Applied Mathematics, **2011, 2, 329-342**
doi:10.4236/am.2011.23039 Published Online March 2011 (http://www.SciRP.org/journal/am)

Vibration and Buckling Approximation of an Axially Loaded Cylindrical Shell with a Three Lobed Cross Section Having Varying Thickness

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Received November 2, 2010; *revised January* 17, 2011; *accepted January* 20, 2011

Abstract

On the basis of the thin-shell theory and on the use of the transfer matrix approach, this paper presents the vibrational response and buckling analysis of three-lobed cross-section cylindrical shells, with circumferentially varying thickness, subjected to uniform axial membrane loads. A Fourier approach is used to separate the variables, and the governing equations of the shell are formulated in terms of eight first-order differential equations in the circumferential coordinate, and by using the transfer matrix of the shell, these equations are written in a matrix differential equation. The transfer matrix is derived from the non-linear differential equations of the cylindrical shells with variable thickness by introducing the trigonometric series in the longitudinal direction and applying a numerical integration in the circumferential direction. The natural frequencies and critical loads beside the mode shapes are calculated numerically in terms of the transfer matrix elements for the symmetrical and antisymmetrical vibration modes. The influences of the thickness variation of cross-section and radius variation at lobed corners of the shell on the natural frequencies, mode shapes and critical loads are examined.

Keywords: Free Vibration, Buckling, Vibration of Continuous System, Noncircular Cylindrical Shell, Transfer Matrix Approach, Variable Thickness, Axial Loads

1. Introduction

In recent years, structural engineers have been gradually concerned with the analysis of vibration and stability problems of circular cylindrical shells which have noncircular profiles because of the greatest important in many engineering applications such as in the design of machines and structures. The vibration and buckling modes of thin elastic shells essentially depend on some determining functions such as the radius of the curvature of the neutral surface, the shell thickness, the shape of the shell edges, etc. In simple cases when these functions are constant, the vibration and buckling modes occupy the entire shell surface. If the determining functions vary from point to point of the neutral surface then localization of the vibration and buckling modes lies near the weakest lines on the shell surface, and this kind from problems is too difficult because the radius of its curvature varies with the circumferential coordinate, closed-form or analytic solutions cannot be obtained, in general, for this class of shells, numerical or approximate techniques are necessary for their analysis. The vibration and stability of circular cylindrical shells have been studied by many researchers since the basic equations for these were established by Flügge [1] and Donnell [2]. Becker and Gerard [3], Gerard [4] Cheng and Ho [5], Jones [6], Stavsky and Friedland [7] and Lei and Cheng [8] studied the elastic stability of orthotropic or composite shells under axial loading. Greenberg and Stavsky [9-12] studied the vibrations and buckling of laminated orthotropic shells under compression. Rosen and Singer [13] have carried out a theoretical and experimental investigation of the vibrations of axially loaded stiffened cylindrical shells. Yamada *et al*. [14,15] studied the vibration and stability of circular cylindrical shells subjected to axial load and also the free vibration of noncircular cylindrical shells with variable circumferential profiles. In recent years, Suzuki and Leissa [16] beside Kumar and Singh [17] studied the approximate vibrational analysis of noncircular cylindrical shells having circumferentially varying thickness by using the finite strip method, but Mitao *et al.* [18] studied the same problem of open cylindrical shells by using the transfer matrix method. Tonin and Bies [19], Bergman *et al.* [20], Irie *et al.* [21], Takahashi *et al.* [22], Koiter *et al.* [23] and Abdullah and Hakan [24] studied the buckling vibration of shells with variable thickness, and in those studies, generally, Kirchoff-Love's first approximation theory has been used and the effect of the thickness variation on the critical parameters has been proved numerically. Eliseeva and Filippov [25] and Filippov *et al.* [26] studied the vibration and buckling of cylindrical shells of variable thickness with slanted and curvelinear edges, respectively, and the asymptotic and finite element methods are used to get the vibration frequencies and critical loads. However, the problem of vibration and buckling of the shell-type structures treated here which are composed of circular cylindrical panels and flat plates with circumferential variable thickness under axial membrane loads does not appear to have been dealt with in the literature.

The purpose of this paper is to present the vibration and buckling analysis of an isotropic cylindrical shell with a three lobed cross section of circumferentially varying thickness, subjected to uniformly distributed axial compressive loads, by using the transfer matrix method. The transfer matrix is derived from the non-linear differential equations for the cylindrical shells by numerical integration. The method is applied to symmetrical and antisymmetrial shell. The natural frequencies and critical buckling loads beside the vibration and buckling modes of the shells are presented. The influences of the thickness variation and radius variation on the frequencies and buckling are examined. The results are cited in tabular and graphical forms.

2. Mathematical Formulation of the Problem

Consider an isotropic, elastic, cylindrical shell of a three-lobed cross-section profile expressed by the equation $r = a\, f\left(\theta\right)$, where a is the reference radius of curvature, chosen to be the radius of a circle having the same circumference as the three-lobed profile, and $f\left(\theta\right)$ is a prescribed function of θ. L_1 and L_2 are the axial and circumferential lengths of the middle surface of the shell, and the thickness $H\left(\theta\right)$ varying continuously in the circumferential direction. The cylindrical coordinates $\left(x, s, z\right)$ are taken to define the position of a point on the middle surface of the shell, as shown in **Figure 1(a)**, and **Figure 1(b)** shows the three-lobed cross-section profile of the middle surface, with the apothem denoted by A_1, and the radius of curvature at the lobed corners by R_1. While u, υ and w are the deflection displacements of the middle surface of the shell in the longitudinal, circumferential and transverse directions, respectively. We suppose

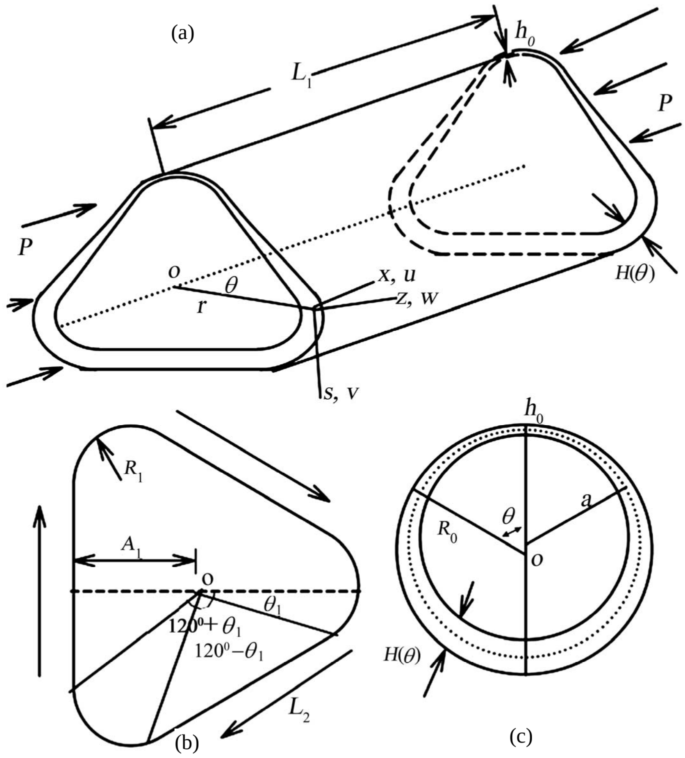

Figure 1. Coordinate system and geometry of a three-lobed cross-section cylindrical shell with circumferential variable thickness.

that the shell thickness H at any point along the circumference is small and depends on the coordinate θ and takes the following form:

$$H(\theta) = h_0\phi(\theta) \qquad (1)$$

where h_0 is a small parameter, chosen to be the average thickness of the shell over the length L_2. For the cylindrical shell which cross-section is obtained by the cutaway the circle of the radius r_0 from the circle of the radius R_0 (see **Figure 1(c)**) function $\phi(\theta)$ have the form: $\phi(\theta)=1+\delta(1-\cos\theta)$ where δ is the amplitude of thickness variation, $\delta = d/h_0$, and d is the distance between the circles centers. In general case $h_0 = H(\theta = 0)$ is the minimum value of $\phi(\theta)$ while $h_m = H(\theta = \pi)$ is the maximum value of $\phi(\theta)$, and in case of $d = 0$ the shell has constant thickness h_0. The dependence of the shell thickness ratio $\eta = h_m/h_0$ on δ has the form $\eta = 1 + 2\delta$.

For studying the free harmonic vibrations and buckling of the shell under consideration, the equilibrium equations of forces for the cylindrical shell, subjected to an axial compressive load P, based on the Goldenveizer-Novozhilov theory are taken from [27,28] as follows:

$$\begin{gathered} N'_x + N^{\bullet}_{sx} - Pu'' + \rho H(\theta)\omega^2 u = 0, \quad N'_{xs} + N^{\bullet}_s + Q_s/R - P\upsilon'' + \rho H(\theta)\omega^2\upsilon = 0, \\ Q'_x + Q^{\bullet}_s - N_s/R - Pw'' + \rho H(\theta)\omega^2 w = 0, \quad M'_x + M^{\bullet}_{sx} - Q_x = 0, \\ M'_{xs} + M^{\bullet}_s - Q_s = 0, \quad S_s - Q_s - M'_{sx} = 0, \quad N_{xs} - N_{sx} - M_{sx}/R = 0, \end{gathered} \qquad (2)$$

where N_x, N_s and Q_x, Q_s are the normal and transverse shearing forces in the x and s directions, respectively, N_{sx} and N_{xs} are the in-plane shearing forces, M_x, M_s and M_{xs}, M_{sx} are the bending moment and the twisting moment, respectively, S_s is the equivalent (Kelvin-Kirchoff) shearing force, P is the axial force per unit length, constant along the circumference, ρ is the mass density, R is the radius of curvature of the middle surface, ω is the angular frequency of vibration, $' \equiv \partial/\partial x$, and $^{\bullet} \equiv \partial/\partial s$.

The relations between strains and deflections for the cylindrical shells used here are taken from [29] as follows:

$$\begin{gathered} \varepsilon_x = u', \quad \varepsilon_s = \upsilon^{\bullet} + w/R, \quad \gamma_{xs} = \upsilon' + u^{\bullet}, \quad \gamma_{xz} = w' + \psi_x = 0, \\ \gamma_{sz} = w^{\bullet} + \psi_s - \upsilon/R = 0, \quad k_x = \psi'_x, \quad k_s = \psi^{\bullet}_s + \left(\upsilon^{\bullet} + w/R\right)/R, k_{sx} = \psi'_s, \quad k_{xs} = \psi^{\bullet}_x + \upsilon'/R \end{gathered} \qquad (3)$$

where ε_x and ε_s are the normal strains of the middle surface of the shell, γ_{xs}, γ_{xz} and γ_{sz} are the shear strains, and the quantities k_x, k_s, k_{sx} and k_{xs} representing the change of curvature and the twist of the middle surface, ψ_x is the bending slope, and ψ_s is the angular rotation.

The components of force and moment resultants in terms of Equation (3) are given as:

$$\begin{gathered} N_x = D(\varepsilon_x + \nu\varepsilon_s), \quad N_s = D(\varepsilon_s + \nu\varepsilon_x), \quad N_{xs} = D(1-\nu)\gamma_{xs}/2, \\ M_x = K(k_x + \nu k_s), \quad M_s = K(k_s + \nu k_x), \quad M_{sx} = k(1-\nu)k_{sx} \end{gathered} \qquad (4)$$

From Equations (2)-(4), with eliminating the variables Q_x, Q_s, N_x, N_{xs}, M_x, M_{xs} and M_{sx} which are not differ-rentiated with respect to s, the system of the partial differential equations for the state variables u, υ, w, ψ_s, M_s, S_s, N_s and N_{sx} of the shell are obtained as follows:

$$\begin{gathered} u^{\bullet} = 2/(D(1-\nu))N_{sx} + \left(H^2/6R\right)\psi'_s - \upsilon', \quad \upsilon^{\bullet} = N_s/D - w/R - \nu u', \quad w^{\bullet} = \upsilon/r - \psi_s, \\ \psi^{\bullet}_s = M_s/K + \nu\psi'_x - N_s/RD + (\nu/R)u', \quad M^{\bullet}_s = S_s - 2K(1-\nu)\psi''_s, \\ S^{\bullet}_s = N_s/R - \nu M''_s + K\left(1-\nu^2\right)w''' + Pw'' - \rho\omega^2 Hw, \quad N^{\bullet}_s = P\upsilon'' - S_s/R - N'_{sx} - \rho\omega^2 H\upsilon, \\ N^{\bullet}_{sx} = D\left(1-\nu^2\right)u'' + Pu'' - \nu N'_s - \rho\omega^2 Hu. \end{gathered} \qquad (5)$$

The quantities D and K, respectively, are the extensional and flexural rigidities expressed in terms of the Young's modulus E, Poisson's ratio ν and the wall thickness $H(\theta)$ as the form: $D = EH/\left(1-\nu^2\right)$ and $K = EH^3/12\left(1-\nu^2\right)$, and under the variable thickness case, using Equation (1), those are written as follows:

$$D = \left(Eh_0 / \left(1-\nu^2\right)\right)\phi(\theta) = D_0\,\phi(\theta) \quad (6)$$

$$K = \left(E\left(h_0\right)^3 / \left(1-\nu^2\right)\right)\phi^3(\theta) = K_0\,\phi^3(\theta) \quad (7)$$

where D_0 and K_0 are the reference extensional and flexural rigidities of the shell, chosen to be the averages on the middle surface of the shell over the length L_2.

For the free vibration of a simply supported shell, the solution of the system of Equation (5) is sought as follows:

$$\begin{aligned}
&u(x,s) = \bar{U}(s)\cos\frac{m\pi}{L_1}x,\quad \left(\upsilon(x,s),\, w(x,s)\right) = \left(\bar{V}(s), \bar{W}(s)\right)\sin\frac{m\pi}{L_1}x,\quad \psi_s(x,s) = \bar{\psi}_s(s)\sin\frac{m\pi}{L_1}x,\\
&\left(N_x(x,s),\, N_s(x,s),\, Q_s(x,s),\, S_s(x,s)\right) = \left(\bar{N}_x(s), \bar{N}_s(s), \bar{Q}_s(s), \bar{S}_s(s)\right)\sin\frac{m\pi}{L_1}x,\\
&\left(N_{xs}(x,s),\, N_{sx}(x,s),\, Q_x(x,s)\right) = \left(\bar{N}_{xs}(s), \bar{N}_{sx}(s), \bar{Q}_x(s)\right)\cos\frac{m\pi}{L_1}x,\\
&\left(M_x(x,s),\, M_s(x,s)\right) = \left(\bar{M}_x(s), \bar{M}_s(s)\right)\sin\frac{m\pi}{L_1}x,\\
&\left(M_{xs}(x,s),\, M_{sx}(x,s)\right) = \left(\bar{M}_{xs}(s), \bar{M}_{sx}(s)\right)\cos\frac{m\pi}{L_1}x,\quad m = 1, 2, \cdots
\end{aligned} \quad (8)$$

where m is the axial half wave number and the quantities $\bar{U}(s), \bar{V}(s)\cdots$ are the state variables and undetermined functions of s.

3. Matrix Form of the Basic Equations

The differential equations as shown previously are modified to a suitable form and solved numerically. Hence, by substituting Equation (8) into Equation (5) and take relations (6) and (7) into account, the system of vibration equations of the shell can be written in non-linear ordinary differential equations referred to the variable *s* only are obtained, in the following matrix form:

$$a\frac{d}{ds}\begin{pmatrix}\tilde{U}\\ \tilde{V}\\ \tilde{W}\\ \tilde{\psi}_s\\ \tilde{M}_s\\ \tilde{S}_s\\ \tilde{N}_s\\ \tilde{N}_{sx}\end{pmatrix} = \begin{bmatrix}0 & V_{12} & 0 & V_{14} & 0 & 0 & 0 & V_{18}\\ V_{21} & 0 & V_{23} & 0 & 0 & 0 & V_{27} & 0\\ 0 & V_{32} & 0 & V_{34} & 0 & 0 & 0 & 0\\ V_{41} & 0 & V_{43} & 0 & V_{45} & 0 & V_{47} & 0\\ 0 & 0 & 0 & V_{54} & 0 & V_{56} & 0 & 0\\ 0 & 0 & V_{63} & 0 & V_{65} & 0 & V_{67} & 0\\ 0 & V_{72} & 0 & 0 & 0 & V_{76} & 0 & V_{78}\\ V_{81} & 0 & 0 & 0 & 0 & 0 & V_{87} & 0\end{bmatrix}\begin{pmatrix}\tilde{U}\\ \tilde{V}\\ \tilde{W}\\ \tilde{\psi}_s\\ \tilde{M}_s\\ \tilde{S}_s\\ \tilde{N}_s\\ \tilde{N}_{sx}\end{pmatrix}. \quad (9)$$

By using the state vector of fundamental unknowns $Z(s)$, system (9) can be written as follows:

$$\left(a\frac{d}{ds}\right)\left(Z(s)\right) = \left[V(s)\right]\left(Z(s)\right) \quad (10)$$

$$Z(s) = \left(\tilde{U}, \tilde{V}, \tilde{W}, \tilde{\psi}_s, \tilde{M}_s, \tilde{S}_s, \tilde{N}_s, \tilde{N}_{sx}\right)^T,$$

$$\left(\tilde{U}, \tilde{V}, \tilde{W}\right) = k_0\left(\bar{U}, \bar{V}, \bar{W}\right),$$

$$\tilde{\psi}_s = \left(k_0/\beta\right)\bar{\psi}_s,\ \tilde{M}_s = \left(1/\beta^2\right)\bar{M}_s,$$

$$\left(\tilde{S}_s, \tilde{N}_s, \tilde{N}_{sx}\right) = \left(1/\beta^3\right)\left(\bar{S}_s, \bar{N}_s, \bar{N}_{sx}\right),\quad \beta = \frac{m\pi}{L_1}.$$

For the noncircular cylindrical shell which cross-section profile is obtained by function ($r = a\,f(\theta)$), the hypotenuse (ds) of a right triangle whose sides are infinitesimal distances along the surface coordinates of the shell takes the form $(ds)^2 = (dr)^2 + (rd\theta)^2$, then we have

$$\frac{ds}{a} = \sqrt{\left(f(\theta)\right)^2 + \left(\frac{df(\theta)}{d\theta}\right)^2}\ d\theta \quad (11)$$

By using Equation (11), the system of vibration Equations (10) takes the form:

$$\left(\frac{d}{d\theta}\right)\left(Z(\theta)\right) = \Psi(\theta)\left[V(\theta)\right]\left(Z(\theta)\right) \quad (12)$$

Where $\Psi(\theta)=\sqrt{\left(f(\theta)\right)^2+\left(\frac{df(\theta)}{d\theta}\right)^2}$, and the coefficients matrix $\left[V(\theta)\right]$ are given as:

$$V_{12}=-(m\pi/l),\ V_{14}=(m\pi/l)^2\left(h^2/6\right)\phi,\ V_{18}=(m\pi/l)^3\left(h^2/6(1-\nu)\phi\right),\ V_{21}=\nu(m\pi/l),$$
$$V_{23}=-1/c,\ V_{27}=(m\pi/l)^3\left(h^2/12\phi\right),\ V_{32}=1/c,\ V_{34}=-(m\pi/l),\ V_{41}=-\nu/c$$
$$V_{43}=-\nu(m\pi/l)^2,\ V_{45}=1/h\phi^3,\ V_{46}=h/12c\phi,\ V_{54}=2(1-\nu)h(m\pi/l)^2\phi^2$$
$$V_{56}=1,\ V_{63}=\left(1-\nu^2\right)(m\pi/l)^4\phi^3/2-\phi\lambda^2(12/h)/(m\pi/l)^3-\bar{P}/(m\pi/l),$$
$$V_{65}=\nu(m\pi/l),\ V_{67}=(m\pi/l)/c,\ V_{72}=-\bar{P}/(m\pi/l)-\phi\lambda^2(12/h)/(m\pi/l)^3,$$
$$V_{76}=-1/c,\ V_{78}=m\pi/l,\ V_{81}=\phi\left(1-\nu^2\right)\left(12/h^2\right)/m\pi/l-\bar{P}/(m\pi/l)-\phi\lambda^2(12/h)/(m\pi/l)^3,\ V_{87}=-\nu(m\pi/l)$$

in terms of the following dimensionless shell parameters: frequency parameter $\lambda^2=\rho h_0 a^2\omega^2/D_0$, load factor $\bar{P}=\left(a^2/K_0\right)/P$, $l=L_1/a$ and $h=h_0/a$.

As the function $f(\theta)$ formulates the profile of a shell with a three lobed cross section, and expressed as in [15]

$$f(\theta)=\begin{cases}2(a_1-c_1)\cos\theta+\sqrt{c_1^2-4(a_1-c_1)^2\sin^2\theta}, & \text{for } 0\le\theta\le\theta_1\\ a_1\operatorname{cosec}\left(\theta+30^0\right),\ c_1=\infty, & \text{for } \theta_1\le\theta\le120^0-\theta_1\\ 2(a_1-c_1)\cos\left(\theta-120^0\right)+\sqrt{c_1^2-4(a_1-c_1)^2\sin^2\left(\theta-120^0\right)}, & \text{for } 120^0-\theta_1\le\theta\le120^0+\theta_1\\ -a_1\sec\theta,\ c_1=\infty, & \text{for } 120^0+\theta_1\le\theta\le180^0\end{cases},$$

$$a_1=A_1/a,\ c_1=R_1/a,\ \zeta=c_1/a_1,\ \theta_1=\tan^{-1}\left\{\sqrt{3}\,c_1/\left(4a_1-3c_1\right)\right\}$$

The state vector $Z(\theta)$ of fundamental unknowns can be expressed as in [30]

$$\left(Z(\theta)\right)=\left[Y(\theta)\right]\left(Z(0)\right) \tag{13}$$

by using the transfer matrix $\left[Y(\theta)\right]$ of the shell, and the substitution of the expression into Equation (10) yields

$$(d/d\theta)\left[Y(\theta)\right]=\Psi(\theta)\left[V(\theta)\right]\left[Y(\theta)\right],\quad \left[Y(0)\right]=[I]. \tag{14}$$

The governing system (14) is too complicated to obtain any closed form solution, and this problem is highly favorable for solving by numerical methods. Hence, the matrix $\left[Y(\theta)\right]$ is obtained by using numerical integration, by use of the Runge-kutta integration method of forth order, with the starting value $\left[Y(0)\right]=[I]$ (unit matrix) which is given by taking $\theta=0$ in Equation (13), and its solution depends only on the geometric and martial properties of the shell, and the same solution can be used for appropriate boundary conditions imposed at the shell circumference.

For a plane passing through the central axis in a shell with structural symmetry, symmetrical and antisymmetrical profiles can be obtained, and consequently, only one-half of the shell circumference is considered with the boundary conditions at the ends taken to be the symmetric or antisymmetric conditions. Therefore, the boundary conditions for symmetrical and antisymmetrical vibrations are:

$$\tilde{V}=\tilde{\psi}_s=0,\ \tilde{S}_s=\tilde{N}_{sx}=0,\quad \tilde{U}=\tilde{W}=0,\ \tilde{N}_s=\tilde{M}_s=0,\ \text{respectively} \tag{15}$$

4. Natural Frequencies and Modes

A shell with ($f(\theta)=\phi(\theta)=1$) represents a circular cylindrical shell of constant thickness. The substitution of Equation (14) into Equation (12) results the frequency equations

$$\begin{bmatrix}Y_{21}&Y_{23}&Y_{25}&Y_{27}\\Y_{41}&Y_{43}&Y_{45}&Y_{47}\\Y_{61}&Y_{63}&Y_{65}&Y_{67}\\Y_{81}&Y_{83}&Y_{85}&Y_{87}\end{bmatrix}_{(\pi)}\begin{pmatrix}\tilde{U}\\\tilde{W}\\\tilde{M}_s\\\tilde{N}_s\end{pmatrix}_{(0)}=0\quad\text{for symmetrical vibration,} \tag{16}$$

$$\begin{bmatrix} Y_{12} & Y_{14} & Y_{16} & Y_{18} \\ Y_{32} & Y_{34} & Y_{36} & Y_{38} \\ Y_{52} & Y_{45} & Y_{56} & Y_{58} \\ Y_{72} & Y_{74} & Y_{76} & Y_{78} \end{bmatrix}_{(\pi)} \begin{pmatrix} \tilde{V} \\ \tilde{\psi}_s \\ \tilde{S}_s \\ \tilde{N}_{sx} \end{pmatrix}_{(0)} = 0 \quad \text{for antisymmetrical vibration.} \tag{17}$$

The matrix [V] depends on the frequency parameter λ and the load factor $\bar{P}$ in addition to θ, $\left[Y(\pi)\right]$ is also a function of these parameters. Equations (16) and (17) give a set of linear homogenous equations with unknown coefficients $\left(\tilde{U},\tilde{W},\tilde{M}_s,\tilde{N}_s\right)^T_{(0)}$ and $\left(\tilde{V},\tilde{\psi}_s,\tilde{S}_s,\tilde{N}_{sx}\right)^T_{(0)}$, respectively, at $\theta=0$. For the existence of a nontrivial solution of these coefficients, the determinant of the coefficient matrix should be vanished. The standard procedures cannot be employed for obtaining the eigenvalues of the frequency equations. The nontrivial solution is found by searching the values λ which make the determinant zero by using Lagrange interpolation procedure. The eigenvalues of λ which make the determinant zero give the natural frequencies of the shell. The mode shapes at any point of the cross-section of the shell, over the length L_2, are determined by calculating the eigenvectors corresponding to the eigenvalues by using Gaussian elimination procedure. For free vibration problems without prestress the natural frequencies result when $\bar{P}=0$, and for vibration problems with prestress the natural frequencies result for non-zero values of $\bar{P}$. For buckling problems, the buckling load results when $\lambda=0$ in Equations (16) and (17), and the lowest values of the buckling loads give the critical loads of the shell.

5. Numerical Results and Discussion

A computer program based on the analysis described herein has been developed to study the symmetrical and antisymmetrical vibrations of the considered shell, and the vibration frequencies and the critical buckling loads of the shell with circumferential variable thickness, are calculated numerically. Results for uniform thickness of circular cylindrical shells ($\eta=1$ and $\zeta=1$) are obtained. The correctness of applied method is cited in [31] with other literature. Our study is divided into two parts:

5.1. Vibrations

The study of vibrations is determined by finding the natural frequencies λ and corresponding mode shapes at $\bar{P}=0$. The numerical results presented herein pertain to the minimum frequencies and the associated mode shapes of the shell.

The effect of variation in thickness on the minimum frequencies of vibration, **Tables 1 and 2** give the funda-

Table 1. The fundamental frequencies λ for symmetric modes of a three-lobed cross-section cylindrical shell with variable thickness, for which (m = 1, ν = 0.3, l = 4, h = 0.02).

	η				
ζ	1	2	3	4	5
0.0	0.019568	0.028312	0.035171	0.041386	0.047378
0.2	0.019736	0.027475	0.034116	0.040315	0.046313
0.4	0.029985	0.035825	0.040712	0.045741	0.051032
0.6	0.039787	0.050925	0.056946	0.061702	0.066208
0.8	0.053618	0.068977	0.079662	0.086458	0.091184
1.0	0.074135	0.087694	0.100293	0.110095	0.115710

Table 2. The fundamental frequencies λ for antisymmetric modes of a three-lobed cross-section cylindrical shell with variable thickness, for which (m = 1, ν = 0.3, l = 4, h = 0.02).

	η				
ζ	1	2	3	4	5
0.0	0.019568	0.026504	0.033007	0.039530	0.046143
0.2	0.019736	0.026235	0.032464	0.038710	0.045019
0.4	0.029985	0.036643	0.042989	0.049309	0.055683
0.6	0.045266	0.054520	0.062245	0.069595	0.076761
0.8	0.070167	0.082516	0.091719	0.100270	0.108014
1.0	0.074135	0.087694	0.100293	0.110095	0.115710

mental frequencies for symmetric and antisymmetric vibrations of a three-lobed cross-section cylindrical shell, with the same circumferential length, versus the radius ratio ζ for different values of η. The results presented in these tables show that the increase of the thickness ratio resulted in an increase in the fundamental frequencies for each value of the radius ratio. These results confirm the fact that the effect of increasing the shell flexural rigidity becomes larger than that of increasing the shell mass when the thickness ratio in creases. Also, the increase of the radius ratio for the chosen values of the thickness ratio results in an increase in the frequencies λ and they become larger at $\zeta = 1$ (circular cylinder). In case of a constant thickness ($\eta = 1$), the symmetric and antisymmetric type vibrations have the same values versus the radius ratio, except for the symmetric and antisymmetric ones with respect to three planes passing through a corner and an axial mid-line of the opposite wall give unidentical values. For a circular cylindrical shell ($\zeta = 1$), the symmetric and antisymmetric type vibrations give the same values versus the thickness ratio. In **Tables 3 and 4**, the first five frequencies λ of symmetric and antisymmetric type vibrations are presented for different values of radius ratios ($\zeta = 0.5, 0.7, 1.0$) and thickness ratios ($\eta = 1, 2, 5$). The numbers in the parentheses are the axial half wave numbers of the mode in the x-direction. An ordering change of the mode in the x-direction can be seen for certain values of ζ and η.

In **Table 5**, the comparison of the fundamental frequencies λ for symmetric and antisymmetric vibrations of a shell with radius ratio ($\zeta = 0.5$) is cited for different axial half wave numbers and thickness ratio. With an increase of axial half wave number aspect the thickness ratio, the fundamental frequencies increase. It is shown by this table that the values of fundamental frequencies λ for symmetric and antisymmetric modes are very close to each other for the large mode number, m.

Figures (2-4) show the first five circumferential mode shapes of a three-lobed cross-section cylindrical shell of variable thickness for symmetric and antisymmetric vibration modes corresponding to the frequencies λ listed in **Tables 3 and 4**. The thick lines show the composi tion of the circumferential and transverse deflections, and the numbers in the parentheses are the axial half wave num-

Table 3. The first five frequencies λ for symmetric modes of the shell, with ($\nu = 0.3$, $l = 4$, $h = 0.02$).

		Modes				
ζ	η	*First*	*Second*	*Third*	*Forth*	*Fifth*
	1	0.036513(1)	0.037069(1)	0.046870(2)	0.052419(2)	0.065142(3)
	[15]	0.03653	0.03716	0.04691	0.05248	-
0.5	2	0.043201(1)	0.056216(1)	0.061026(2)	0.082699(3)	0.086711(2)
	5	0.057253(1)	0.092093(2)	0.100173(1)	0.125091(3)	0.162623(2)
0.7	2	0.059096(1)	0.075956(1)	0.086524(2)	0.116652(3)	0.121253(2)
	5	0.077638(1)	0.116732(1)	0.125442(2)	0.170122(3)	0.200812(1)
1.0	2	0.087641(1)	0.114102(1)	0.130261(1)	0.177653(2)	0.201403(1)
	5	0.114922(1)	0.133672(1)	0.210313(2)	0.218261(1)	0.246320(1)

Table 4. The first five frequencies λ for antisymmetric modes of the shell, with ($\nu = 0.3$, $l = 4$, $h = 0.02$).

		Modes				
ζ	η	*First*	*Second*	*Third*	*Forth*	*Fifth*
	1	0.037069(1)	0.052419(2)	0.064586(1)	0.068488(3)	0.089686(1)
	[15]	0.03716	0.05240	0.06470	-	0.8982
0.5	2	0.044870(1)	0.066191(2)	0.080338(1)	0.086261(3)	0.124450(1)
	5	0.065098(1)	0.099283(2)	0.121120(1)	0.131662(3)	0.194172(1)
0.7	2	0.066803(1)	0.092671(2)	0.093671(1)	0.117432(3)	0.139351(1)
	5	0.091152(1)	0.131372(1)	0.136280(2)	0.172388(3)	0.201140(1)
1.0	2	0.087659(1)	0.114122(1)	0.130263(1)	0.177656(2)	0.201405(1)
	5	0.115203(1)	0.133701(1)	0.210395(2)	0.218264(1)	0.246371(1)

Table 5. Comparison of fundamental frequencies for symmetric and antisymmetric modes of the shell, with ($\zeta = 0.5, \nu = 0.3, l = 4, h = 0.02$)

	Symmetric modes			Antisymmetric modes		
	η			η		
m	1	2	3	1	2	3
1	0.036513	0.043201	0.057253	0.037069	0.044870	0.065098
2	0.046870	0.061026	0.092093	0.052419	0.066191	0.099283
3	0.065142	0.082699	0.125091	0.068488	0.086261	0.131662
4	0.089935	0.112703	0.168702	0.091517	0.114570	0.173476
5	0.121390	0.151092	0.222726	0.122015	0.151953	0.225840
6	0.159833	0.197871	0.285843	0.160012	0.198190	0.287713
7	0.205271	0.252890	0.357090	0.205380	0.252920	0.358031
8	0.258043	0.315931	0.435938	0.258211	0.315819	0.436230
9	0.318039	0.386746	0.522161	0.318140	0.386557	0.522053
10	0.385308	0.464975	0.616405	0.385370	0.464749	0.615964

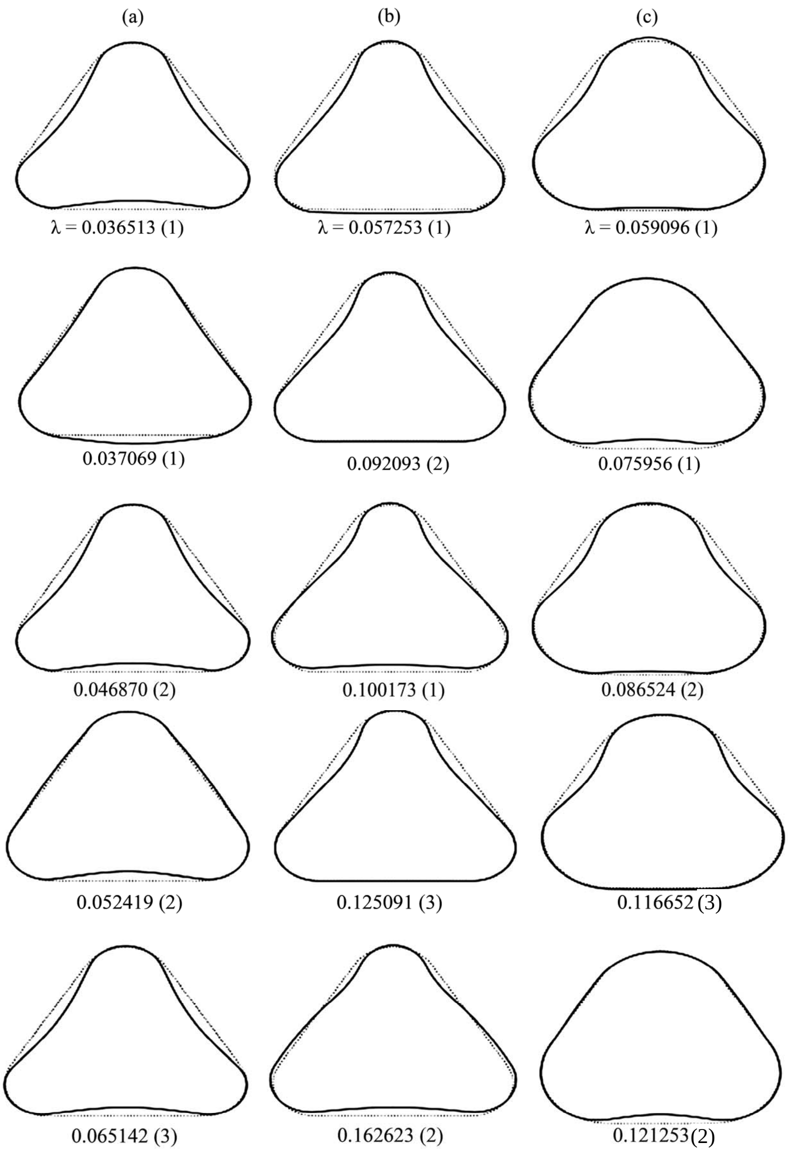

Figure 2. The first five symmetrical mode shapes of a three-lobed cross-section cylindrical shell with variable thickness. {(a) $\eta = 1$, (b) $\eta = 5$, $\zeta = 0.5$ }; {(c) $\eta = 2$, $\zeta = 0.7$ }.

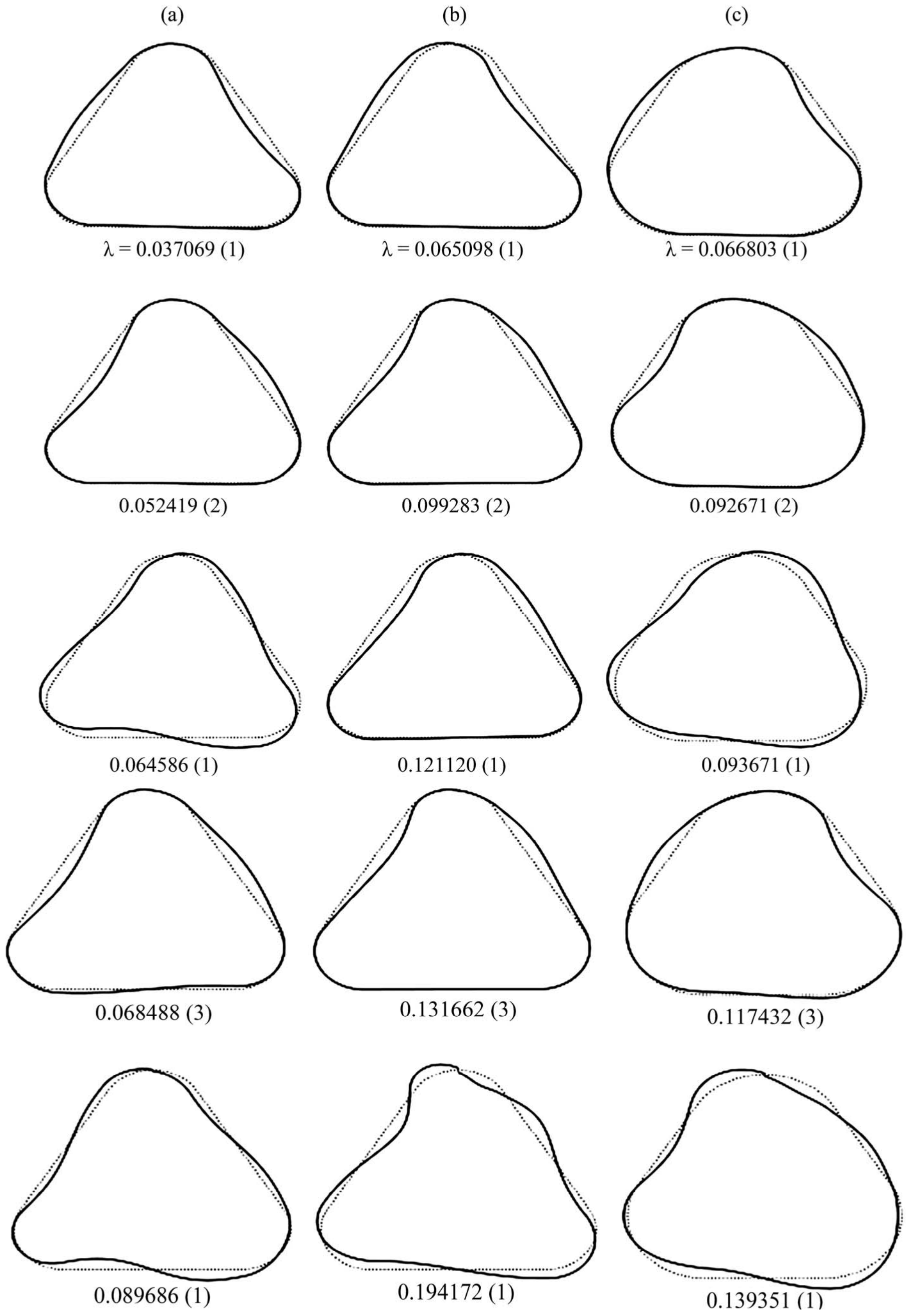

Figure 3. The first five antisymmetrical mode shapes of a three-lobed cross-section cylindrical shell with variable thickness. {(a) $\eta = 1$, (b) $\eta = 5$, $\zeta = 0.5$}; {(c) $\eta = 2$, $\zeta = 0.7$}.

ber corresponding to the fundamental frequencies λ. There are considerable differences between the modes of $\eta = 1$ and $\eta > 1$ for symmetric and antisymmetric modes. For $\eta > 1$, the majority of symmetric and antisymmetric modes, the displacements at the thinner edge are larger than those at the thicker edge *i.e.* the vibration modes are localized near the top generatrix $\theta = 0$. It is also shown by these figures that ordering changes of the modes, which have the same axial half wave number, and the mode shapes are similar in the sets of the vibration modes having *m* = 1, 2 and 3 for $\eta \geq 1$.

5.2. Buckling

Consider the buckling of a three-lobed cross-section cylindrical shell with circumferential variable thickness under axial compressive loads, constant over the length L_2. To obtain the buckling loads we will search the zero va-

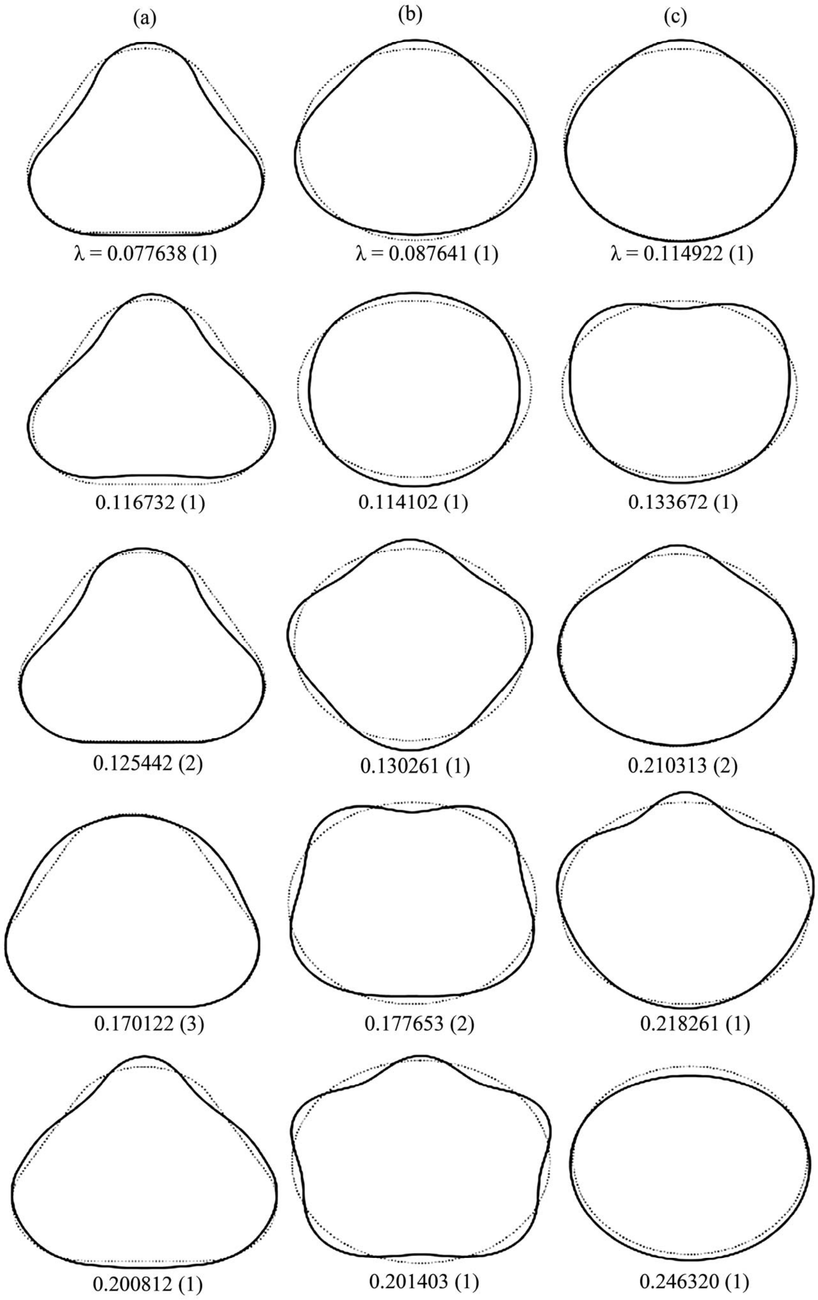

Figure 4. The first five symmetrical mode shapes of a circular cylindrical shell with variable thickness. {(a) $\eta = 5$, $\zeta = 0.7$}; {(b) $\eta = 2$, (c) $\eta = 5$, $\zeta = 1.0$}.

lue of the eigenvalues λ, and to obtain the critical loads we will search the lowest values of the buckling loads.

In **Figures (5-7)**, the eigenvalues of vibration λ versus the axial compression loads $\overline{P}$ are shown of a three-lobed cross-section cylindrical shell, with variable thickness, for symmetric and antisymmetric modes. The number in the parentheses is the axial half wave number corresponding to each one of the first five eigenvalues listed in **Tables 3 and 4**. With an increase in axial load, each eigenvalue monotonically decreases and finally becomes zero. The values on the ordinate represent the eigenvalues of the shell without the action of axial load. The loads that make the eigenvalues zero are called buckling loads, beyond which the shell becomes unstable and then can not keep its shape. It is shown from theses figures that the buckling loads increase with an increase of the thickness ratio η for each value of ζ, and also for each value of η the buckling loads increase with an increase of ζ, and for $\zeta = 1$ they become larger. The critical loads for symmetric and antisymmetric vibrations are occurred with $m = 3$.

Figure 7 shows the eigenvalues of vibration λ versus the axial compression loads for a circular cylindrical shell with circumferential variable thickness. In **Figure 7(a)**, the plotted eigencurve gives the critical buckling load of a circular cylindrical shell of a constant thickness, ($\bar{P} = 266.579$), and good agreement result with [14], and it is occurred with $m = 1$. While the critical buckling loads of a circular cylindrical shell having variable thickness are occurred with $m = 2$.

Figure 8 shows the buckling loads versus the axial half wave number of a shell with radius ratio ($\zeta = 0.5$) for symmetric and antisymmetric modes. It is shown from this figure, symmetric case gives lower buckling loads than antisymmetric one, and the buckling loads are very close to each other for the large mode number m. Also, with an increase of axial half wave number, the buckling loads increase after once decreasing. For each value of η, the critical buckling loads of the shell are occurred with $m = 3$ for the symmetric and antisymmetric modes.

6. Conclusions

An approximate analysis for studying the free vibration and buckling of a three-lobed cross-section cylindrical shell having circumferential varying thickness under axial membrane loads is presented. The numerical results presented herein pertain to the fundamental frequencies and buckling loads as well as the associated mode shapes

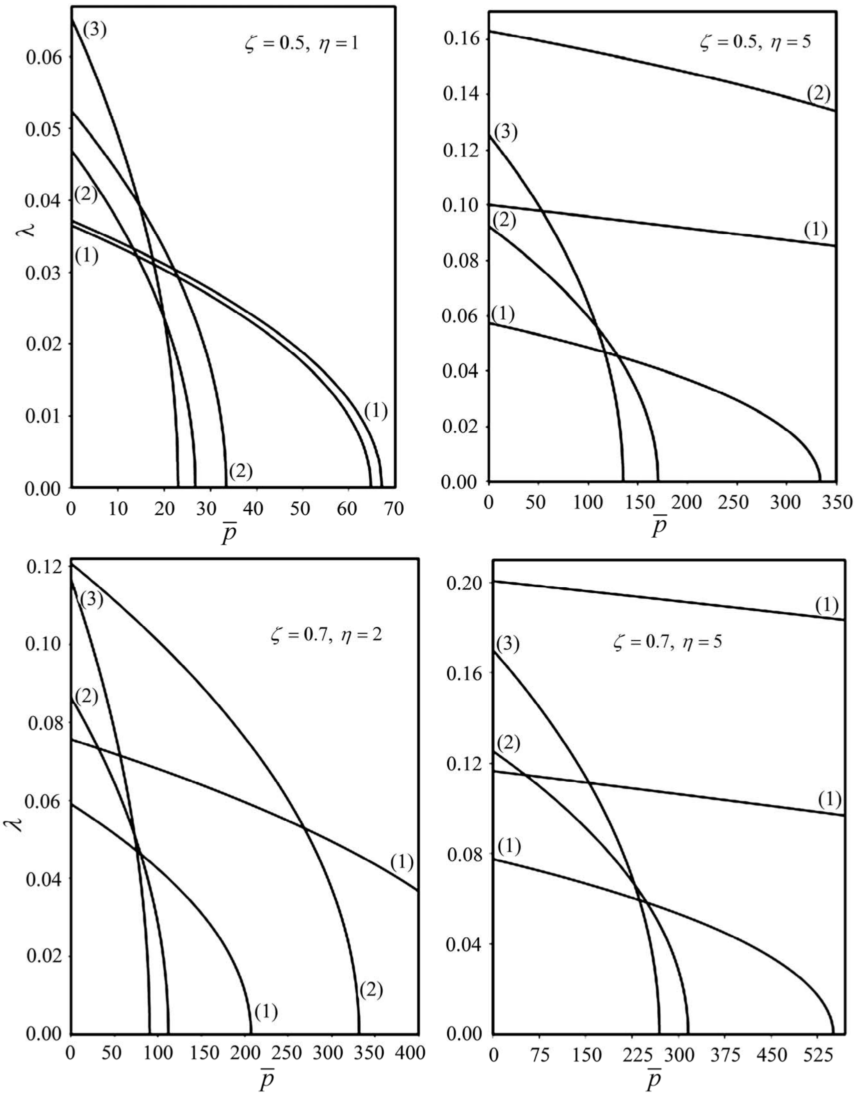

Figure 5. Eigenvalues of vibration versus axial load of a three-lobed cross-section cylindrical shell with variable thickness, symmetric case. (ν = 0.3, h = 0.02).

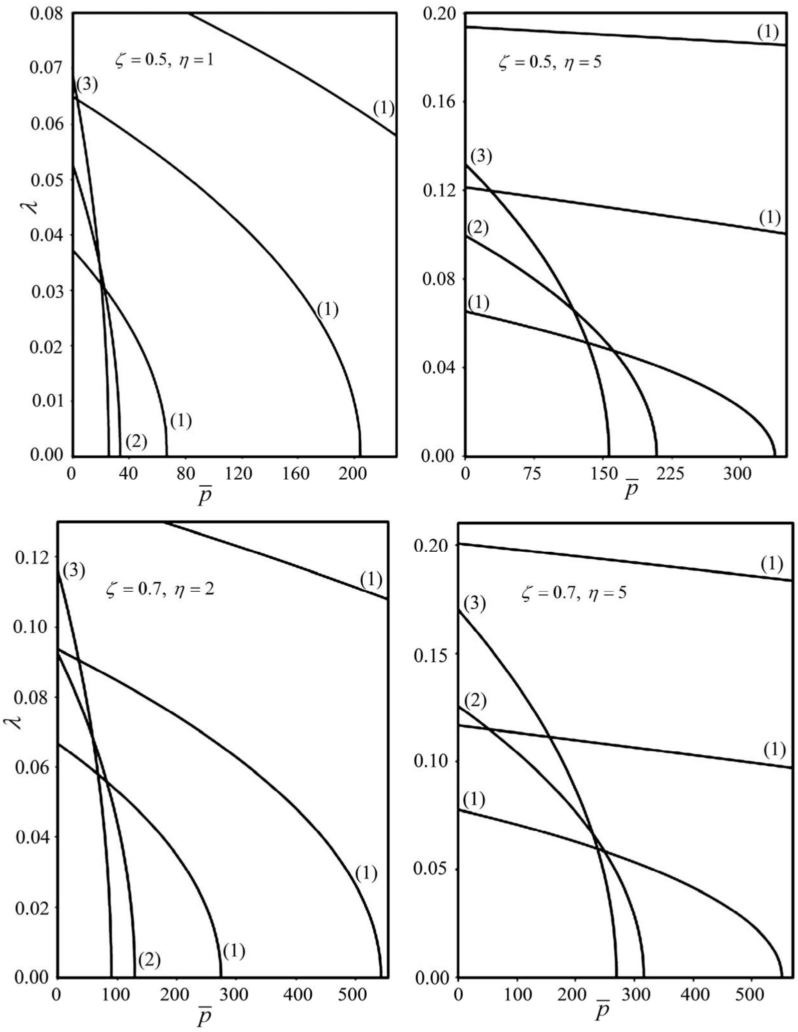

Figure 6. Eigenvalues of vibration versus axial load of a three-lobed cross-section cylindrical shell with variable thickness, antisymmetric case. (ν = 0.3, h = 0.02).

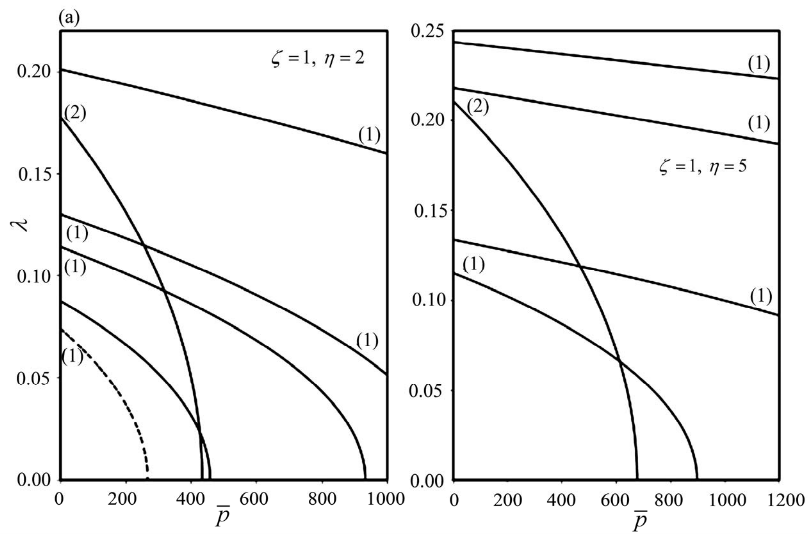

Figure 7. Eigenvalues of vibration versus axial compression load of a circular cylindrical shell with circumferential variable thickness. (ν = 0.3, h = 0.02).

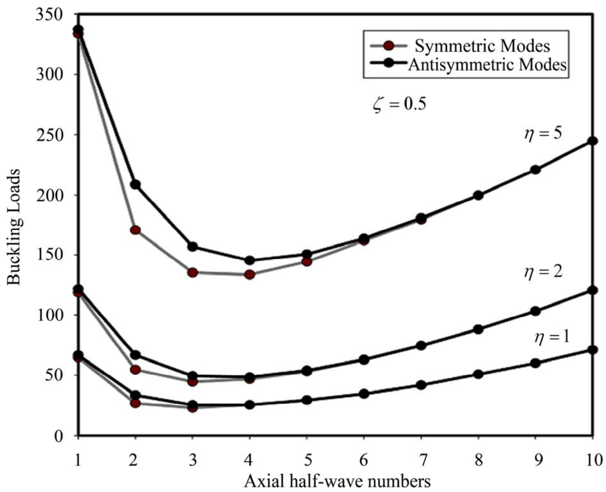

Figure 8. Buckling loads versus axial half wave number of a three-lobed cross-section cylindrical shell with variable thickness. (ν = 0.3, h = 0.02).

by using the transfer matrix method. The method is based on thin shell theory and applied to a shell of symmetric and antisymmetric vibration modes, and the analysis is formulated to overcome the mathematical difficulties associated with mode coupling caused by variable shell wall curvature and thickness. The first five fundamental frequencies and mode shapes as well as critical buckling loads have been presented, and the effects of the thickness ratio of the cross-section on the natural frequencies, mode shapes, and buckling loads were examined. For the thickness ratio $\eta = 1$, the vibration modes are distributed regularly over the shell surface, but for $\eta > 1$ the modes are localized near the weakest generatrix $\theta = 0$, (thinner edge), in most cases of the vibration modes. However, the critical buckling loads increase with either increasing radius ratio or increasing thickness ratio and become larger for a circular cylindrical shell. For the cylindrical shell of ($\zeta = 1$ and $\eta > 1$), the critical loads are occurred with $m = 2$, but for the shell under consideration of ($\zeta < 1$ and $\eta \geq 1$) they occurred with $m = 3$ for the symmetric and antisymmetric vibration modes.

7. Acknowledgments

The author is grateful to anonymous reviewers for their good efforts and valuable comments which helped improve the quality of this paper.

8. References

[1] W. Flügge, "Die Stabilität der Kreiszylinderschale," *Ingenieur Archiv*, Vol. 3, No. 5, 1932, pp. 463-506.

[2] L. H. Donnell, "Stability of Thin-Walled Tubes under Torsion," *NACA Report*, No. NACA TR-479, 1933.

[3] H. Becker and G. Gerard, "Elastic Stability of Orthotropic Shells," *Journal of Aerospace Science*, Vol. 29, No. 5, 1962, pp. 505-512.

[4] G. Gerard, "Compressive Stability of Orthotropic Cylinders," *Journal of Aerospace Science*, Vol. 29, 1962, pp. 1171-1179.

[5] S. Cheng and B. Ho, "Stability of Heterogeneous Aeolotropic Cylindrical Shells under Combined Loading," *AIAA Journal*, Vol. 1, No. 40, 1963, pp. 892-898.

[6] R. M. Jones, "Buckling of Circular Cylindrical Shells with Multiple Orthotropic Layers and Eccentric Stiffeners," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2301-2305. doi:10.2514/3.4986

[7] Y. Stavsky and S. Friedl, "Stability of Heterogeneous Orthotropic Cylindrical Shells in Axial Compression," *Israel Journal of Technology*, Vol. 7, 1969, pp. 111-119.

[8] M. M. Lei and S. Cheng, "Buckling of Composite and Homogeneous Isotropic Cylindrical Shells under Axial and Radial Loading," *Journal of Applied Mechanics*, Vol. 36, No. 4, 1969, pp. 791-798.

[9] J. B. Greenberg and Y. Stavsky, "Buckling and Vibration of Orthotropic Composite Cylindrical Shells," *Acta Mechanica*, Vol. 36, No. 12, 1980, pp. 15-29. doi:10.1007/BF01178233

[10] J. B. Greenberg and Y. Stavsky, "Vibrations of Axially Compressed Laminated Orthotropic Cylindrical Shells, Including Transverse Shear Deformation," *Acta Mechanica*, Vol. 37, No. 1-2, 1980, pp. 13-28. doi:10.1007/BF01441240

[11] J. B. Greenberg and Y. Stavsky, "Stability and Vibrations of Compressed Aeolotropic Composite Cylindrical Shells," *Journal of Applied Mechanics*, Vol. 49, No. 4, 1982, pp. 843-848. doi:10.1115/1.3162625

[12] J. B. Greenberg and Y. Stavsky, "Vibrations and Buckling of Composite Orthotropic Cylindrical Shells with Nonuniform Axial Loads," *Composites Part B: Engineering*, Vol. 29, No. 6, 1998, pp. 695-702. doi:10.1016/S1359-8368(98)00029-8

[13] A. Rosen and J. Singer, "Vibrations of Axially Loaded Stiffened Cylindrical Shells," *Journal of Sound and Vibration*, Vol. 34, No. 3, 1974, pp. 357-378. doi:10.1016/S0022-460X(74)80317-2

[14] G. Yamada, T. Irie and M. Tsushima, "Vibration and Stability of Orthotropic Circular Cylindrical Shells Subjected to Axial Load," *Journal of Acoustical Society of America*, Vol. 75, No. 3, 1984, pp. 842-848. doi:10.1121/1.390594

[15] G. Yamada, T. Irie and Y. Tagawa, "Free Vibration of Non-Circular Cylindrical Shells with Variable Circumferential Profile," *Journal of Sound and Vibration*, Vol. 95, No. 1, 1984, pp. 117-126. doi:10.1016/0022-460X(84)90264-5

[16] K. Suzuki and A. W. Leissa, "Free Vibrations of Noncircular Cylindrical Shells Having Circumferentially Varying Thickness," *Journal of Applied Mechanics*, Vol. 52,

No. 1, 1985, pp. 149-154. doi:10.1115/1.3168986

[17] V. Kumar and A. V. Singh, "Approximate Vibrational Analysis of Noncircular Cylinders Having Varying Thickness," *AIAA Journal*, Vol. 30, No. 7, 1991, pp. 1929-1931. doi:10.2514/3.11161

[18] O. Mitao, T. Hideki and S. Tsunemi, "Vibration Analysis of Curved Panels with Variable Thickness," *Engineering Computations*, Vol. 13, No. 2, 1996, pp. 226-239. doi:10.1108/02644409610114549

[19] R. F. Tonin and D. A. Bies, "Free Vibration of Circular Cylinders of Variable Thickness," *Journal of Sound and Vibration*, Vol. 62, No. 2, 1979, pp. 165-180. doi:10.1016/0022-460X(79)90019-1

[20] R. M. Bergman, S. A. Sidorin and P. E. Tovstik, "Construction of Solutions of the Equations for Free Vibration of a Cylindrical Shell of Variable Thickness along the Directrix," *Mechanics of Solids*, Vol. 14, No. 4, 1979, pp. 127-134.

[21] T. Irie, G. Yamada and Y. Kaneko, "Free Vibration of a Conical Shell with Variable Thickness," *Journal of Sound and Vibration*, Vol. 82, No. 1, 1982, pp. 83-94. doi:10.1016/0022-460X(82)90544-2

[22] S. Takahashi, K. Suzuki and T. Kosawada, "Vibrations of Conical Shells with Varying Thickness," *Japan Society of Mechanical Engineering*, Vol. 28, No. 235, 1985, pp. 117-123.

[23] W. I. Koiter, I. Elishakoff, Y. W. Li and J. H. Starness, "Buckling of an Axially Compressed Cylindrical Shell of Variable Thickness," *International Journal of Solids and Structures*, Vol. 31, No. 6, 1994, pp. 797-805. doi:10.1016/0020-7683(94)90078-7

[24] H. Abdullah and H. Erdem, "The Stability of Non-Homogenous Elastic Cylindrical Thin Shells with Variable Thickness under a Dynamic External Pressure," *Turkish Journal of Engineering and Environmental Sciences*, In Turkish, Vol. 26, No. 2, 2002, pp. 155-164.

[25] S. L. Eliseeva and S. B. Filippov, "Buckling and Vibrations of Cylindrical Shell of Variable Thickness with Slanted Edge," *Vestnik Sankt-Peterskogo Universiteta*, In Russian, No. 3, 2003, pp. 84-91.

[26] S. B. Filippov, D. N. Ivanov and N. V. Naumova, "Free Vibrations and Buckling of a Thin Cylindrical Shell of Variable Thickness with Curelinear Edge," *Technische Mechanik*, Vol. 25, No. 1, 2005, pp. 1-8.

[27] A. L. Goldenveizer, "Theory of Thin Shells," Pergamon Press, New York, 1961.

[28] V. V. Novozhilov, "The Theory of Thin Elastic Shells," P. Noordhoff Ltd., Groningen, 1964.

[29] R. Uhrig, "Elastostatik und Elastokinetik in Matrizenschreibweise," Springer-Verlag, Berlin, 1973.

[30] A. Tesar and L. Fillo, "Transfer Matrix Method," Kluwer Academic, Dordrecht, 1988.

[31] M. Khalifa, "A Study of Free Vibration of a Circumferentially Non-Uniform Cylindrical Shell with a Four Lobed Cross Section," In Press, *Journal of Vibration and Control*, 2010.

