Advances in Linear Algebra & Matrix Theory, **2013, 3, 69-72**
Published Online December 2013 (http://www.scirp.org/journal/alamt)
http://dx.doi.org/10.4236/alamt.2013.34013

Commuting Outer Inverses

Muneo Chō[1], Gabriel Kantún-Montiel[2]
[1]Department of Mathematics, Faculty of Science, Kanagawa University, Kanagawa, Japan
[2]Department of Research and Graduate Studies, Linda Vista University, Pueblo Nuevo Solistahuacán, Mexico
Email: chiyom01@kanagawa-u.ac.jp, gkantun@ulv.edu.mx

Received November 12, 2013; revised December 9, 2013; accepted December 16, 2013

ABSTRACT

The group, Drazin and Koliha-Drazin inverses are particular classes of commuting outer inverses. In this note, we use the inverse along an element to study some spectral conditions related to these inverses in the case of bounded linear operators on a Banach space.

Keywords: Generalized Inverse; Koliha-Drazin Inverse; Outer Inverse

1. Introduction

Several of the useful properties of the group, Drazin and Koliha-Drazin inverses can be related to their spectral characterizations. Some of these can be traced to the property of being commuting outer inverses.

Let $B(X)$ be the set of bounded linear operators on a Banach space X, and let $A \in B(X)$. We denote the range by $\mathcal{R}(A)$ and the null space of A by $\mathcal{N}(A)$.

Let M and N be closed subspaces of X. The outer inverse with prescribed range M and null space N, denoted $A^{(2)}_{M,N}$ is the unique operator B which satisfies:

$$B = BAB, \quad \mathcal{R}(B) = M, \quad \mathcal{N}(B) = N.$$

There is some advantage in prescribing the null space and range of an outer inverse by means of a third operator. In doing so, we will use the notion of invertibility along an element introduced by X. Mary ([1]). We say A is invertible along T if there exists $B \in B(X)$ such that

$$B = BAB, \quad \mathcal{R}(B) = \mathcal{R}(T), \quad \mathcal{N}(B) = \mathcal{N}(T).$$

In this case, the inverse along T is unique and we write $B = A^{\parallel T}$.

From $B = BAB$ we have that BA and AB are projections such that $\mathcal{R}(B) = \mathcal{R}(BA)$ and $\mathcal{N}(B) = \mathcal{N}(AB)$. Thus, we are effectively prescribing the range of the projection BA and the null space of the projection AB.

One of the useful properties of a generalized inverse is that, although the operator is not invertible, there is a subspace for which the reduction of the operator to that subspace is indeed invertible:

Theorem 1. ([2, Theorem 2]) Let $A, T \in B(X)$ be nonzero operators. The following statements are equivalent.

1. A is invertible along T.
2. $\mathcal{R}(T)$ is a closed and complemented subspace of X, $A(\mathcal{R}(T)) = \mathcal{R}(AT)$ is closed such that $\mathcal{R}(AT) \oplus \mathcal{N}(T) = X$ and the reduction $A|_{\mathcal{R}(T)} : \mathcal{R}(T) \to \mathcal{R}(AT)$ is invertible.

Recall an operator $A \in B(X)$ is said to be group invertible if there exists $B \in B(X)$ such that

$$A = ABA, \quad B = BAB, \quad AB = BA.$$

In this case, such B is unique and we write $B = A^{\sharp}$ for the group inverse of A.

Proposition 2. ([2, Theorem 3]) If $A \in B(X)$ is invertible along $T \in B(X)$, then AT and TA are group invertible and $A^{\parallel T} = T(AT)^{\sharp} = (TA)^{\sharp} T$.

Example 3. Let $X = \ell_2$ the space of square-summable sequences. Let $A, T \in B(X)$ be defined by $Ax := (x_2, x_3, x_4, x_5, \cdots)$ and $Tx := (0, x_1, x_2, x_3, x_4, \cdots)$. Then A is invertible along T with $A^{\parallel T} = T$.

In the following section, we study an operator A such that A is invertible along T with $AT = TA$. Then, in Section 3 we study some projections related to the outer inverse with prescribed range and null space. Finally, in Section 4 we specialize to spectral projections, covering results from Dajić and Koliha ([3]).

2. Invertibility along a Commuting Operator

Proposition 4. Let A be invertible along T. If $AT = TA$, then $AA^{\|T} = A^{\|T}A$.

Proof. From Proposition 2 we have:

$$AA^{\|T} = AT(AT)^{\sharp} = (AT)^{\sharp} AT = (TA)^{\sharp} TA = A^{\|T}A.$$

If A is invertible along T, then we have the following matrix form ([2]):

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{M} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(AT) \\ \mathcal{N}(T) \end{bmatrix},$$

where A_1 is invertible and $\mathcal{M}$ is a complement of $\mathcal{R}(T)$, that is, $X = \mathcal{R}(T) \oplus \mathcal{M}$.

When A and T commute, we can say a little more:

Theorem 5. Let A be invertible along T and $AT = TA$. Then there exist an invertible operator A_1 on $\mathcal{R}(T)$ and an operator A_2 on $\mathcal{N}(T)$ such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix}$$

and $A^{\|T} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$

Proof. Suppose A is invertible along T and $AT = TA$. Then by Proposition 4 $AA^{\|T} = A^{\|T}A$. Thus, since $AA^{\|T} = A^{\|T}A$ is a projection, we have that

$$X = \mathcal{R}\left(A^{\|T}A\right) \oplus \mathcal{N}\left(AA^{\|T}\right).$$

Since

$$\mathcal{R}(T) = \mathcal{R}\left(A^{\|T}\right) = \mathcal{R}\left(A^{\|T}A\right)$$

and

$$\mathcal{N}(T) = \mathcal{N}\left(A^{\|T}\right) = \mathcal{N}\left(A^{\|T}A\right)$$

we also have

$$X = \mathcal{R}(T) \oplus \mathcal{N}(T),$$

and hence we can consider the following matrix decomposition of A:

$$A = \begin{bmatrix} A_1 & A_3 \\ A_4 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix}.$$

In this case, $A_1 : \mathcal{R}(T) \to \mathcal{R}(T)$, $A_1 x = Ax$, is invertible. Indeed, to see that it is onto note that since $\mathcal{R}(AT) = \mathcal{R}(TA) \subset \mathcal{R}(T)$ and $\mathcal{R}(T) = \mathcal{R}\left(AA^{\|T}\right) = \mathcal{R}\left(AT(AT)^{\sharp}\right) \subseteq \mathcal{R}(AT)$, we have $\mathcal{R}(T) = \mathcal{R}(AT)$ and hence $\mathcal{R}(A_1) = \mathcal{R}(AT) = \mathcal{R}(T)$. To see that it is also 1-1, let $x \in \mathcal{N}(A) \cap \mathcal{R}(T)$. Since $\mathcal{R}(T) = \mathcal{R}\left(A^{\|T}\right)$, there exists $y \in X$ such that $x = A^{\|T}y$. Then, $0 = A^{\|T}Ax = A^{\|T}AA^{\|T}y = A^{\|T}y = x$.

Moreover, since $AT = TA$, subspaces $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are A-invariant and A maps $\mathcal{R}(T)$ onto $\mathcal{R}(T) = \mathcal{R}(AT)$, we get $A_3 = A_4 = 0$. Thus, $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix}$ and clearly $A^{\|T} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$.

Example 6. Let $X = \ell_2$ the space of square-summable sequences. Let $A, T \in B(X)$ be defined by

$$Ax := (x_2, 2x_1, 0, 0, \cdots),$$

$$Tx := (x_2, x_1, 0, 0, \cdots).$$

Then it is easy to verify that $A^{\|T}$ is the operator such that

$$A^{\|T}x = \left(\frac{1}{2}x_2, x_1, 0, 0, \cdots\right).$$

It is clear that $AA^{\|T} = A^{\|T}A$, but we have

$$ATx = (x_1, 2x_2, 0, 0, \cdots) \neq (2x_1, x_2, 0, 0, \cdots) = TAx.$$

3. Projections

Commuting outer inverses are naturally linked to projections.

Proposition 7. Let $A, T \in B(X)$. If A is invertible along T and $AT = TA$, then there exists a bounded projection $P \in B(X)$ such that A is invertible along P.

Proof. From Theorem 5 we have that if A is invertible along T and $AT = TA$, then

$$X = \mathcal{R}(T) \oplus \mathcal{N}(T).$$

Thus, there exists a bounded projection $P \in B(X)$ such that $\mathcal{R}(P) = \mathcal{R}(T)$ and $\mathcal{N}(P) = \mathcal{N}(T)$. Hence, A is invertible along P.

For a sort of converse, we give a necessary condition in Theorem 9.

Example 8. An operator A and a projection P such that A is invertible along P and $AP \neq PA$.

Let $M_2(\mathbb{R})$ be the set of two by two matrices with real entries. Let $A \in M_2(\mathbb{R})$ be the (rotation) matrix defined by

$$A := \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

and let $P \in M_2(\mathbb{R})$ be the projection defined by

$$P := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, it is easy to check that $AP = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ is group invertible, with group inverse $(AP)^\sharp = \begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2} & 0 \end{bmatrix}$.

We have

$$A^{\parallel P} = P(AP)^\sharp = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, A is invertible along P but since $PA = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}$, we see $AP \neq PA$. Notice that $\mathcal{R}(AP) \not\subset \mathcal{R}(P)$.

Theorem 9. Let $P \in B(X)$ be a projection and suppose $A \in B(X)$ is invertible along P. If $\mathcal{R}(AP) \subset \mathcal{R}(P)$, then $AP = PA$.

Proof. Since $A \in B(X)$ is invertible along P, A has the following matrix form [2, Corollary 1]:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{M} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(AP) \\ \mathcal{N}(P) \end{bmatrix},$$

where A_1 is invertible and $\mathcal{M}$ is the complement of $\mathcal{R}(P)$, that is, $X = \mathcal{R}(P) \oplus \mathcal{M}$.

Since $\mathcal{N}(P)$ is a complement for $\mathcal{R}(P)$, we can take $\mathcal{M} = \mathcal{N}(P)$. Then, we have

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(AP) \\ \mathcal{N}(P) \end{bmatrix},$$

Now, suppose $\mathcal{R}(AP) \subset \mathcal{R}(P)$. From Theorem 1 we know that $A_1 = A|_{\mathcal{R}(P)} : \mathcal{R}(P) \to \mathcal{R}(AP)$ is invertible, which implies $\mathcal{R}(AP) = \mathcal{R}(P)$. Thus,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{bmatrix}.$$

It follows $AP = PA$.

Note that the theorem above together with Proposition 4 implies that if $\mathcal{R}(AP) \subset \mathcal{R}(P)$, then $A^{\parallel P}P = PA^{\parallel P}$. However, we can prove:

Proposition 10. Let $P \in B(X)$ be a projection, and suppose A is invertible along P. Then $A^{\parallel P}P = PA^{\parallel P}$.

Proof. Using Proposition 4,

$$\begin{aligned} A^{\parallel P}P &= (PA)^\sharp PP = (PA)^\sharp P \\ &= P(AP)^\sharp = PP(AP)^\sharp = PA^{\parallel P}. \end{aligned}$$

Example 11. An operator $A \in B(X)$ and a projection $P \in B(X)$ such that $AP = PA$ but A is not invertible along P.

Let $X = \ell_2$ and let $A, P \in B(X)$ be defined by

$$Ax = (0, x_2, x_3, x_4, \cdots),$$
$$Px = (x_1, x_2, 0, 0, \cdots).$$

Then

$$APx = (0, x_2, 0, 0, \cdots),$$
$$PAx = (0, x_2, 0, 0, \cdots).$$

However, the reduction $A|_{\mathcal{R}(P)} : \mathcal{R}(P) \to \mathcal{R}(AP)$ can not be invertible, and by Theorem 1 A is not invertible along P.

Theorem 12. Let $P \in B(X)$ be a projection and $A \in B(X)$ be such that $AP = PA$. Then, A is invertible along P if and only if $\mathcal{R}(P) \subset \mathcal{R}(A)$ and $\mathcal{N}(A) \subset \mathcal{N}(P)$.

Proof. Suppose A is invertible along P. Since $\mathcal{R}(A^{\parallel P}) = \mathcal{R}(A^{\parallel P}AA^{\parallel P}) \subset \mathcal{R}(A^{\parallel P}A) \subset \mathcal{R}(A^{\parallel P})$ and from Proposition 4 and the definition of $A^{\parallel P}$ we have

$$\mathcal{R}(P) = \mathcal{R}(A^{\parallel P}) = \mathcal{R}(A^{\parallel P}A) = \mathcal{R}(AA^{\parallel P}) \subset \mathcal{R}(A).$$

Similarly, from

$$\mathcal{N}(A^{\parallel P}) = \mathcal{N}(A^{\parallel P}AA^{\parallel P}) \supset \mathcal{N}(AA^{\parallel P}) \supset \mathcal{N}(A^{\parallel P}),$$

Proposition 4 and the definition of $A^{\parallel P}$ we get

$$\mathcal{N}(P) = \mathcal{N}(A^{\parallel P}) = \mathcal{N}(AA^{\parallel P}) = \mathcal{N}(A^{\parallel P}A) \supset \mathcal{N}(A).$$

Conversely, suppose $\mathcal{R}(P) \subset \mathcal{R}(A)$ and $\mathcal{N}(P) \supset \mathcal{N}(A)$. We use Theorem 1. The reduction $A|_{\mathcal{R}(P)} : \mathcal{R}(P) \to \mathcal{R}(AP)$ is clearly onto. From $\mathcal{N}(A) \subset \mathcal{N}(P)$ and $\mathcal{N}(P) \cap \mathcal{R}(P) = \{0\}$ we get that it is also 1-1.To see that $\mathcal{R}(AP)$ is closed and $\mathcal{R}(AP) \oplus \mathcal{N}(P) = X$ we will show $\mathcal{R}(AP) = \mathcal{R}(P)$. It is clear that $\mathcal{R}(AP) = \mathcal{R}(PA) \subset \mathcal{R}(P)$. For the other inclusion, let $x \in \mathcal{R}(P)$. Since $\mathcal{R}(P) \subset \mathcal{R}(A)$, there exists $y \in X$ such that $x = Ay$. Then, from $x = Px = PAy = APy$ it follows that $x \in \mathcal{R}(AP)$. Thus, $\mathcal{R}(P) \subset \mathcal{R}(AP)$. Finally, it is clear that $\mathcal{R}(P)$ is closed and complemented. Hence A is invertible along P.

4. Spectral Projections

Recall the spectrum $\sigma(A)$ of an operator $A \in B(X)$ is the set $\sigma(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$.

Suppose A is invertible along T and $AT = TA$. Then we have the matrix form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T) \end{bmatrix}.$$

From $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are invariant under A, we have:

$$\sigma(A) = \sigma(A_1) \cup \sigma(A_2).$$

Since A_1 is invertible but A is not, we know that $0 \in \sigma(A_2)$.

Note that from $X = \mathcal{R}(T) \oplus \mathcal{N}(T)$ and the matrix form of A, there exists a projection P such that $\mathcal{R}(P) = \mathcal{R}(T)$, $\mathcal{N}(P) = \mathcal{N}(T)$ and $PA = AP$. Thus, without loss of generality, we can suppose A is invertible along a projection P. A very important class of projections which commutes with A is a class of spectral projections, which we now discuss.

The resolvent set $\rho(A)$ of A is $\rho(A) := \mathbb{C} \setminus \sigma(A)$ and for $\lambda \in \rho(A)$ the resolvent function $R_\lambda(A)$ is

$$R_\lambda(A) := (A - \lambda I)^{-1}.$$

A subset $\Lambda \subset \sigma(A)$ is said to be a spectral set of A if Λ and $\sigma(A) \setminus \Lambda$ are both closed in $\mathbb{C}$. For a spectral set Λ of A, $P_\Lambda(A)$ the spectral projection associated with A is defined by

$$P_\Lambda(A) := -\frac{1}{2\pi i}\int_C R_\lambda(A)\,\mathrm{d}\lambda,$$

where C is a Cauchy contour that separates Λ from $\sigma(A) \setminus \Lambda$.

If 0 is a point of the resolvent set or an isolated point of the spectrum $\sigma(A)$, then the operator A is called quasipolar. Let A be quasipolar and let P_0 be the spectral projection associated with the spectral set $\{0\}$, then ([4]):

$$X = \mathcal{R}(P_0) \oplus \mathcal{N}(P_0)$$

Let $Q \in B(X)$. If $\sigma(Q) = \{0\}$, then we say that Q is a quasinilpotent operator. Recall that Q is nilpotent if $Q^n = O$ for some $n \in \mathbb{N}$, and nilpotent operators are quasinilpotent.

Quasipolar operators are generalized invertible in the sense of Koliha: an operator $A \in B(X)$ is Koliha-Drazin invertible if there exists $B \in B(X)$ such that $A(I - AB)$ is quasinilpotent, $B = BAB$, $AB = BA$.

In this case, by Lemma 2.4 of [5], Koliha-Drazin inverse is unique and we write $B = A^D$.

An operator A is Koliha-Drazin invertible if and only if 0 is an isolated point of $\sigma(A)$. If $0 \in \sigma(A)$ is a pole of the resolvent of order n, then A is Drazin invertible with Drazin index n. If 0 is a simple pole then it is group invertible.

As noted above, the Koliha-Drazin is a particular case when we consider the spectral set $\Lambda = \{0\}$. For the general case when Λ is a spectral set such that $0 \in \Lambda$, Dajić and Koliha have defined a generalized inverse and studied its properties ([3]).

Theorem 13. Let $A \in B(X)$ and Λ be a spectral set for A. If $0 \notin \Lambda$ then A is invertible along $P_\Lambda(A)$.

Proof. Let $P = P_\Lambda(A)$. Then $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are closed and $X = \mathcal{R}(P) \oplus \mathcal{N}(P)$. Now, since $\mathcal{R}(P)$ is A-invariant, $\sigma\left(A|_{\mathcal{R}(P)}\right) = \Lambda$ and $0 \notin \Lambda$ we have that $A|_{\mathcal{R}(P)} : \mathcal{R}(P) \to \mathcal{R}(P)$ is invertible. Thus, $\mathcal{R}(AP) = \mathcal{R}(P)$ is closed, $\mathcal{R}(AP) \oplus N(P) = X$ and $A|_{\mathcal{R}(P)} : \mathcal{R}(P) \to \mathcal{R}(AP)$ is invertible. Therefore, by Theorem 1 A is invertible along P.

Corollary 14. Let $A \in B(X)$ and Λ be a spectral set for A. If $0 \in \Lambda$ then A is invertible along $I - P_\Lambda(A)$.

Proof. If $0 \in \Lambda$, then $0 \notin \sigma(A) \setminus \Lambda$. From the theorem above, A is invertible along $P_{\sigma(A)\setminus\Lambda}(A) = I - P_\Lambda(A)$.

5. Acknowledgements

This research is partially supported by Grant-in-Aid Scientific Research No.24540195.

REFERENCES

[1] X. Mary, "On Generalized Inverses and Green's Relations," *Linear Algebra Applications*, Vol. 434, No. 8, 2011, pp. 1836-1844. http://dx.doi.org/10.1016/j.laa.2010.11.045

[2] G. Kantún-Montiel and S. V. Djordjević, "Invertibility along an Operator," unpublished.

[3] A. Dajić and J. J. Koliha, "The Sigma-g-Drazin Inverse and the Generalized Mbekhta Decomposition," *Integral Equations and Operator Theory*, Vol. 57, No. 3, 2007, pp. 309-326. http://dx.doi.org/10.1007/s00020-006-1454-0

[4] P. Aiena, "Fredholm and Local Spectral Theory, with Applications to Multipliers," Kluwer Academic Publishers, Dordrecht-Boston-London, 2004.

[5] J. J. Koliha, "A Generalized Drazin Inverse," *Glasgow Mathematical Journal*, Vol. 38, No. 3, 1996, pp. 367-381. http://dx.doi.org/10.1017/S0017089500031803