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A Generating-Function Perspective on a Nonrealizable Trace-Zero Spectrum of Nonnegative 5 × 5 Matrices

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Abstract

We study a classical trace-zero spectrum that has played a central role in the analysis of the 5 × 5 nonnegative inverse eigenvalue problem. This spectrum is particularly illustrative because it fails to be realizable at the unperturbed parameter value, yet becomes realizable precisely once the symmetric perturbation exceeds a unique critical threshold. Using an exponential generating-function representation of power sums, we show that the refined Johnson-Loewy-London inequality is governed by a strictly increasing functional whose derivative is a polynomial in the perturbation parameter. This yields a transparent structural explanation of the sharp realizability threshold and recovers, in a unified way, earlier results of Salzmann, Friedland, and Laffey-Meehan. The method extends naturally to higher-order Johnson-Loewy-London inequalities and provides a convenient framework for analyzing parametrized families in the nonnegative inverse eigenvalue problem.

Keywords

Nonnegative Inverse Eigenvalue Problem, Trace-Zero Spectrum, Johnson-Loewy-London Inequality, Generating Functions

1. Introduction

The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions under which a prescribed list of complex numbers

$$\sigma = (\lambda_1, \cdots, \lambda_n)$$

occurs as the spectrum of a real nonnegative matrix $A \in \mathbb{R}^{n \times n}$. When such a matrix exists, the list σ is said to be realizable, and A is called a realizing matrix for

σ.

Associated with σ are the power sums

$$s_k := \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k, \ k = 1, 2, 3, \cdots.$$

If σ is realizable, then for any realizing matrix A one has

$$s_k = \operatorname{Tr}\left(A^k\right) \geq 0, \ k = 1, 2, 3, \cdots.$$

Further necessary conditions were established by Johnson [1] and by Loewy-London [2], who proved that

$$n^{m-1} s_{km} \geq s_k^m, \ k, m = 1, 2, 3, \cdots.$$

When n is odd and $s_1 = 0$, Laffey and Meehan [3] obtained a refinement of these inequalities, showing in particular that

$$(n-1) s_4 \geq s_2^2.$$

In addition, the Perron-Frobenius theorem [4] [5] implies that any realizable spectrum must contain a dominant eigenvalue that is real, nonnegative, and of maximal modulus.

The NIEP is completely resolved for $n = 3$ and $n = 4$, and for $n = 5$ under the trace-zero condition $s_1 = 0$. We refer the reader to the monographs of Berman-Plemmons and Mine [6] [7], as well as the work of Laffey and Meehan [8], for background and further references. We also note that nonnegative matrix spectra and related inverse problems arise naturally in applied settings such as stochastic models and queueing theory [9] [10].

In this paper, we focus on the family of spectra

$$\sigma = \left(\frac{3}{2}\mu, \frac{3}{2}\mu, -\mu, -\mu, -\mu\right), \ \mu > 0,$$

and its symmetric perturbation

$$\sigma_t = \left(\frac{3}{2}\mu + t, \frac{3}{2}\mu - t, -\mu, -\mu, -\mu\right), \ t > 0.$$

The unperturbed spectrum σ is unrealizable, while for each $\mu > 0$ there exists a minimal parameter $t_{\min} > 0$ beyond which the perturbed spectrum σ_t satisfies the refined Johnson-Loewy-London inequality. This family of spectra has appeared repeatedly in the literature: the unrealizability of the unperturbed case was observed by Salzmann for particular values of μ, further fixed-parameter cases were studied by Friedland, and for a fixed μ the minimal symmetric perturbation required for realizability was computed explicitly by Laffey and Meehan.

While generating functions have previously appeared in the study of the nonnegative inverse eigenvalue problem (e.g., Johnson-Paparella [11]), the present work uses an exponential generating function to reveal a monotonicity structure governing Johnson-Loewy-London functionals under symmetric perturbations. Although realizability thresholds for the spectrum

$$\left(\frac{3}{2}\mu, \frac{3}{2}\mu, -\mu, -\mu, -\mu\right)$$

have been computed previously for specific values of μ (notably by Salzmann, Friedland, and Laffey-Meehan),the contribution of the present work is not the existence of such thresholds by itself, but rather a conceptual explanation of their origin and uniqueness. By introducing an exponential generating-function framework, we show that the refined Johnson-Loewy-London functional governing realizability is monotone along symmetric perturbations. This yields, for all $\mu > 0$, a unified and transparent explanation of the sharp threshold phenomenon without resorting to case-specific computations.

We now introduce the generating-function formulation that underlies this explanation.

2. Generating-Function Formulation

For the perturbed spectrum σ_t, define the exponential generating function

$$M(u;t) = \sum_{i=1}^{5} \mathrm{e}^{u\lambda_i(t)}.$$

Then the power sums are recovered via

$$s_k(t) = \frac{\partial^k}{\partial u^k} M(u,t)\bigg|_{u=0}, \quad k \geq 1.$$

This formulation allows derivatives with respect to the perturbation parameter to be expressed explicitly in terms of low-degree polynomials. The symmetric pair $(a+t, a-t)$ contributes

$$\mathrm{e}^{u(a+t)} + \mathrm{e}^{u(a-t)} = 2\mathrm{e}^{ua}\cosh(ut),$$

which implies that each power sum $s_k(t)$ is an even polynomial in t. Writing $x = t^2$, we may therefore express

$$s_k(t) = h_k(x)$$

for suitable polynomials h_k. Differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} s_k(t) = 2t h_k'(x).$$

3. Main Results

We now state the central result explaining the sharp realizability threshold for the perturbed spectrum.

Theorem 1 (Generating—function explanation of a realizability threshold) *Let* $\mu > 0$ *and consider the trace-zero spectrum*

$$\sigma = \left(\frac{3}{2}\mu, \frac{3}{2}\mu, -\mu, -\mu, -\mu\right)$$

together with its symmetric perturbation

$$\sigma_t = \left(\frac{3}{2}\mu + t, \frac{3}{2}\mu - t, -\mu, -\mu, -\mu\right), \quad t \geq 0.$$

Then, σ is not realizable by a nonnegative 5×5 matrix. Moreover, there exists a unique threshold $t_{\min} > 0$ such that σ_t satisfies the refined Johnson-Loewy-London inequality (and hence satisfies this necessary condition for realizability) if and only if $t \geq t_{\min}$. Equivalently, the function

$$G(t) := 4s_4(t) - s_2(t)^2$$

is strictly increasing on $(0, \infty)$, satisfies $G(0) < 0$, and therefore admits a unique zero $t_{\min} > 0$.

Proof. For $n = 5$ under the trace-zero condition $s_1 = 0$, realizability of a spectrum by a nonnegative matrix requires the refined Johnson-Loewy-London inequality

$$4s_4 \geq s_2^2$$

to hold. For the parametrized family σ_t, we study the function

$$G(t) = 4s_4(t) - s_2(t)^2.$$

Using the exponential generating-function formulation developed earlier, each power sum $s_k(t)$ may be written as $s_k(t) = h_k(t^2)$ for a polynomial h_k. Differentiation yields

$$G'(t) = 2t\left(4h_4'(x) - 2h_2(x)h_2'(x)\right) \quad x = t^2.$$

For the present family, a direct computation gives

$$h_2(x) = \frac{15}{2}\mu^2 + 2x, \quad h_2'(x) = 2, \quad h_4'(x) = 27\mu^2 + 4x.$$

Substituting into the expression above, we obtain

$$G'(t) = 4t\left(39\mu^2 + 4x\right),$$

which is strictly positive for all $t > 0$. Hence $G(t)$ is strictly increasing on $(0, \infty)$.

Evaluating at $t = 0$ yields

$$G(0) = 4s_4(0) - s_2(0)^2 = -\frac{15}{4}\mu^4 < 0,$$

showing that the unperturbed spectrum σ is unrealizable. Since $G(t) \to +\infty$ as $t \to \infty$, there exists a unique $t_{\min} > 0$ such that $G(t_{\min}) = 0$. Consequently, σ_t satisfies the refined Johnson-Loewy-London inequality if and only if $t \geq t_{\min}$.

The explicit value of $t_{\min}$ follows from solving $G(t) = 0$ and can be computed in closed form.

The strict monotonicity above explains the existence and uniqueness of the realizability threshold independently of any case-by-case analysis.

4. Higher-Order JLL Inequalities

The generating-function approach extends naturally to the general JLL expressions

$$G_{k,m}(t) = n^{m-1}s_{km}(t) - s_k(t)^m.$$

For a parametrized family σ_t, we say that a JLL obstruction persists at parameter t if $G_{k,m}(t) < 0$ for some $k, m \geq 1$. Writing $s_k(t) = h_k(x)$ again yields

$$G'_{k,m}(t)=2t\left(n^{m-1}h'_{km}(x)-mh_k(x)^{m-1}h'_k(x)\right).$$

Hence, the monotonicity of $G_{k,m}$ reduces to the sign of the polynomial

$$B_{k,m}(x)=n^{m-1}h'_{km}(x)-mh_k(x)^{m-1}h'_k(x).$$

Whenever $B_{k,m}(x)\ge 0$ for all $x\ge 0$, the associated JLL functional $G_{k,m}(t)$ is monotone nondecreasing along symmetric perturbations. In cases where $G_{k,m}(0)<0$ and $\lim_{t\to\infty}G_{k,m}(t)=+\infty$, this monotonicity implies the existence of a unique parameter value at which the JLL obstruction vanishes. This provides a general symbolic framework for analyzing parametrized families in the nonnegative inverse eigenvalue problem. As a simple illustration, for the family σ_t considered in Section 3 and the case $(k,m)=(2,2)$, the polynomial $B_{2,2}(x)$ reduces to a positive linear function of x, recovering the strict monotonicity of the refined JLL functional analyzed in Theorem 1. We emphasize that positivity of $B_{k,m}$ is not guaranteed in general and depends on both the spectral configuration and the choice of (k,m). The present framework does not assert universal monotonicity, but rather provides a symbolic mechanism for identifying when such monotonicity occurs.

5. Conclusion

We have shown that a classical trace-zero nonrealizability phenomenon for 5 × 5 nonnegative matrices admits a transparent explanation through exponential generating functions. The refined JLL inequality becomes strictly monotone under symmetric perturbations, yielding a sharp and unique realizability threshold. The approach extends naturally to higher-order inequalities and suggests a general method for studying threshold phenomena arising from necessary conditions in the NIEP. The key contribution of this work is the identification of monotonicity along symmetric perturbations as the underlying mechanism driving sharp realizability thresholds in the NIEP.

6. Remark

The unrealizability of the spectrum σ for $\mu=\frac{2}{3}$ was first observed by Salzmann [12], while the case $\mu=2$ was analyzed by Friedland [13]. For $\mu=2$, Laffey and Meehan [3] computed explicitly the minimal symmetric perturbation required for realizability by applying the refined Johnson-Loewy-London inequality.

The generating-function approach developed here recovers this threshold as a special case and shows, moreover, that the existence and uniqueness of such a threshold is a structural consequence of monotonicity along symmetric perturbations. In particular, solving the equation $G(t)=0$ yields the explicit formula after simplification

$$t_{\min}=\mu\sqrt{\frac{\sqrt{98304}-312}{32}},$$

which reduces to the value obtained in [3] when $\mu = 2$.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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