Applied Mathematics, **2011, 2, 1356-1358**
doi:10.4236/am.2011.211189 Published Online November 2011 (http://www.SciRP.org/journal/am)

Scientific Research

On the Zeros of a Polynomial

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Received February 27, 2011; *revised May* 25, 2011; *accepted June* 3, 2011

Abstract

In this paper we consider the problem of finding the estimate of maximum number of zeros in a prescribed region and the results which we obtain generalizes and improves upon some well known results.

Keywords: Polynomial, Zeros, Complex Number, Prescribed Region

1. Introduction

Let $p(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0 > 0$$

then according to a well known result of Enstrom and Kakeya, the polynomial $p(z)$, does not vanish in $|z|>1$. concerning the number of zeros of the polynomial in the region $|z| \le \frac{1}{2}$, the following result is due to Mohammad [1].

Theorem A. Let $p(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0 > 0,$$

then the number of zeros of $p(z)$ in $|z| \le \frac{1}{2}$, does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_o}.$$

Dewan [2] generalized Theorem A to the polynomials with complex coefficients and obtained the following result.

Theorem B. If $p(z)=\sum_{i=0}^{n} a_i z^i$ is a polynomial of degree n with complex coefficients such that

$|\arg a_i - \beta| \le \alpha \le \frac{\pi}{2}, i=0,1,2,\cdots,n$ for some real β and

$$|a_n| \ge |a_{n-1}| \ge \cdots \ge |a_1| \ge |a_0|,$$

then the number of zeros of $p(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{i=0}^{n-1}|a_i|}{|a_0|}.$$

Theorem C. Let $p(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree *n* with complex coefficients. If $\operatorname{Re} a_i = \alpha_i$, $\operatorname{Im} a_i = \beta_i$, for $i=0,1,\cdots,n$ and $\alpha_n \ge \alpha_{n-1} \ge \cdots \ge \alpha_1 \ge \alpha_0 > 0$, then the number of zeros of $p(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{\alpha_n+\sum_{i=0}^{n}|\beta_i|}{|a_0|}.$$

In this paper we generalize Theorem B and Theorem C under less restrictive conditions on the coefficients, which also improve upon them. More precisely, we prove the following.

Theorem 1. Let $p(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with complex coefficients, such that

$|\arg a_i - \beta| \le \alpha \le \frac{\pi}{2}, i=0,1,2,\cdots,n.$ for some real β and

$$|a_n| \ge |a_{n-1}| \ge \cdots \ge |a_1| \ge |a_0| > 0$$

then the number of zeros of $p(z)$ in $|z|<\delta$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\left(\sum_{i=0}^{n-1}|a_i|\right)}{|a_0|}$$

where $0<\delta<1$.

Theorem 2. Let $p(z)=\sum_{i=0}^{n}a_iz^i$ be a polynomial of degree *n* with complex coefficients. If $\operatorname{Re}a_i=\alpha_i$, $\operatorname{Im}a_i=\beta_i$, for $i=0,1,\cdots,n$ and $\alpha_n\ge\alpha_{n-1}\ge\cdots\ge\alpha_1\ge\alpha_0>0$, $\alpha_n>0$, then the number of zeros of $p(z)$ in $|z|\le\delta, 0<\delta<1$ does not exceed

$$1+\frac{1}{\log\frac{1}{\delta}}\log\frac{\alpha_n+\sum_{i=0}^{n}|\beta_i|}{|a_0|}.$$

2. Lemma

We need the following lemma for proof of the theorems.

Lemma. Let $p(z)=\sum_{i=0}^{n}a_iz^i$ be a polynomial of degree n such that

$$|\arg a_i-\beta|\le\alpha\le\frac{\pi}{2};|a_i|\ge|a_{i-1}| \quad \text{for some} \quad i=0,1,2,\cdots,n,$$

then

$$|a_i-a_{i-1}|\le(|a_i|-|a_{i-1}|)\cos\alpha+(|a_i|+|a_{i-1}|)\sin\alpha.$$

The proof of the above lemma is omitted as it follows from the lemma in [3].

3. Proof of the Theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned}g(z)&=(1-z)p(z)=(1-z)\left(a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0\right)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n\\&\quad+(a_{n-1}-a_{n-2})z^{n-1}+\cdots+(a_1-a_0)z+a_0\end{aligned}$$

For $|z|\le1$, we have

$$\begin{aligned}g(z)&=|a_n|+\sum_{i=1}^{n}|a_i-a_{i-1}|+|a_0|\\&\le|a_n|+\sum_{i=1}^{n}(|a_i|-|a_{i-1}|)\cos\alpha+\sum_{i=1}^{n}(|a_i|+|a_{i-1}|)\sin\alpha+|a_0|\\&\qquad\qquad(\text{by using Lemma})\\&=|a_n|(\cos\alpha+\sin\alpha+1)+2\left(\sum_{i=0}^{n-1}|a_i|\right)\sin\alpha\\&\quad-|a_0|(\cos\alpha+\sin\alpha-1)\\&\le|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{i=0}^{n-1}|a_i|.\end{aligned}$$

If $f(z)$ is regular, $f(0)\ne0$ and $f(z)\le M$ in $|z|\le1$, then ([4], p.171) the number of zeros of $f(z)$ in $|z|\le\delta, 0<\delta<1$ does not exceed $\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|f(0)|}$.

Apply this result to $g(z)$ in $|z|\le\delta$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\left(\sum_{i=0}^{n-1}|a_i|\right)}{|a_0|}.$$

All the number of zeros of $p(z)$ in $|z|\le\delta$ is also equal to the number of zeros of $g(z)$ in $|z|\le\delta$. This completes proof of Theorem 1.

Proof of Theorem 2.

Consider

$$g(z)=(1-z)p(z)=a_nz^{n+1}+\sum_{i=1}^{n}(a_i-a_{i-1})z^i+a_0$$

For $|z|\le1$,

$$\begin{aligned}|g(z)|&\le|a_n|+\sum_{i=1}^{n}|a_i-a_{i-1}|+|a_0|\\&\le\alpha_n+|\beta_n|+\sum_{i=1}^{n}\{|\alpha_i-\alpha_{i-1}|+|\beta_i-\beta_{i-1}|\}+\alpha_0+|\beta_0|\\&\le\alpha_n+|\beta_n|+\sum_{i=1}^{n}(\alpha_i-\alpha_{i-1})+\sum_{i=1}^{n}(|\beta_i|+|\beta_{i-1}|)+|\beta_0|\\&=2\left(\alpha_n+\sum_{i=0}^{n}|\beta_i|\right)\end{aligned}$$

and using the same argument as in proof of Theorem 1, the proof of Theorem 2 follows.

Remark 1. For $\delta=\frac{1}{2}$, Theorem 1 is a refinement of Theorem B and for $\delta=\frac{1}{2}$, and $\alpha=\beta=0$, it gives a refinement of Theorem A.

Remark 2. Theorem C can be deduced as a particular case of Theorem 2 by putting $\delta=\frac{1}{2}$. If we put $\beta_i=0,\ 0\le i\le n$ in Theorem 2, we can deduce Theorem A.

Corollary 1. Let $p(z)=\sum_{i=0}^{n}a_iz^i$ be a polynomial of degree *n*, such that

$$\alpha_n\ge\alpha_{n-1}\ge\cdots\ge\alpha_1\ge\alpha_0,$$

then the number of zeros of *p*(*z*) in $|z|\le\delta, 0<\delta<1$, does not exceed

$$1+\frac{1}{\log\frac{1}{\delta}}\log\frac{\alpha_n}{|a_0|}.$$

4. Acknowledgements

Author is highly thankful to the referees for their valuable suggestions.

5. References

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