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Higher-Order Duality for Minimax Fractional Type Programming Involving Symmetric Matrices

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Abstract

Convexity and generalized convexity play important roles in optimization theory. With the development of programming problem, there has been a growing interest in the higher-order dual problem and a lot of related generalized convexities are given. In this paper, we give the convexity of $(F,\alpha,\rho,d,b,\phi)_\beta$ vector-pseudoquasi-Type I and formulate a higher-order duality for minimax fractional type programming involving symmetric matrices, and give the weak, strong and strict converse duality theorems under the condition of higher-order $(F,\alpha,\rho,d,b,\phi)_\beta$ vector-pseudoquasi-Type I.

Keywords: Higher-Order $(F,\alpha,\rho,d,b,\phi)_\beta$ Vector-Pseudoquasi-Type I, Higher-Order Duality, Minimax Fractional Type Programming, Positive Semidefinite Symmetric Matrix

1. Introduction

In this paper, we focus on the following nondifferentiable minimax fractional programming problem:

$$\min_{x\in R^n}\sup_{y\in Y}\frac{f(x,y)+(x^TBx)^{\frac{1}{2}}}{h(x,y)-(x^TCx)^{\frac{1}{2}}} \qquad (P)$$

subject to $g(x)\le 0,\ x\in R^n,$

where Y is a compact subset of R^l, $f,h:R^n\times R^l\to R$ and $g:R^n\to R^m$ are continuously differentiable functions on $R^n\times R^l$ and R^n, respectively, and

$f(x,y)+(x^TBx)^{\frac{1}{2}}\ge 0,$

$h(x,y)-(x^TCx)^{\frac{1}{2}}>0,\ \forall(x,y)\in R^n\times R^l$, B and C are two positive semidefinite $n\times n$ symmetric matrices.

When $B=C=0$, (*P*) is a differentiable minimax fractional programming problem.

The duality of programming problem involving symmetric matrix has been investigated widely. Schmitendorf [1] established necessary and sufficient optimality conditions for a particular case of the following problem (P^*) under convexity conditions.

$$\min\sup_{y\in Y} f(x,y)+(x^TBx)^{\frac{1}{2}} \qquad (P^*)$$

subject to $g(x)\le 0.$

Under the optimality conditions of [1], Tanimoto [2] defined a first-order dual problem of (P^*), which generalized the duality theorems for convex minimax programming problems considered by Weir [3] and relaxed the convexity assumptions in the sufficient optimality of [1]. Mishra and Rueda [4] introduced generalized second-order type I functions and considered the minimax programming problem (P^*) involving those functions and established second-order duality theorems for problem (P^*). Husian, Anurag Jaysural and Ahmad [5] established two types of second-order dual models for problem (P^*), which extends some previously known results on minimax programming.

With the development of programming problem, there has been a growing interest in the higher-order dual problem. Mangasarian [6] first formulated a class of second- and higher-order dual problems for a nonlinear programming problem involving twice differentiable functions. In [7], Zhang considered the following nondifferentiable mathematical programming problem:

$$\text{Minimize}\ \ f(x)+(x^TBx)^{\frac{1}{2}} \qquad (P^{**})$$

subject to $g(x)\ge 0,$

under higher-order invexity assumptions. Mishra and Rueda [8] generalized the results of Zhang [7] to higher-order type I functions.

In [9], Ahmad, Husain and Sharma considered the nondifferentiable minimax programming problem (P^*), and formulated a unified higher-order dual of (P^*), and established weak, strong and strict converse duality theorems under higher-order (F,α,ρ,d)-Type I assumptions. In [10], Jayswal and Stancu-Minasian formulated the weak, strong and strict converse duality of (P^*) under generalized convexity of higher-order (F,α,ρ,d)-Type I.

For problem (*P*), H. C. Lai and K. Tanaka gave the necessary and sufficient conditions under the conditions of pseudo-convex, strictly pseudo-convex and quasi-convex [11].

In this paper, we will establish a higher-order dual of (P) and give the weak, strong and strict converse duality theorems under $(F,\alpha,\rho,d,b,\phi)_\beta$ vector-pseudoquasi-Type I assumptions. The convexity conditions in this paper generalized the convexity in [8], and hence, presents an answer of a question raised in [10].

2. Preliminaries

Let R^n be the *n*-dimensional Euclidean space, R_+^n be its nonegative orthant and X be an open subset of R^n. Let S be the set of all feasible solutions of (*P*). Denote $M=\{1,2,\cdots,m\}$. For each $(x,y)\in S\times Y$, we define

$$J(x)=\left\{j\in M: g_j(x)=0\right\}$$

$$Y(x)=\left\{y\in Y:\frac{f(x,y)+(x^TBx)^{\frac{1}{2}}}{h(x,y)-(x^TCx)^{\frac{1}{2}}}=\sup_{z\in Y}\frac{f(x,z)+(x^TBx)^{\frac{1}{2}}}{h(x,z)-(x^TCx)^{\frac{1}{2}}}\right\}$$

$$K(x)=\Big\{(s,t,\tilde{y})\in N\times R_+^s\times R^{ls}:1\le s\le n+1,\ t=(t_1,t_2,\cdots,t_s)\in R_+^s \text{ with } \sum_{i=1}^s t_i=1,\ \tilde{y}=(\bar{y}_1,\bar{y}_2,\cdots,\bar{y}_s) \text{ and } \bar{y}_i\in Y(x),\ i=1,2,\cdots,s\Big\}$$

Definition 1: A function $F:X\times X\times R^n\to R$ is said to be sublinear in its third argument, if $\forall x,\bar{x}\in X$,

1) (subadditivity) $\forall a_1,a_2\in R^n$,

$$F(x,\bar{x};a_1+a_2)\le F(x,\bar{x};a_1)+F(x,\bar{x};a_2);$$

2) (positive homogeneous) $\forall\ \alpha\in R_+,\ a\in R^n$,

$$F(x,\bar{x};\alpha a)=\alpha F(x,\bar{x};a).$$

Let $f:R^n\times R^l\to R$, $\rho=(\rho_1,\rho_2)\in R^2$, $\beta\in R^n$, $\alpha_1,\alpha_2:X\times X\to R_+\setminus\{0\}$, $b,d:X\times X\to R$, $e:R^n\to R$ and $\phi:R\to R$. Let $w:R^n\times R^l\times R^n\to R$, $l:R^n\times R^n\to$ and $k:R^n\times R^n\to R^m$ be three differentiable functions. We assume that F is a sublinear functional throughout this paper.

Definition 2: (f,e) is said to be higher-order $(F,\alpha,\rho,d,b,\phi)_\beta$ vector-pseudoquasi-Type I at $\bar{x}\in X$ with respect to $p\in R^n$, if for all $x\in S$ and $\bar{y}\in Y(x)$,

$$\begin{aligned}&b(x,\bar{x})\phi\Big[f(x,\bar{y})+x^T\beta-f(\bar{x},\bar{y})-\bar{x}^T\beta-w(\bar{x},\bar{y},p)\\&\qquad+p^T\nabla_p w(\bar{x},\bar{y},p)\Big]<0\\&\Rightarrow F(x,\bar{x},\alpha_1(x,\bar{x})(\nabla_p w(\bar{x},\bar{y},p)+\beta))\\&\quad<-\rho_1 d^2(x,\bar{x})\\&\quad-\Big[e(\bar{x})+l(\bar{x},p)-p^T\nabla_p l(\bar{x},p)\Big]\le 0\\&\Rightarrow F(x,\bar{x},\alpha_2(x,\bar{x})(\nabla_p l(\bar{x},p)))\le-\rho_2 d^2(x,\bar{x}).\end{aligned}$$

Definition 3: (f,e) is said to be strictly higher-order $(F,\alpha,\rho,d,b,\phi)_\beta$ vector-pseudoquasi-Type I at $\bar{x}\in X$ with respect to $p\in R^n$, if for all $\bar{x}\ne x\in S$ and $\bar{y}\in Y(x)$,

$$\begin{aligned}&b(x,\bar{x})\phi\Big[f(x,\bar{y})+x^T\beta-f(\bar{x},\bar{y})-\bar{x}^T\beta-w(\bar{x},\bar{y},p)\\&\qquad+p^T\nabla_p w(\bar{x},\bar{y},p)\Big]\le 0\\&\Rightarrow F(x,\bar{x},\alpha_1(x,\bar{x})(\nabla_p w(\bar{x},\bar{y},p)+\beta))\\&\quad<-\rho_1 d^2(x,\bar{x})\\&\quad-\Big[e(\bar{x})+l(\bar{x},p)-p^T\nabla_p l(\bar{x},p)\Big]\le 0\\&\Rightarrow F(x,\bar{x},\alpha_2(x,\bar{x})(\nabla_p l(\bar{x},p)))\le-\rho_2 d^2(x,\bar{x}).\end{aligned}$$

Obviously, when ϕ is subadditive function and satisfies $a\le 0\Rightarrow\phi(a)\le 0$, higher-order $(F,\alpha,\rho,d,b,\phi)_{Bu}$ vector-pseudoquasi-Type I is the convexity condition of Theorem 3.1 in [10].

In the following section, we will use Lemma 1 and Lemma 2 which were given in [11].

Lemma 1: (Necessary Condition) If x^* is an optimal solution of problem (*P*) satisfying $x^{*T}Bx^*>0, x^{*T}Cx^*>0$ and $\nabla g_j(x^*),\ j\in J(x^*)$ are linear independent, then there exist $(s^*,t^*,\bar{y}^*)\in K(x^*)$, $u^*,v^*\in R^n$ and $\mu^*\in R_+^m,\lambda^*\in R_+$ such that

$$\sum_{i=1}^{s^*}t_i^*\left\{\nabla f(x^*,\bar{y}_i^*)-\lambda^*(\nabla h(x^*,\bar{y}_i^*))+Bu^*+\lambda^*Cv^*\right\}+\nabla\sum_{j=1}^m\mu_j^*g_j(x^*)=0 \tag{2.1}$$

$$f(x^*,\bar{y}_i^*)+(x^{*T}Bx^*)^{\frac{1}{2}}-\lambda^*h(x^*,\bar{y}_i^*)+\lambda^*(x^{*T}Cx^*)^{\frac{1}{2}}=0,\quad i=1,2,\cdots,s^* \tag{2.2}$$

$$\sum_{i=1}^m\mu_j^*g_j(x^*)=0 \tag{2.3}$$

$$\sum_{j=1}^{s^*} t_i^* = 1, t_i^* \geq 0, i = 1, 2, \cdots, s^* \tag{2.4}$$

$$\begin{cases} u^{*T} Bu^* \leq 1, v^{*T} Cv^* \leq 1, \\ x^{*T} Bu^* = (x^{*T} Bx^*)^{\frac{1}{2}}, \\ x^{*T} Cv^* = (x^{*T} Bx^*)^{\frac{1}{2}}. \end{cases} \tag{2.5}$$

Lemma 2: Let B be a positive semidefinite symmetric matrix of order n. Then for all $x, u \in R^n$,

$$x^T Bu \leq (x^T Bx)^{\frac{1}{2}} \times (u^T Bu)^{\frac{1}{2}}.$$

The equality holds when $Bx = \gamma Bu$ for some $\gamma \geq 0$.

Evidently, if $(u^T Bu)^{\frac{1}{2}} \leq 1$, we have

$$x^T Bu \leq (x^T Bx)^{\frac{1}{2}}.$$

3. Duality Model

We consider the following dual model (WD).

$$\max_{(s,t,\tilde{y}) \in K(z)} \sup_{(z,u,v,\lambda,\mu,p) \in H(s,t,\tilde{y})} \lambda,$$

where $H(s,t,\tilde{y})$ denote the set of all $(z,u,v,\lambda,\mu,p) \in R^n \times R^n \times R^n \times R_+ \times R_+^m \times R^n$ satisfying

$$\sum_{i=1}^{s} t_i \nabla_p w(z, \overline{y}_i, p) + Bu + \lambda Cv + \sum_{j=1}^{m} \nabla_p (\mu_j k_j(z,p)) = 0 \tag{3.11}$$

$$u^T Bu \leq 1, v^T Cv \leq 1, \tag{3.12}$$

$$\begin{aligned} &\sum_{i=1}^{s} t_i \left\{ f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i) \right\} + z^T Bu + \lambda z^T Cv, \\ &+ \sum_{i=1}^{s} t_i \left[w(z,\overline{y}_i,p) - p^T \nabla_p w(z,\overline{y}_i,p) \right] \geq 0, \end{aligned} \tag{3.13}$$

$$\sum_{j=1}^{m} \left\{ \mu_j g_j(z) + \mu_j k_j(z,p) - p^T \nabla_p (\mu_j k_j(z,p)) \right\} \geq 0. \tag{3.14}$$

If for a triplet $(s,t,\tilde{y}) \in K(z)$, the set $H(s,t,\tilde{y}) = \varnothing$, then we define the supremum over $H(s,t,\tilde{y})$ to be $-\infty$.

Next, we establish the duality of type (WD).

Theorem 3.1 (Weak Duality) Let x and $(z,u,v,\lambda,\mu,s,t,\tilde{y},p)$ be feasible solutions of (P) and (WD), respectively. Assume that

1) $\left[\sum_{i=1}^{s} t_i \left(f(\cdot,\overline{y}_i) - \lambda h(\cdot,\overline{y}_i) \right), \sum_{j=1}^{m} \mu_j g_j(\cdot) \right]$ is higher-order $(F,\alpha,\rho,d,b,\phi)_{Bu+\lambda Cv}$ vector-pseudoquasi Type I at z,

2) $\phi(a) \geq 0 \Rightarrow a \geq 0, \quad b(x,z) > 0,$

$$\frac{\rho_1}{\alpha_1(x,z)} + \frac{\rho_2}{\alpha_2(x,z)} \geq 0.$$

Then

$$\sup_{y \in Y} \frac{f(x,y) + (x^T Bx)^{\frac{1}{2}}}{h(x,y) - (x^T Cx)^{\frac{1}{2}}} \geq \lambda.$$

Proof: From (3.14), we know that

$$-\left[\sum_{j=1}^{m} \left\{ \mu_j g_j(z) + \mu_j k_j(z,p) - p^T \nabla_p (\mu_j k_j(z,p)) \right\} \right] \leq 0,$$

then follows form 1) and $\alpha_2(x,z) > 0$, we have

$$F\left(x,z; \sum_{j=1}^{m} \nabla_p (\mu_j k_j(z,p)) \right) \leq -\frac{\rho_2}{\alpha_2(x,z)} d^2(x,z).$$

Since F is sublinear in its third argument, by (3.11) we can get

$$\begin{aligned} 0 &= F\left(x,z; \sum_{i=1}^{s} t_i \nabla_p w(z,\overline{y}_i,p) + Bu \right. \\ &\quad \left. + \lambda Cv + \sum_{j=1}^{m} \nabla_p (\mu_j k_j(z,p)) \right) \\ &= F\left(x,z; \sum_{i=1}^{s} t_i \nabla_p w(z,\overline{y}_i,p) + Bu + \lambda Cv \right) \\ &\quad + F\left(x,z; \sum_{j=1}^{m} \nabla_p (\mu_j k_j(z,p)) \right) \\ &\leq F\left(x,z; \sum_{i=1}^{s} t_i \nabla_p w(z,\overline{y}_i,p) + Bu + \lambda Cv \right) \\ &\quad - \frac{\rho_2}{\alpha_2(x,z)} d^2(x,z). \end{aligned}$$

Furthermore, by $\frac{\rho_1}{\alpha_1(x,z)} + \frac{\rho_2}{\alpha_2(x,z)} \geq 0$ and $\alpha_1(x,z) > 0,$ we have

$$\begin{aligned} &F\left(x,z; \alpha_1(x,z) \left(\sum_{i=1}^{s} t_i \nabla_p w(z,\overline{y}_i,p) + Bu + \lambda Cv \right) \right) \\ &\geq -\rho_1 d^2(x,z), \end{aligned}$$

which implies that

$$\begin{aligned} b(x,z)\phi\Bigg(&\left(\sum_{i=1}^{s} t_i \left\{ f(x,\overline{y}_i) - \lambda h(x,\overline{y}_i) \right\} + x^T Bu + \lambda x^T Cv \right) \\ &- \left(\sum_{i=1}^{s} t_i \left\{ f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i) \right\} + z^T Bu + \lambda (z^T Cv) \right) \\ &- \sum_{i=1}^{s} t_i \left[w(z,\overline{y}_i,p) - p^T \nabla_p w(z,\overline{y}_i,p) \right] \Bigg) \\ &\geq 0. \end{aligned}$$

From 2) and (3.13), we can get

$$\sum_{i=1}^{s} t_i \left\{ f(x,\bar{y}_i) - \lambda h(x,\bar{y}_i) \right\} + x^T Bu + \lambda x^T Cv$$
$$\geq \sum_{i=1}^{s} t_i \left\{ f(z,\bar{y}_i) - \lambda h(z,\bar{y}_i) \right\} + z^T Bu + \lambda (z^T Cv)$$
$$+ \sum_{i=1}^{s} t_i \left[w(z,\bar{y}_i,p) - p^T \nabla_p w(z,\bar{y}_i,p) \right]$$
$$\geq 0.$$

Therefore, following from (3.12) and Lemma 2,

$$\sum_{i=1}^{s} t_i \left[f(x,\bar{y}_i) + (x^T Bx)^{\frac{1}{2}} - \lambda \left(h(x,\bar{y}_i) - (x^T Cx)^{\frac{1}{2}} \right) \right]$$
$$\geq \sum_{i=1}^{s} t_i \left\{ f(x,\bar{y}_i) - \lambda h(x,\bar{y}_i) \right\} + x^T Bu + \lambda x^T Cv$$
$$\geq 0.$$

Since $t = (t_1, t_2, \cdots, t_s) \neq 0$, $t_i \geq 0, i = 1, 2, \cdots, s$, $\bar{y}_i \in Y$ and $h(x,y) - (x^T Cx)^{\frac{1}{2}} > 0, \forall (x,y) \in R^n \times R^l$, at least exists one $q \in \{1, 2, \cdots, s\}$, such that

$$\frac{f(x,\bar{y}_q) + (x^T Bx)^{\frac{1}{2}}}{h(x,\bar{y}_q) - (x^T Cx)^{\frac{1}{2}}} \geq \lambda,$$

which implies that

$$\sup_{y \in Y} \frac{f(x,y) + (x^T Bx)^{\frac{1}{2}}}{h(x,y) - (x^T Cx)^{\frac{1}{2}}} \geq \lambda.$$

Theorem 3.2 (Strong duality) Let x^* be an optimal solution of (P) satisfying $x^{*T} Bx^* > 0, x^{*T} Cx^* > 0$ and let $\nabla g_j(x^*), j \in J(x^*)$ be linear independent. Assume that for any $i = 1, 2, \cdots, s^*$

$$w(x^*, \bar{y}_i^*, 0) = 0,$$
$$\nabla_p w(x^*, \bar{y}_i^*, 0) = \nabla f(x^*, \bar{y}_i^*) - \lambda^* \left(\nabla h(x^*, \bar{y}_i^*) \right),$$

and for any $j \in J(x^*)$

$$k_j(x^*, 0) = 0, \qquad \nabla_p k_j(x^*, 0) = \nabla g_j(x^*).$$

Then there exist $(s^*, t^*, \tilde{y}^*) \in K(x^*)$ and $(x^*, u^*, v^*, \lambda^*, \mu^*, p^*) \in H(s^*, t^*, \tilde{y}^*)$ such that $(x^*, u^*, v^*, \lambda^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is a feasible solution of (WD) and the two objectives have the same values. Furthermore, if the assumptions of weak duality hold for all feasible solutions of (*P*) and (WD), then $(x^*, u^*, v^*, \lambda^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is an optimal solutions of (WD).

Proof: Since x^* is an optimal solution of (P) satisfying $x^{*T} Bx^* > 0, x^{*T} Cx^* > 0$ and $\nabla g_j(x^*), j \in J(x^*)$ is linearly independent, by Lemma 1, there exist $(s^*, t^*, \tilde{y}^*) \in K(x^*)$, $u^*, v^* \in R^n$ and $\mu^* \in R_+^m, \lambda^* \in R_+$ such that

$$\sum_{i=1}^{s^*} t_i^* \left\{ \nabla f(x^*, \bar{y}_i^*) - \lambda^* \left(\nabla h(x^*, \bar{y}_i^*) \right) + Bu^* + \lambda^* Cv^* \right\} + \nabla \sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0 \tag{2.1}$$

$$f(x^*, \bar{y}_i^*) + (x^{*T} Bx^*)^{\frac{1}{2}} - \lambda^* h(x^*, \bar{y}_i^*) + \lambda^* (x^{*T} Cx^*)^{\frac{1}{2}} = 0, \quad i = 1, 2, \cdots, s^* \tag{2.2}$$

$$\sum_{i=1}^{m} \mu_j^* g_j(x^*) = 0 \tag{2.3}$$

$$\sum_{j=1}^{s^*} t_i^* = 1, t_i^* \geq 0, i = 1, 2, \cdots, s^* \tag{2.4}$$

$$\begin{cases} u^{*T} Bu^* \leq 1, v^{*T} Cv^* \leq 1, \\ x^{*T} Bu^* = (x^{*T} Bx^*)^{\frac{1}{2}}, \\ x^{*T} Cv^* = (x^{*T} Bx^*)^{\frac{1}{2}}. \end{cases} \tag{2.5}$$

By (2.1) (2.2) (2.3) (2.5) and the conditions of theorem 3.2, we know that $(x^*, u^*, v^*, \lambda^*, \mu^*, p^* = 0) \in H(s^*, t^*, \tilde{y}^*)$, that is $(x^*, u^*, v^*, \lambda^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is an feasible solutions of (WD). Furthermore, (3.2) implies that $(x^*, u^*, v^*, \lambda^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is an optimal solutions of (WD).

Theorem 3.3 (Strict Converse Duality) Let x^* be a feasible solution of (*P*) and $(z^*, u^*, v^*, \lambda^*, \mu^*, p^*)$ be a feasible solution of (WD). Suppose that

1)'

$$\sup_{y^* \in Y} \frac{f(x^*, y^*) + (x^{*T} Bx^*)^{\frac{1}{2}}}{h(x^*, y^*) - (x^{*T} Cx^*)^{\frac{1}{2}}} = \lambda^*;$$

2)' $\left[\sum_{i=1}^{s^*} t_i^* \left(f(\cdot, \bar{y}_i^*) - \lambda^* h(\cdot, \bar{y}_i^*) \right), \sum_{j=1}^{m} \mu_j^* g_j(\cdot) \right]$ is strictly higher-order $(F, \alpha, \rho, d, b, \phi)_{Bu^* + \lambda^* Cv^*}$ vector-pseudoquasi-Type I at z^* and;

3)' $\phi(a) > 0 \Rightarrow a > 0$, $b(x^*, z^*) > 0$,

$$\frac{\rho_1}{\alpha_1(x^*, z^*)} + \frac{\rho_2}{\alpha_2(x^*, z^*)} \geq 0$$

Then

$$z^* = x^*;$$

that is z^* is an optimal solution of (WD).

Proof: Suppose that the contradiction is not true, that is $z^* \neq x^*$. Similar to the proof of Theorem 3.1 we obtain

$$F\left(x^*,z^*;\alpha_1(x^*,z^*)\left(\sum_{i=1}^{s^*}t_i^*\nabla_p w(z^*,\bar{y}_i^*,p^*)+Bu^*+\lambda^*Cv^*\right)\right)$$
$$\geq -\rho_1 d^2(x^*,z^*),$$

which implies that

$$b(x^*,z^*)\phi\Bigg(\Bigg(\sum_{i=1}^{s^*}t_i^*\left\{f(x^*,\bar{y}_i^*)-\lambda^*h(x^*,\bar{y}_i^*)\right\}$$
$$+x^{*T}Bu^*+\lambda^*x^{*T}Cv^*\Bigg)$$
$$-\Bigg(\sum_{i=1}^{s^*}t_i^*\left\{f(z^*,\bar{y}_i^*)-\lambda^*h(z^*,\bar{y}_i^*)\right\}$$
$$+z^{*T}Bu^*+\lambda^*(z^{*T}Cv^*)\Bigg)$$
$$-\sum_{i=1}^{s^*}t_i^*\left[w(z^*,\bar{y}_i^*,p^*)-p^{*T}\nabla_p w(z^*,\bar{y}_i^*,p^*)\right]\Bigg)$$
$$>0.$$

From 3)' and (3.13), we can get

$$\sum_{i=1}^{s^*}t_i^*\left\{f(x^*,\bar{y}_i^*)-\lambda^*h(x^*,\bar{y}_i^*)\right\}+x^{*T}Bu^*+\lambda^*x^{*T}Cv^*$$
$$>\sum_{i=1}^{s^*}t_i^*\left\{f(z^*,\bar{y}_i^*)-\lambda^*h(z^*,\bar{y}_i^*)\right\}+z^{*T}Bu^*$$
$$+\lambda^*(z^{*T}Cv^*)+\sum_{i=1}^{s^*}t_i^*\left[w(z^*,\bar{y}_i^*,p^*)-p^{*T}\nabla_p w(z^*,\bar{y}_i^*,p^*)\right]$$
$$\geq 0.$$

Therefore, following from (3.12) and Lemma 2,

$$\sum_{i=1}^{s^*}t_i^*\left[f(x^*,\bar{y}_i^*)+(x^{*T}Bx^*)^{\frac{1}{2}}-\lambda^*\left(h(x^*,\bar{y}_i^*)-(x^{*T}Cx^*)^{\frac{1}{2}}\right)\right]$$
$$\geq\sum_{i=1}^{s^*}t_i^*\left\{f(x^*,\bar{y}_i^*)-\lambda^*h(x^*,\bar{y}_i^*)\right\}+x^{*T}Bu^*+\lambda^*x^{*T}Cv^*$$
$$>0.$$

Since $t^*=(t_1^*,t_2^*,\cdots,t_s^*)\neq 0$, $t_i^*\geq 0, i=1,2,\cdots,s^*$, $\bar{y}_i^*\in Y$ and $h(x^*,y^*)-(x^{*T}Cx^*)^{\frac{1}{2}}>0, \forall\,(x^*,y^*)\in R^n\times R^l$, at least exists one $q^*\in\{1,2,\cdots,s\}$, such that

$$\frac{f(x^*,\bar{y}_{q^*}^*)+(x^{*T}Bx^*)^{\frac{1}{2}}}{h(x^*,\bar{y}_{q^*}^*)-(x^{*T}Cx^*)^{\frac{1}{2}}}>\lambda^*,$$

which implies that

$$\sup_{y^*\in Y}\frac{f(x^*,y)+(x^{*T}Bx^*)^{\frac{1}{2}}}{h(x^*,y)-(x^{*T}Cx^*)^{\frac{1}{2}}}>\lambda^*$$

which contradicts with 1)'.

Remark: If we take place condition 2)' of this theorem by

2)" $\left[\sum_{i=1}^{s^*}t_i^*\left(f(\cdot,\bar{y}_i^*)-\lambda^*h(\cdot,\bar{y}_i^*)\right),\sum_{j=1}^{m}\mu_j^*g_j(\cdot)\right]$ is higher-order $(F,\alpha,\rho,d,b,\phi)_{Bu^*+\lambda^*Cv^*}$ vector-pseudoquasi-Type I at z^*, and take place condition 3)' by

3)" $\phi(a)\geq 0\Rightarrow a>0,\quad b(x^*,z^*)>0,$

$\dfrac{\rho_1}{\alpha_1(x^*,z^*)}+\dfrac{\rho_2}{\alpha_2(x^*,z^*)}\geq 0,$ the strict converse duality holds too.

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