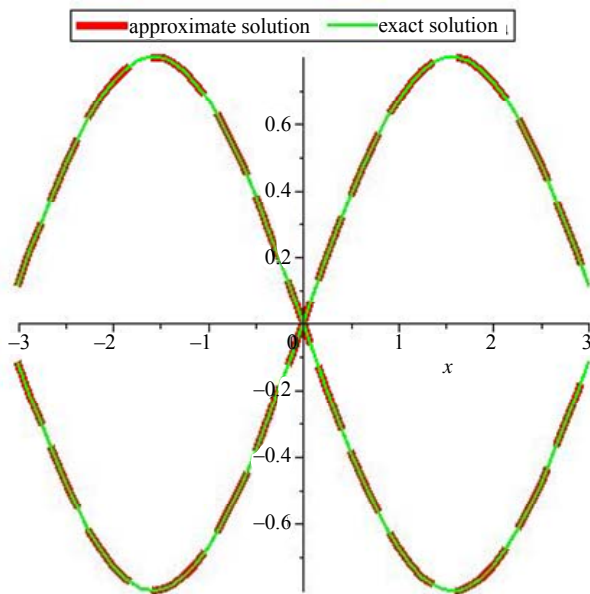


Figure 2. The graphs of approximated and exact solution for Example 3.3.

$$y(0) = 0, z(0) = 0, y(1) = 1, z(1) = 1. \quad (28)$$

With exact solution $(x) = \frac{7x - 5x^2}{2}$, $z(x) = x$, see [33]. By having the following auxiliary functional:

$$L = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z + \lambda(y'^2 - xy' - z'^2)) dx$$

The system of Euler-Lagrange equations is in the form:

$$\frac{d}{dx}(2y' + 2\lambda y' - \lambda x) = 0, \quad 4 + \frac{d}{dx}(2z' - 4x - 2\lambda z') = 0.$$

So

$$(2 + 2\lambda)y'' - \lambda = 0, (2 - 2\lambda)z'' = 0.$$

By using Homotopy variational iterative method gives:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda_1 ((2 + 2\lambda)y_n''(s) - \lambda) ds,$$

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda_2 ((2 - 2\lambda)z_n''(s)) ds.$$

Now

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) + \left[\lambda_1 ((2 + 2\lambda)\delta y_n'(s)) \right]_{s=x} \\ &\quad - \left[\lambda_1' ((2 + 2\lambda)\delta y_n(s)) \right]_{s=x} \\ &\quad + \int_0^x \lambda_1'' ((2 + 2\lambda)\delta y_n(s)) ds = 0. \end{aligned}$$

therefore

$$\begin{aligned} [1 - (2 + 2\lambda)\lambda_1']_{s=x} &= 0, \\ [\lambda_1(2 + 2\lambda)]_{s=x} &= 0, \\ [\lambda_1''(2 + 2\lambda)]_{s=x} &= 0. \end{aligned}$$

Hence

$$\lambda_1 = \frac{s - x}{2 + 2\lambda}$$

and

$$\begin{aligned} \delta z_{n+1}(x) &= \delta z_n(x) + \left[\lambda_2 ((2 - 2\lambda)\delta z_n'(s)) \right]_{s=x} \\ &\quad - \left[\lambda_2' ((2 - 2\lambda)\delta z_n(s)) \right]_{s=x} \\ &\quad + \int_0^x \lambda_2'' ((2 - 2\lambda)\delta z_n(s)) ds = 0 \end{aligned}$$

so

$$\begin{aligned} [1 - (2 - 2\lambda)\lambda_2']_{s=x} &= 0, \\ [\lambda_2(2 - 2\lambda)]_{s=x} &= 0, \\ [\lambda_2''(2 - 2\lambda)]_{s=x} &= 0. \end{aligned}$$

So λ_2 is obtained as:

$$\lambda_2 = \frac{s-x}{2-2\lambda}$$

and the following iterative equations are obtained:

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{s-x}{2+2\lambda} ((2+2\lambda)y_n''(s) - \lambda) ds,$$

$$z_{n+1}(x) = z_n(x) + \int_0^x \frac{s-x}{2-2\lambda} ((2-2\lambda)z_n''(s)) ds.$$

By choosing $y_0 = ax + b, z_0 = cx + d$:

$$\begin{aligned} y_1(x) &= ax + b + \int_0^x \frac{s-x}{2+2\lambda} (-\lambda) ds \\ &= ax + b + \frac{\lambda x^2}{4(1+\lambda)}, \end{aligned}$$

$$z_1 = cx + d$$

And by imposing (28) on this functions:

$$a = 1 - \frac{\lambda}{4(1+\lambda)}, \quad b = 0, \quad c = 1, \quad d = 0,$$

$$y_1(x) = \left(1 - \frac{\lambda}{4(1+\lambda)}\right)x + \frac{\lambda x^2}{4(1+\lambda)},$$

$$z_1 = x$$

from (27):

$$\lambda_1 = -\frac{10}{11}, \lambda_2 = -\frac{12}{11}$$

And consequently:

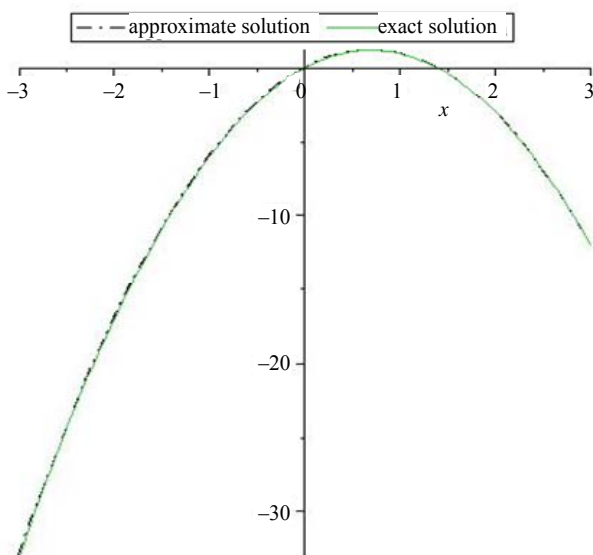


Figure 3. The graphs of approximated and exact solution for Example 3.4.

$$y(x) = \frac{7x-5x^2}{2}, z(x) = x.$$

which is the exact solution (see **Figure 3**).

4. Conclusion

The He’s variational iterative method is an efficient method for solving various kinds of problems. In this paper variational iterative method is employed for finding the minimum of a functional with moving boundaries and isoperimetric problems. Using He’s variational iterative method the solution of the problem is provided in a closed form. Since this method does not need to the discretize of the variables, there is no computational round off error. Moreover, only a few numbers of iterations are needed to obtain a satisfactory result.

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