American Journal of Operations Research, 2026, 16(2), 55-78
https://www.scirp.org/journal/ajor
ISSN Online: 2160-8849
ISSN Print: 2160-8830

Sustainable Inventory System for Decaying Products with Advertisement, Price, and Time-Sensitive Demand, and Expiration Date

Sushil Kumar, Manish Kumar, Shivani Singh

Department of Mathematics & Astronomy, University of Lucknow, Lucknow, India
Email: sushilmath4444@gmail.com, manishmsw9@gmail.com, shivanisinghfd@gmail.com

How to cite this paper: Kumar, S., Kumar, M. and Singh, S. (2026) Sustainable Inventory System for Decaying Products with Advertisement, Price, and Time-Sensitive Demand, and Expiration Date. *American Journal of Operations Research*, **16**, 55-78.
https://doi.org/10.4236/ajor.2026.162003

Received: December 23, 2025
Accepted: March 9, 2026
Published: March 12, 2026

Abstract

Sustainability is essential for growing environmental concern especially those related to carbon emissions by the supply chain, and production. In today's era, mitigating carbon emissions, and waste has become a major concern for a manufacturing firm or industry. Different tax policies are employed for high-emission industrial, and commercial activities. This study proposes a sustainable inventory system for decaying products that are socially, economically, and environmentally savvy. It is assumed that the demand is advertisement, price, and time-sensitive. The price and advertisement of a product attract the customers to buy more or less. Most of the products have a fixed life span or expiration date for maintaining their quality in original properties. Therefore, the customers prefer the products of maximum life span. This model is designed with a sustainable goal that the incurred total cost is minimized over a planning horizon, together with the determination of optimal price, time period of positive inventory, and the cycle length. The practical utility, and a better understanding of this model are shown with the help of a numerical example. A sensitivity analysis of some crucial parameters is conducted, and it is based on their variations.

Keywords

Inventory, Deterioration, Expiration Date, Advertisement, Price, Time-Reliant Demand

1. Introduction

A major concern for the industries or businesses is based to reduce their carbon emissions. For this, they redesign or restructure their inventory planning so that the carbon emission of the system is as low as possible. This is fulfilled by adopting the concept of sustainable development. Sustainable development is defined as an economic development that is not destroying the environment. At first, Sustainable development was adopted by United Nations. Later, it is adopting by each of the national, and international governments.

Generally, sustainability is used in global sense, and the goal of sustainability is global. Therefore, sustainability is used in this paper, and its direct impacts are as follows,

- Business impact: Here, consumers support to sustainable businesses. They are reducing their wastage, and energy consumption.
- Environmental impact: Here, businesses protect the environment and conserve natural resources. The exploitation of natural resources is made as low as possible.
- Social impact: Here, industries or businesses know their importance for the welfare of peoples, and societies.

Panda *et al.* [1] proposed an economic order quantity (EOQ) model for decaying products with stock-varying demand, and discounted selling price. Sarkar and Sarkar [2] explored an improved inventory model for ameliorating items with stock-dependent demand, and time-varying deterioration rate. Shah *et al.* [3] examined an optimal inventory and marketing policy for non-instantaneous decaying products. Chen *et al.* [4] addressing a carbon constrained EOQ model for perishable items. Choudhury *et al.* [5] created an inventory model for decaying products. They included the stock-dependent demand, time-varying holding cost, and allowed shortages. Hovelaque and Bironneau [6] constructed an EOQ model for perishable items using carbon-emission dependent demand.

Advertisement and price of products have a crucial role in increasing the demand. These attract the customers to buy more or less within your budget. Shaikh [7] created an inventory model for deteriorating items using selling-price varying demand, advertisement, and mixed type trade-credit policy.

In real-world inventory control systems, the deterioration of products has a significant issue. The deterioration is reduced with the help of preservation technology. Preservation technology is defined as an essential component used in minimizing the deterioration rate. It is also used to measure and control the deterioration rate simultaneously. Nowadays, preservation technology has been adopted by every manufacturing/business firm. Because control on deterioration rate of bakery products, foods grains, soft drinks, medicines, vaccines etc. is a big challenge.

Mishra *et al.* [8] studied an inventory model with price, and stock-dependent demand with controllable deterioration rate using preservation technology investment. Mishra [9] optimized a three-rates-of-production inventory model for deteriorating items involving selling price, advertisement-dependent demand, and shortages. Kazami *et al.* [10] analyzed an EOQ model for decaying products with imperfect quality, and carbon-emission constraints. Shaikh *et al.* [11] addressed an EOQ model for decaying items with stock-dependent demand, price discount facility, and shortages. Tiwari *et al.* [12] presented sustainable ordering policies for an inventory model of non-instantaneous deteriorating items with multi-trade-credit policies, and carbon emissions. Taleizadeh *et al.* [13] examined an inventory model with joint pricing and inventory decision policy for deteriorating items under carbon emission. Mishra *et al.* [14] discussed an optimum sustainable management policy for a back-ordered inventory model of deteriorating items under controllable carbon emissions. Daryanto *et al.* [15] revised sustainable EOQ model for deteriorating items considering carbon emissions. San-Jose *et al.* [16] presented an optimal policy for an inventory model for perishable items with price, time, and advertisement dependent demand. Sarkar *et al.* [17] constructed a supply chain inventory model. They showed the contributed effect of carbon emission, and production quality improvement for fixed lifetime products. Mishra *et al.* [18] addressed a supply chain inventory model for decaying items with controlled deterioration, and carbon emission in a greenhouse firm. Das *et al.* [19] explored a multi-objective solid transportation-location problem incorporating carbon emission in inventory management. Taleizadeh *et al.* [20] created a sustainable inventory model for deteriorating items with price-sensitive demand. This model consists of carbon emission, partial trade-credit policy, and partial backlogging. Mishra and Mishra [21] focused on a sustainable inventory model for non-instantaneous deteriorating items with quality assessment. This model includes carbon emissions, and shortages. Kumar *et al.* [22] constructed a production inventory model for perishable items with advertisement-dependent demand, and supply chain management under carbon emission. Kugele *et al.* [23] created a production system, and analyzed a geometric programming solution of second degree difficulty for carbon ejection. Sankari *et al.* [24] studied a sustainable inventory model for growing items incorporating carbon emissions, product expiry, and profit sharing policy. Magfura *et al.* [25] analyzed a sustainable inventory model for non-instantaneous deteriorating items with composite demand. Sobia *et al.* [26] considered a deterministic inventory model for constant deteriorating items with a generalized exponential diminishing demand, and stable holding cost. Kumar *et al.* [27] developed a sustainable inventory system for decaying products with expiration date, carbon emission, and price-sensitive exponentially decreasing demand.

1.1. Research Gap

The previous inventory models have been studied and compared with this model. Sustainability is adopted by several researchers in their models. Guo and Zhang [28] examined an inventory model for decaying products with stock dependent demand, and variable holding cost rate. Rangaranjan *et al.* [29] explored a sustainable production model. They assumed the power-pattern demand, carbon emissions, and self life considerations in their model. Alharbi [30] employs a dragonfly algorithm in their model. This model investigates the controlled non-instantaneous deterioration for green products.

1.1.1. Contribution

This study considers the sustainability factors, which are summarized as follows,

- The advertisement, price-sensitive, and time varying demand correspond to the finance sustainability.
- The carbon emission generated by several operational-activities linked with the inventory is assumed regarding environmental sustainability.
- The expiration dates, and partial backlogging of products are used for the welfare of society.

1.1.2. Objective

This study has the following objectives,

1) To analyze the impact of backlogging parameter on the total cost.

2) To analyze the impact of expiration date of products on the total cost.

3) To analyze the impact of advertising, and holding cost parameters on the total cost.

2. Assumptions and Notations

The following assumptions and notations are linked with this model,

1) The demand rate is $D(s,t)=\begin{cases} A_c(a-bs+ct), & 0\le t\le T_1 \\ A_C(a-bs), & T_1\le t\le T \end{cases}$, where A_C, and s are the advertisement, and price of product and $a,b,c\ge 0$ are constants.

2) The products deplete with time and cannot be sold to the customers as the expiration date is reached. The expiration date dependent deterioration rate is $\theta(t)=\frac{1}{1+m-t}$, $0\le t\le T\le m$, where m is the expiration date. At the starting (as $t\to 0$) the deterioration rate is minimum, and when $t\to m$ the deterioration rate is 1 which means all the products deteriorate at its expiration date (as in Wu *et al.* 2018).

3) The products are not reworked.

4) The lead time is zero.

5) The backlogging rate is, $B(t)=\frac{1}{1+\delta(T-t)}$, where δ is a backlogging parameter, and t is the waiting time.

6) The ordering cost per order is o_c.

7) The holding cost per unit per unit time is h_c.

8) The shortage cost per unit is s_c.

9) The purchase cost per unit is p_c.

10) The lost sales cost per unit is ls_c.

11) The carbon emission rate for placing the orders is γ_1.

12) The carbon emission rate for holding the orders is γ_2.

13) The carbon emission rate for shipping the orders is γ_3.

14) The shipping price per unit is sh_c.

15) The carbon tax per unit is c_T.

16) The time period of positive inventory level is T_1.

17) The replenishment cycle length is T.

18) The decision variables are s, T_1, and T.

19) The total inventory cost per unit per unit time is $TC(s, T_1, T)$.

3. Statement of the Problem

Mathematical Derivation of Model

Here, the inventory system under assumptions is given by Figure 1. The inventory system consists of Q units of the product in the beginning of each cycle. The inventory level Q is gradually becoming depleted, due to demand and deterioration in the time-interval $[0, T_1]$, and becomes zero at time $t = T_1$. Just after the time $t = T_1$ shortage starts. In the time-interval $[T_1, T]$ shortages are backlogged at the rate $B(t) = \frac{1}{1+\delta(T-t)}$, where δ is a backlogging parameter, and t is a waiting time.

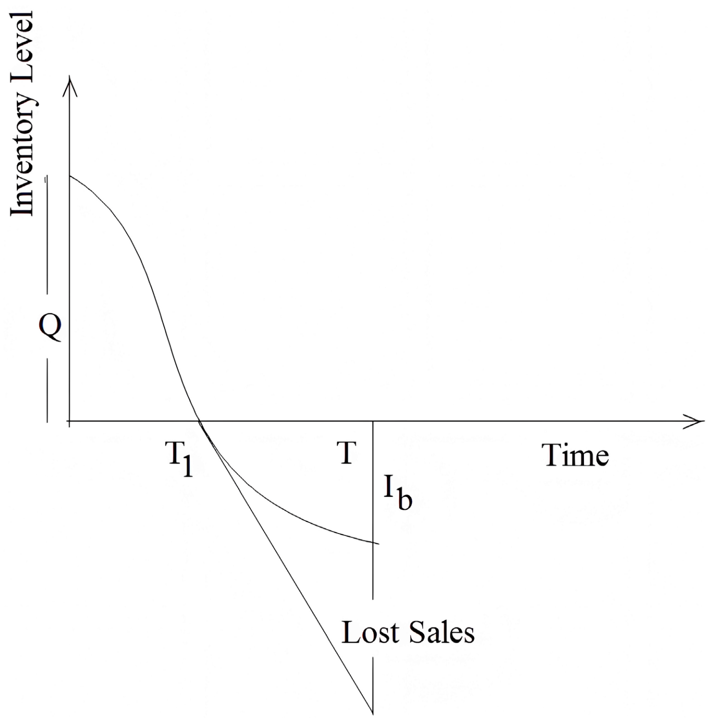

Figure 1. Inventory model.

The instantaneous inventory level at any time t in the time-interval $[0, T]$ is given by the following differential equations,

$$\frac{dI_1}{dt} + \frac{1}{1+m-t} I_1 = -\{A_C(a-bs)+ct\}, \quad 0 \le t \le T_1 \tag{1}$$

Together with, an initial condition, $I_1(T_1) = 0$

$$\frac{dI_2}{dt} = -\frac{A_C(a-bs)}{1+\delta(T-t)}, \quad T_1 \le t \le T \tag{2}$$

Together with, an initial condition, $I_2(T_1)=0$

The solutions of Equations (1) & (2) are given by Equations (3) & (4) respectively,

$$I_1 = (1+m-t)\left[A_C(a-bs)\log\left\{\frac{1+m-t}{1+m-T_1}\right\}-ct-cT_1+c(1+m)\log\left\{\frac{1+m-t}{1+m-T_1}\right\}\right]$$

Or

$$\begin{aligned}I_1 =& -\left\{c(1+m)+A_C(a-bs)(1-m)+c\left(1-m^2\right)(1-m)\right\}t+\left\{-c(1+m)\right.\\&\left.+A_C(a-bs)(1-m)+c\left(1-m^2\right)(1-m)\right\}T_1+\frac{1}{2}\left\{2c-A_C(a-bs)\right.\\&\left.+2A_C(a-bs)(1-m)+c\left(1-m^2\right)\right\}t^2+\frac{1}{2}\left\{A_C(a-bs)(1-m)+c\left(1-m^2\right)\right\}T_1^2\\&+\left\{c-A_C(a-bs)(1-m)-c\left(1-m^2\right)\right\}tT_1+\frac{1}{2}\left\{A_C(a-bs)+c(1+m)\right\}t^3\\&-\frac{1}{2}\left\{A_C(a-bs)+c(1+m)\right\}tT_1^2,\quad 0\le t\le T_1\end{aligned}\tag{3}$$

$$I_2=\frac{A_C(a-bs)}{\delta}\log\left\{\frac{1+\delta(T-t)}{1+\delta(T-T_1)}\right\}$$

Or

$$I_2=-\frac{A_C(a-bs)}{2}\left[2t-2T_1+\delta t^2-\delta T_1^2-2\delta Tt+2\delta TT_1\right],\quad T_1\le t\le T\tag{4}$$

The initial inventory Q is calculated by substituting $I_1(0)=Q$ in Equation (3), so

$$\begin{aligned}Q=&\left\{-c(1+m)+A_C(a-bs)(1-m)+c\left(1-m^2\right)(1-m)\right\}T_1\\&+\frac{1}{2}\left\{A_C(a-bs)+c\left(1-m^2\right)\right\}\end{aligned}\tag{5}$$

The back order quantity I_B is calculated by substituting $I_2(T)=I_B$ in Equation (4), so

$$I_B=-\frac{A_C(a-bs)}{2}\left[2T-2T_1-\delta T^2-\delta T_1^2+2\delta TT_1\right]\tag{6}$$

The total variable inventory cost per unit per unit time is given by

$$TC(s,T_1,T)=\frac{1}{T}\left[O_C+H_C+S_C+P_C+LS_C+SH_C+CT_C\right]\tag{7}$$

where, $O_C, H_C, S_C, P_C, LS_C, SH_C$, and CT_C are the ordering cost, holding cost, shortage cost, purchase cost, lost sales cost, shipping cost, and carbon tax cost per unit per unit time respectively. The respective costs are calculated by the following equations.

The ordering cost per unit cycle is calculated by,

$$O_C=\frac{o_c}{T}\tag{8}$$

The holding cost per unit cycle is calculated by,

$$H_C = \frac{h_c}{T}\int_0^{T_1} I_1(t)\,\mathrm{d}t$$

Putting the value of $I_1(t)$ given by the Equation (3), we obtain

$$\begin{aligned} H_C = \Big[& \frac{1}{2}\left\{-3c(1+m) + A_C(a-bs)(1-m) + c(1-m)^2(1+m)\right\} T_1^2 \\ & + \frac{1}{6}\left\{5c + 2A_C(a-bs) - A_C(a-bs)(1-m) + c\left(1-m^2\right)\right\} T_1^3 \\ & - \frac{1}{8}\left\{A_C(a-bs) + c(1+m)\right\} T_1^4 \Big] \end{aligned} \quad (9)$$

The shortage cost per unit cycle is calculated by,

$$S_C = -\frac{s_c}{T}\int_{T_1}^{T} I_2(t)\,\mathrm{d}t$$

Putting the value of $I_2(t)$ given by Equation (4), we have

$$S_C = \frac{s_c A_C(a-bs)}{2T}\left[T^2 + T_1^2 - 2TT_1 - \frac{2\delta}{3}T^3 + \frac{2\delta}{3}T_1^3 + 2\delta T^2 T_1 - 2\delta T T_1^2\right] \quad (10)$$

The purchase cost per unit cycle is calculated by,

$$P_C = \frac{p_c}{T}\left[Q + I_B\right]$$

Putting the values of Q and I_B given by Equations (6) and (5), we have

$$\begin{aligned} P_C = \frac{p_c}{T}\Big[& \left\{-c(1+m) + A_C(a-bs)(1-m) + c(1-m)^2(1+m)\right\} T_1 \\ & - \frac{A_C(a-bs)}{2}\left(2T - 2T_1 - \delta T^2 - \delta T_1^2 + 2\delta T T_1\right) \\ & + \frac{1}{2}\left\{A_C(a-bs) + c\left(1-m^2\right)\right\} T_1^2 \Big] \end{aligned} \quad (11)$$

The lost sales cost per unit cycle is calculated by,

$$LS_C = \frac{ls_c}{T}\int_{T_1}^{T}\left[1 - \frac{1}{1+\delta(T-t)}\right] A_C(a-bs)\,\mathrm{d}t$$

After simplifying, we obtain

$$\begin{aligned} LS_C = \frac{ls_c A_C(a-bs)}{6T} & \left[3\delta T^2 + 3\delta T_1^2 - 6\delta T T_1 + 6\delta^2 T^2 T_1 - 6\delta^2 T T_1^2 \right. \\ & \left. - 2\delta^2 T^3 + 2\delta^2 T_1^3\right] \end{aligned} \quad (12)$$

The carbon emission associated in placing, holding, and shipping the orders is,

$$C_E = \gamma_1 + \gamma_2\left\{\int_0^{T_1} I_1(t)\,\mathrm{d}t + \int_{T_1}^{T} I_2(t)\,\mathrm{d}t\right\} + \gamma_3 Q$$

After simplifying, we have

$$\begin{aligned} C_E = \gamma_1 + \gamma_2\Big[& \frac{1}{2}\left\{-3c(1+m) + A_C(a-bs)(1-m) + c(1-m)^2(1+m)\right\} T_1^2 \\ & + \frac{1}{6}\left\{5c + 2A_C(a-bs) - A_C(a-bs)(1-m) + c\left(1-m^2\right)\right\} T_1^3 \end{aligned}$$

$$+\frac{A_C(a-bs)}{2}\left(T^2+T_1^2-2TT_1-\frac{2\delta}{3}T^3+\frac{2\delta}{3}T_1^3+2\delta T^2T_1-2\delta TT_1^2\right)$$
$$-\frac{1}{8}\left\{A_C(a-bs)+c(1-m)\right\}T_1^4\Bigg]+\gamma_3\left[\frac{1}{2}\left\{A_C(a-bs)+c\left(1-m^2\right)\right\}T_1^2\right.$$
$$+\left\{-c(1+m)+A_C(a-bs)(1-m)+c(1-m)^2(1+m)\right\}$$
$$\left.-\frac{A_C(a-bs)}{2}\left(2T-2T_1-\delta T^2-\delta T_1^2+2\delta TT_1\right)\right] \quad (13)$$

The carbon tax cost per unit cycle is calculated by,

$$CT_C=\frac{c_T\times C_E}{T}$$

Putting the value of C_E, given by the Equation (13), It becomes

$$CT_C=\frac{\gamma_1c_T}{T}+\frac{\gamma_2c_T}{T}\left[\frac{1}{2}\left\{-3c(1+m)+A_C(a-bs)(1-m)+c(1-m)^2(1+m)\right\}T_1^2\right.$$
$$+\frac{1}{6}\left\{5c+2A_C(a-bs)-A_C(a-bs)(1-m)+c\left(1-m^2\right)\right\}T_1^3$$
$$+\frac{A_C(a-bs)}{2}\left(T^2+T_1^2-2TT_1-\frac{2\delta}{3}T^3+\frac{2\delta}{3}T_1^3+2\delta T^2T_1-2\delta TT_1^2\right)$$
$$-\frac{1}{8}\left\{A_C(a-bs)+c(1-m)\right\}T_1^4\Bigg]+\frac{\gamma_3c_T}{T}\left[\frac{1}{2}\left\{A_C(a-bs)+c\left(1-m^2\right)\right\}T_1^2\right.$$
$$+\left\{-c(1+m)+A_C(a-bs)(1-m)+c(1-m)^2(1+m)\right\}$$
$$\left.-\frac{A_C(a-bs)}{2}\left(2T-2T_1-\delta T^2-\delta T_1^2+2\delta TT_1\right)\right] \quad (14)$$

The shipping cost per unit cycle is calculated by,

$$SH_C=\frac{sh_c\times(Q+I_B)}{T}$$

Putting the values of Q and I_B, given by Equations (6), and (5), we have

$$SH_C=\frac{sh_c}{T}\left[\left\{-c(1+m)+A_C(a-bs)(1-m)+c(1-m)^2(1+m)\right\}T_1\right.$$
$$-\frac{A_C(a-bs)}{2}\left(2T-2T_1-\delta T^2-\delta T_1^2+2\delta TT_1\right)$$
$$\left.+\frac{1}{2}\left\{A_C(a-bs)+c\left(1-m^2\right)\right\}T_1^2\right] \quad (15)$$

Putting the values of above calculated respective inventory costs in Equation (7), we obtain

$$TC(s,T_1,T)=\frac{1}{T}\Big[(o_c+c_T\gamma_1)+(p_c+sh_c+c_T\gamma_3)\left\{-c(1+m)+c(1-m)^2(1+m)\right.$$
$$\left.+A_C(a-bs)(2-m)\right\}T_1-A_C(a-bs)(p_c+sh_c+c_T\gamma_3)T$$
$$+\frac{1}{2}\Big[h_c\left\{-3c(1+m)+c(1+m)(1-m)^2+A_C(a-bs)(1-m)\right\}$$
$$-A_C(a-bs)(s_c-p_c-2\delta p_c)+p_cc\left(1-m^2\right)+ls_c\delta A_C(a-bs)$$

$$
\begin{aligned}
&+sh_c A_C(a-bs)(1+\delta)+sh_c c(1-m^2)+c_T\gamma_2\{-3c(1+m)\\
&+c(1-m)^2(1+m)+A_C(a-bs)(2-m)\}+c_T\gamma_3\{c(1-m^2)\\
&+A_C(a-bs)(1+\delta)\}\big]T_1^2+\frac{A_C(a-bs)}{2}(-s_c+p_c+ls_c\delta+sh_c\delta\\
&+c_T\gamma_2+c_T\gamma_3\delta)T^2+\{A_C(a-bs)(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta)\\
&+2\delta p_c\}TT_1+\frac{1}{6}\Big[\{5c+c(1-m^2)+A_C(a-bs)(1+m-2\delta s_c\\
&+2\delta^2 ls_c)\}+c_T\gamma_2\{5c+c(1-m^2)+A_C(a-bs)(1+m+2\delta)\}\Big]T_1^3\\
&+\frac{A_C(a-bs)\delta}{3}(s_c-ls_c\delta-c_T\gamma_2)T^3+A_C(a-bs)\delta(-s_c+ls_c\delta\\
&+c_T\gamma_2)T^2T_1+A_C(a-bs)\delta(s_c-ls_c\delta-c_T\gamma_2)TT_1^2\\
&-\frac{h_c}{8}\{A_C(a-bs)+c(1+m)\}T_1^4\Big]
\end{aligned}
\tag{16}
$$

The total cost function $TC(s,T_1,T)$ will be minimum, if the first order derivatives of $TC(s,T_1,T)$ are satisfying,

$$
\frac{\partial TC(s,T_1,T)}{\partial s}=0,\quad \frac{\partial TC(s,T_1,T)}{\partial T_1}=0,\quad \frac{\partial TC(s,T_1,T)}{\partial T}=0 \tag{17}
$$

After solving equations $\frac{\partial TC(s,T_1,T)}{\partial s}=0$, $\frac{\partial TC(s,T_1,T)}{\partial T_1}=0$, $\frac{\partial TC(s,T_1,T)}{\partial T}=0$, we obtain the optimum values of decision variables s,T_1 and T for which the total cost function $TC(s,T_1,T)$ is minimum.

To find, the first order derivatives of $TC(s,T_1,T)$, we differentiate Equation (16) partially with respect to the decision variables s,T_1, and T. We have,

$$
\begin{aligned}
\frac{\partial TC}{\partial T_1}=\frac{1}{T}\Big[&(p_c+sh_c+c_T\gamma_3)\{-cm(1+m)(2-m)+A_C(a-bs)(2-m)\}\\
&+h_c\{-3c(1+m)+c(1+m)(1-m)^2+A_C(a-bs)(1-m)\}T_1\\
&+\{-A_C(a-bs)(s_c-p_c-2\delta p_c)+(p_c+sh_c)c(1-m^2)\}T_1\\
&+A_C(a-bs)\{ls_c\delta+sh_c(1+\delta)\}T_1+c_T\gamma_3\{c(1-m^2)+A_C(a-bs)(1+\delta)\}T_1\\
&+c_T\gamma_2\{-3c(1+m)+c(1+m)(1-m)^2+A_C(a-bs)(2-m)\}T_1\\
&+\{A_C(a-bs)(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta)+2p_c\delta\}T\\
&+\frac{1}{2}\{5c+c(1-m^2)+A_C(a-bs)(1+m-2s_c\delta+2ls_c\delta^2)\}T_1^2\\
&+\frac{c_T\gamma_2}{2}\{5c+c(1-m^2)+A_C(a-bs)(1+m+2\delta)\}T_1^2\\
&+A_C(a-bs)\delta(-s_c+ls_c\delta+c_T\gamma_2)T^2+2A_C(a-bs)\delta(s_c-ls_c\delta-c_T\gamma_2)TT_1\\
&-\frac{h_c}{2}\{A_C(a-bs)+c(1+m)\}T_1^3\Big]
\end{aligned}
\tag{18}
$$

$$\begin{aligned}\frac{\partial TC}{\partial s}=&\frac{1}{T}\Big[-A_Cb(2-m)(p_c+sh_c+c_T\gamma_3)T_1+A_Cb(p_c+sh_c+c_T\gamma_3)T\\&+\frac{A_Cb}{2}\{-h_c(1-m)+s_c-p_c-2p_c\delta-ls_c\delta-sh_c(1+\delta)-c_T\gamma_2(2-m)\}T_1^2\\&-\frac{A_Cb}{2}c_T\gamma_3(1+\delta)T_1^2-\frac{A_Cb}{2}(-s_c+p_c+ls_c\delta+sh_c\delta+c_T\gamma_2+c_T\gamma_3\delta)T^2\\&-A_Cb(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta)TT_1-\frac{A_Cb\delta}{3}(s_c-ls_c\delta-c_T\gamma_2)T^3\\&-\frac{A_Cb}{6}\{1+m-2s_c\delta+2ls_c\delta^2+c_T\gamma_2(1+m+2\delta)\}T_1^3\\&-A_Cb\delta(-s_c+ls_c\delta+c_T\gamma_2)T^2T_1-A_Cb\delta(s_c-ls_c\delta-c_T\gamma_2)TT_1^2\\&+\frac{A_Cbh_c}{8}T_1^4\Big]\end{aligned}\tag{19}$$

$$\begin{aligned}\frac{\partial TC}{\partial T}=&\frac{1}{T}\Big[-A_C(a-bs)(p_c+sh_c+c_T\gamma_3)+A_C(a-bs)(-s_c+p_c+ls_c\delta+sh_c\delta\\&+c_T\gamma_2+c_T\gamma_3\delta)T+\{A_C(a-bs)(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta)\\&+2p_c\delta\}T_1+A_C(a-bs)\delta(s_c-ls_c\delta-c_T\gamma_2)T^2+2A_C(a-bs)\delta(-s_c\\&+ls_c\delta+c_T\gamma_2)TT_1+A_C(a-bs)\delta(s_c-ls_c\delta-c_T\gamma_2)T_1^2\Big]\\&-\frac{1}{T^2}\Big[(o_c+c_T\gamma_1)+(p_c+sh_c+c_T\gamma_3)\{-c(1+m)+c(1-m)^2(1+m)\\&+A_C(a-bs)(2-m)\}T_1-A_C(a-bs)(p_c+sh_c+c_T\gamma_3)T\\&+\frac{1}{2}\Big[h_c\{-3c(1+m)+c(1+m)(1-m)^2+A_C(a-bs)(1-m)\}\\&-A_C(a-bs)(s_c-p_c-2\delta p_c)+p_cc(1-m^2)+ls_c\delta A_C(a-bs)\\&+sh_cA_C(a-bs)(1+\delta)+sh_cc(1-m^2)+c_T\gamma_2\{-3c(1+m)\\&+c(1-m)^2(1+m)+A_C(a-bs)(2-m)\}+c_T\gamma_3\{c(1-m^2)\\&+A_C(a-bs)(1+\delta)\}\Big]T_1^2+\frac{A_C(a-bs)}{2}(-s_c+p_c+ls_c\delta+sh_c\delta\\&+c_T\gamma_2+c_T\gamma_3\delta)T^2+\{A_C(a-bs)(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta)\\&+2\delta p_c\}TT_1+\frac{1}{6}\Big[\{5c+c(1-m^2)+A_C(a-bs)(1+m-2\delta s_c\\&+2\delta^2ls_c)\}+c_T\gamma_2\{5c+c(1-m^2)+A_C(a-bs)(1+m+2\delta)\}\Big]T_1^3\\&+\frac{A_C(a-bs)\delta}{3}(s_c-ls_c\delta-c_T\gamma_2)T^3+A_C(a-bs)\delta(-s_c+ls_c\delta\\&+c_T\gamma_2)T^2T_1+A_C(a-bs)\delta(s_c-ls_c\delta-c_T\gamma_2)TT_1^2\\&-\frac{h_c}{8}\{A_C(a-bs)+c(1+m)\}T_1^4\Big]\end{aligned}\tag{20}$$

4. Numerical Example

Let us consider a numerical example consisting of the following data for the referred parameters of the system in appropriate units as follows,

$$a = 500, b = 0.4, c = 0.5, o_c = 100, h_c = 8, s_c = 4, m = 5, \delta = 0.1, \gamma_1 = 0.1,$$
$$\gamma_2 = 0.3, \gamma_3 = 0.2, sh_c = 0.6, A_C = 0.25, c_T = 0.7, ls_c = 0.8, p_c = 15$$

Table 1 shows the variation in backlogging parameter δ corresponding to the optimal values of decision variables s, T_1 and T. The remaining parameters and variables are assumed fixed.

Table 1. Variation in total cost concerning backlogging parameter.

δ	s	T_1	T	$TC(s,T_1,T)$
0.1	1182.66413	2.55155	4.53248	1454.18203
0.3	1178.81855	2.62155	4.37810	1579.58727
0.5	1172.73116	2.67873	4.20091	1682.20222
0.7	1163.72911	2.72389	3.99832	1761.56469
0.9	1150.96747	2.76293	3.77296	1822.56140

In view of **Table 1**, **Figure 2** gives the pictorial depiction of total cost $TC(s,T_1,T)$ with respect to the backlogging parameter δ.

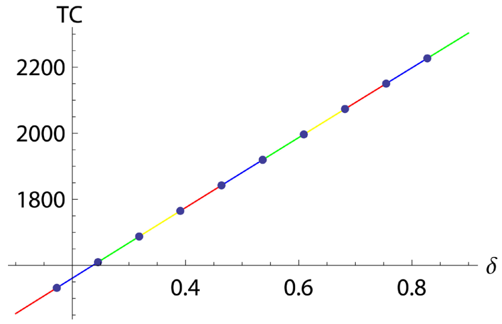

Figure 2. Variation in total cost w. r. to backlogging parameter.

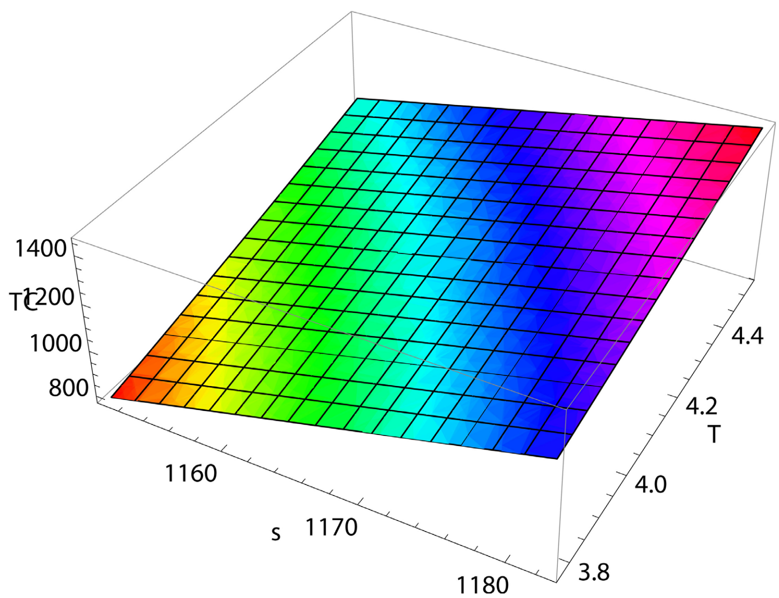

Figure 3. Variation in total cost w. r. to s and T.

In the reference of Table 1, Figure 3 depicts the three dimensions depiction of total cost $TC(s,T_1,T)$ with respect to the decision variables s and T.

In context of Table 1, Figure 4 shows the three dimensions pictorial depiction of total cost $TC(s,T_1,T)$ with respect to the decision variables s and T_1.

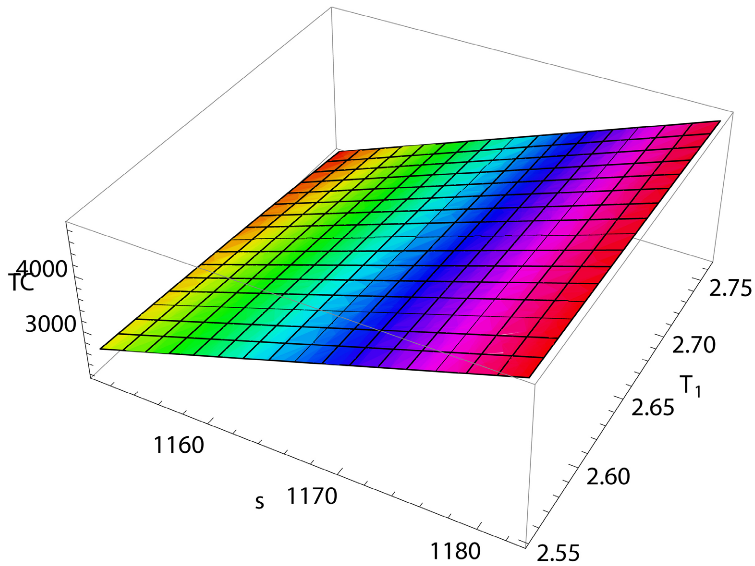

Figure 4. Variation in total cost w. r. to s and T_1.

Table 2 gives the variation in expiration date parameter m corresponding to the optimal values of decision variables s, T_1 and T. The rest parameters are assumed fixed.

Table 2. Variation in total cost concerning expiration date parameter.

m	s	T_1	T	$TC(s,T_1,T)$
5	1182.66413	2.55155	4.53248	1454.18203
7	1127.38467	3.64978	7.53855	8293.33036
9	1057.90749	4.59818	10.72909	28355.73998
12	936.51501	5.94753	15.99766	113214.26354
15	805.58369	7.31171	22.02163	334609.47855

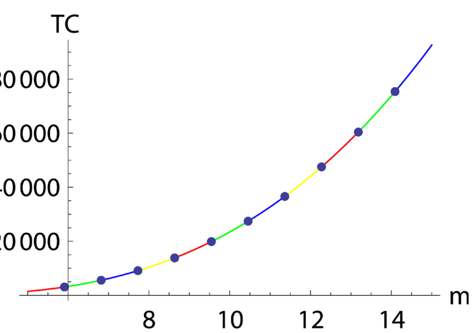

Figure 5. Variation in total cost w. r. to expiration date.

Regarding Table 2, Figure 5 shows the depiction in total cost with respect to the expiration date parameter m.

In view of Table 2, Figure 6 shows the three dimensions pictorial depiction in total cost $TC(s,T_1,T)$ regarding the decision variables s and T.

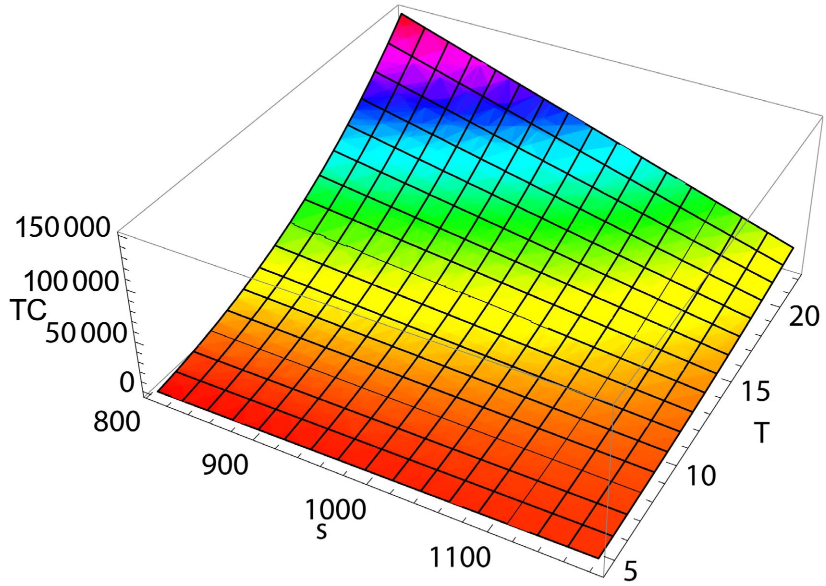

Figure 6. Variation in total cost w. r. to s and T.

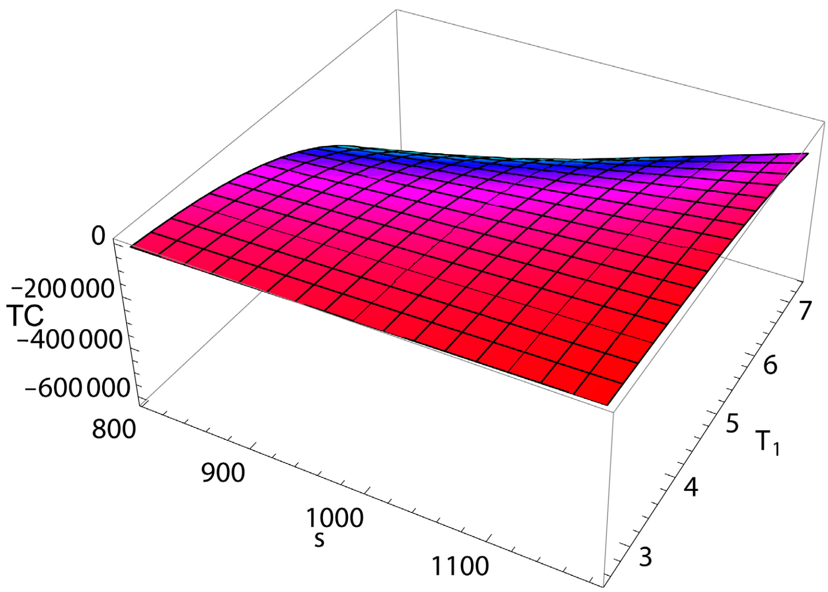

Figure 7. Variation in total cost w. r. to s and T_1.

In context of Table 2, Figure 7 shows the three dimensions pictorial depiction in total cost $TC(s,T_1,T)$ in compliance of decision variables s and T_1.

Table 3 presents the variation in the advertisement cost parameter A_C regarding the optimal values of decision variables s,T_1 and T. The rest parameters and variables have the fixed values.

Table 3. Variation in total cost concerning advertisement cost parameter.

A_C	s	T_1	T	$TC(s,T_1,T)$
0.25	1182.66413	2.55155	4.53248	1454.18203
0.35	1201.89369	2.55096	4.53169	1453.18201
0.45	1212.58316	2.55094	4.53168	1453.18200
0.65	1224.09038	2.55092	4.53165	1453.18195
0.85	1230.19545	2.55090	4.53160	1453.18192

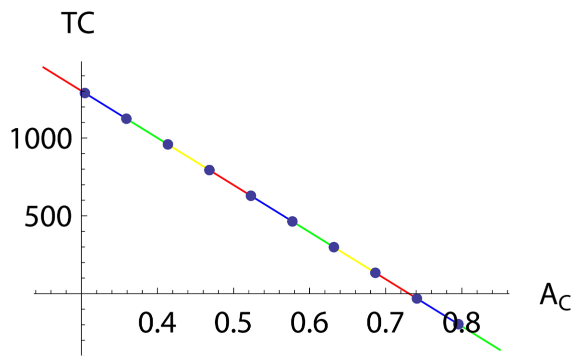

Figure 8. Variation in total cost w. r. to advertisement cost.

In consonance of **Table 3**, **Figure 8** depicts the pictorial depiction in total cost $TC(s,T_1,T)$ regarding the advertisement cost parameter A_C.

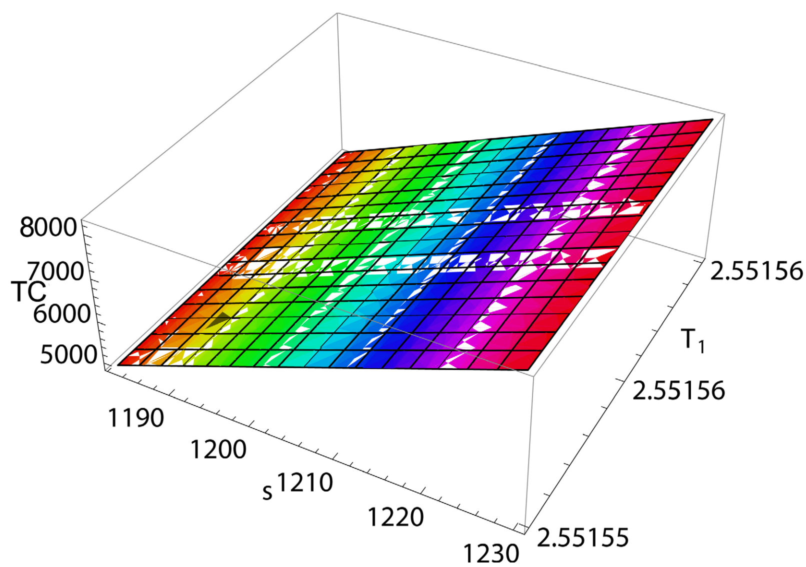

Figure 9. Variation in total cost w. r. to s and T_1.

Regarding **Table 3**, **Figure 9** shows the three dimensions pictorial depiction in total cost $TC(s,T_1,T)$ corresponding to the decision variables s and T_1.

Table 4 stats the variation in holding cost parameter h_c regarding the optimal values of decision variables s,T_1 and T. The rest parameters and variables are assumed to be fixed.

Table 4. Variation in total cost concerning holding cost parameter.

h_c	s	T_1	T	$TC(s,T_1,T)$
8	1182.66413	2.55155	4.53248	1454.18203
10	1186.11561	2.49642	4.80273	1567.02604
12	1188.91814	2.46906	5.07216	1692.18843
15	1192.30752	2.45509	5.47396	1897.02883
20	1196.55715	2.45506	6.13379	2272.72164

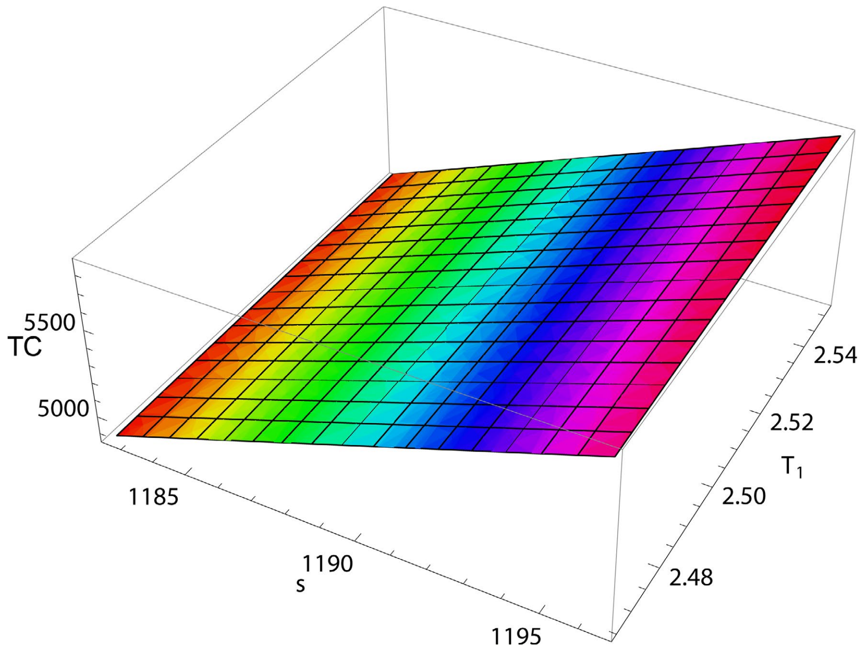

Figure 10. Variation in total cost w. r. to s and T_1.

In the context of **Table 4**, **Figure 10** shows the three dimensions pictorial depiction in consonance of decision variables s, T_1 and T.

Table 5 shows the variation in carbon tax cost parameter s_T with respect to the optimal values of decision variables s, T_1 and T. The rest parameters and variables are considered fixed.

Table 5. Variation in total cost concerning carbon tax cost parameter.

c_T	s	T_1	T	$TC(s,T_1,T)$
0.70	1182.66413	2.55155	4.53248	1454.18203
0.75	1182.11561	2.55546	4.53932	1458.99073
0.80	1182.91814	2.55936	5.54613	1463.79692
0.85	1182.30752	2.56329	5.55306	1468.66776
0.90	1182.55715	2.56721	6.55992	1473.51695

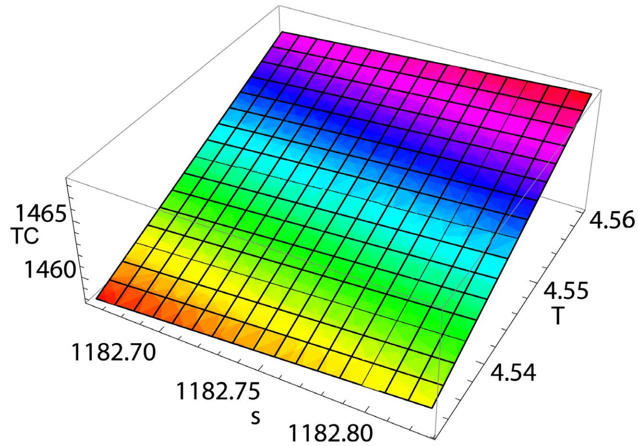

Figure 11. Variation in total cost w. r. to s and T.

In consonance of **Table 5**, **Figure 11** shows the three dimensions pictorial depiction of total cost with respect to the decision variables s and T.

5. Sensitivity Analysis

The sensitivity analysis of some crucial parameters is as follows,

1) Regarding **Table 1**, as the backlogging parameter δ increases, then the total cost function increases. With regards to the decision variables, the increased rate of backlogging also increases the decision variable T_1, and the remaining variables selling price s, and planning horizon/cycle length T are decreased. The cause is that most of the customers are willing to wait for the arrival of new orders.

2) In context of **Table 2**, as the expiration date parameter m increases, then the total cost function increases. With regards to the decision variables, the increased rate of expiration also increases the cycle length T, and time-period T_1 of positive inventory are increased, and the remaining selling-price variable s is decreased. The cause of it is that the system consists of the better preserving and holding facilities.

3) In view of **Table 3**, as the advertising cost parameter A_C increases, then the total cost function decreases. With regards to the decision variables, the increased rate of advertisement also decreases the cycle length T, and time-period T_1 of positive inventory, and the remaining selling variable s is increased. The reason is that the customers are not wishing to wait the arrival of new orders.

4) According to **Table 4**, as the holding cost parameter h_c increases, then the total cost function increases. With regards to the decision variables, the increased rate of holding cost also increases the selling-price variable s, and cycle length T, and the remaining variable T_1 is decreased. The reason is that the ordering cost may be increasing.

5) In the light of **Table 5**, as the carbon tax cost parameter c_T increases, then the total cost function increases. With regards to the decision variables, the increased rate of carbon tax cost also increases the remaining parameters s, T_1, and T. The cause is that the shipping, and holding costs may be increasing.

6. Conclusion

This study consists of a sustainable inventory model based on real-life situations. It considers advertising, price, and time-dependent demand together with carbon emissions. After the transportation of commodities, the retailer has a large need of ordering to utilize their purchase appropriately. During the ordering, holding, and shipping of commodities the retailer wants to reduce carbon emission. Globally, carbon emissions affect the environment, and it becomes a major concern day by day for every country. The analysis shows that the total cost is highly impacted by the expiration date, rather than backlogging, advertising, and holding cost. And the cycle length is deeply affected by the expiration date. This paper may be useful for several businesses or industries, where such types of commodities are produced. In future, several realistic formats of uncertainty upon demand, and backlogging can be considered for the refinement of this model.

Acknowledgements

There is no grant from any funding agency for this research. The respective authors would like to pay thanks to the editor and anonymous reviewers for their conspicuous comments and suggestions to improve the quality of this research.

Conflicts of Interest

It is confirmed that there is no conflict of interest among authors about this publication.

References

[1] Panda, S., Saha, S. and Basu, M. (2008) An EOQ Model for Perishable Products with Discounted Selling-Price, and Stock-Dependent Demand. *Central European Journal of Operations Research*, **17**, 31-53. https://doi.org/10.1007/s10100-008-0073-z

[2] Sarkar, B. and Sarkar, S. (2013) An Improved Inventory Model for Deteriorating Items with Stock-Dependent Demand, Time-Varying Deterioration, and Partial-Backlogging. *Economic Modeling Journal*, **30**, 924-932. https://doi.org/10.1016/j.econmod.2012.09.049

[3] Shah, N.H., Soni, H.N. and Patel, K.A. (2013) Optimizing Inventory and Marketing Policy for Non-Instantaneous Deteriorating Items with Generalized Type Deterioration Rate, and Holding Cost. *Omega*, **41**, 421-430. https://doi.org/10.1016/j.omega.2012.03.002

[4] Chen, X., Benjaafar, S. and Elomri, A. (2013) The Carbon-Constrained EOQ Model for Deteriorating Items. *Operations Research Letters*, **41**, 172-179. https://doi.org/10.1016/j.orl.2012.12.003

[5] Choudhury, K.D., Karmakar, B., Das, M. and Dutta, T.K. (2015) An Inventory Model for Deteriorating Items with Stock-Dependent Demand, Time-Varying Holding Cost, and Shortages. *Opsearch*, **52**, 55-74. https://doi.org/10.1007/s12597-013-0166-x

[6] Hovelaque, V. and Bironneau, L. (2015) The Carbon-Constrained EOQ Model with Carbon Emission Dependent Demand. *International Journal of Production Economics*, **164**, 285-291. https://doi.org/10.1016/j.ijpe.2014.11.022

[7] Shaikh, A.A. (2017) An Inventory Model for Deteriorating Items with Frequency of Advertisement, and Selling Price Dependent Demand Under Mixed Type of Trade-Credit Policy. *International Journal of Logistics System and Management*, **28**, 375-395. https://doi.org/10.1504/IJLSM.2017.086949

[8] Mishra, U., Cardenas-Barron, L.E., Tiwari, S., Shaikh, A.A. and Trevino-Garza, G. (2017) An Inventory Model with Price and Stock-Dependent Demand for Controllable Deterioration Rate, Shortages Under Preservation Technology Investment. *Annals of Operations Research*, **254**, 165-190. https://doi.org/10.1007/s10479-017-2419-1

[9] Mishra, U. (2018) Optimizing A Three-Rates-of-Production Inventory Model for Deteriorating Items Under Market Selling-Price and Advertisement Cost. International *Journal of Management Science, and Engineering Management*, **13**, 295-305. https://doi.org/10.1080/17509653.2018.1445046

[10] Kazemi, N., Abdul-Rashid, S.H., Ghazilla, R.A.R., Shekarian, E. and Zanoni, S. (2018) EOQ Models for Deteriorating Items with Imperfect Quality and Carbon Emission. *International Journal of Systems Science Operations & Logistics*, **5**, 99-115. https://doi.org/10.1080/23302674.2016.1240254

[11] Shaikh, A.A., Khan, A.A., Panda, G.C. and Konstantaras, I. (2019) An EOQ Model for Decaying Items with Price-Discounted Facility and Stock-Varying Demand and Partial Backlogging. *International Transactions in Operations Research*, **26**, 1365-1395. https://doi.org/10.1111/itor.12632

[12] Tiwari, S., Ahmed, W. and Sarkar, B. (2019) Sustainable Ordering Policies for Non-Instantaneous Deteriorating Items Under Carbon Emission and Multi Trade-Credit Policies. *Journal of Cleaner Production*, **240**, Article 118183. https://doi.org/10.1016/j.jclepro.2019.118183

[13] Taleizadeh, A.A., Hazarkhani, B. and Moon, I. (2020) Joint Pricing and Inventory Decisions with Carbon Emission Considerations, Partial Backlogging, and Planned Discount. *Annals of Operations Research*, **290**, 95-113. https://doi.org/10.1007/s10479-018-2968-y

[14] Mishra, U., Wu, J.Z. and Sarkar, B. (2021) Optimum Sustainable Inventory Management with Back Order, and Deterioration Under Controllable Carbon Emission. *Journal of Cleaner Production*, **279**, 1-18. https://doi.org/10.1016/j.jclepro.2020.123699

[15] Daryanto, Y., Wee, H.M. and Wu, K.H. (2021) Revising Sustainable EOQ Model for Decaying Products Under Carbon Emission. *International Journal of Manufacturing Technology and Management*, **35**, 1-11. https://doi.org/10.1504/IJMTM.2021.114697

[16] San-Jose, L.A., Sicilia, J. and Abdul-Jalbar, B. (2021) Optimal Policy for an Inventory System with Price, Time and Frequency of Advertisement-Dependent Demand. *Computers and Operations Research*, **128**, 1-13. https://doi.org/10.1016/j.cor.2020.105169

[17] Sarkar, B., Sarkar, M., Ganguly, B. and Cardenas-Barron, L.E. (2021) Combined Effect of Carbon Emission and Production Quantity Improvement for Fixed Lifetime Production in A Supply Chain Management. *International Journal of Production Economics*, **231**, 1-14. https://doi.org/10.1016/j.ijpe.2020.107867

[18] Mishra, U., Mashud, A.H.M., Tseng, M.L. and Wu, J.Z. (2021) Optimizing a Sustainable Supply Chain Inventory Model for Controllable Deterioration, and Carbon Emission Rates in a Greenhouse Farm. *Mathematics*, **9**, Article 495. https://doi.org/10.3390/math9050495

[19] Das, S.K., Pervin, M., Roy, S.K. and Weber, G.W. (2021) A Hybrid Approach for a

Multi-Objective Solid Transportation Location Problem with Variable Carbon Emission in Inventory Management. *Annals of Operations Research*, **1**, Article 27.

[20] Taleizadeh, A.A., Aliabadi, L. and Thaichon, P. (2022) A Sustainable Inventory System with Price-Sensitive Demand, Carbon Emissions, Partial Trade-Credit Policy, and Partial Backordering. *Operations Research*, **22**, 4471-4516. https://doi.org/10.1007/s12351-021-00683-w

[21] Mishra, R.K. and Mishra, V.K. (2022) An Optimum Sustainable Inventory Model for Non-Instantaneous Deterioration and Quality Assessment Under Carbon Emissions and Complete Back Ordering Shortage. *Arab Journal for Science, and Engineering*, **47**, 3929-3944. https://doi.org/10.1007/s13369-021-06402-z

[22] Kumar, S., Sigroha, M., Kumar, K. and Sarkar, B. (2022) Manufacturing Re-Manufacturing Based Supply Chain Management Under Advertisement and Carbon Emission Process. *Rairo Operations Research*, **56**, 831-851. https://doi.org/10.1051/ro/2021189

[23] Kugele, A.S.H., Ahmad, W. and Sarkar, B. (2022) Geometric Programming Solution of Second Degree Difficulty for Carbon Emission Controlled Reliable Smart Production System. *Rairo Operations Research*, **56**, 1013-1029. https://doi.org/10.1051/ro/2022028

[24] Sankari, J., Mohan, C. and Ramasamy, U. (2023) A Sustainable Inventory Model for Growing Items Considering Carbon Emissions Product Expiry and Profit Sharing Policy. *Journal of Future Sustainability*, **3**, 201-222. https://doi.org/10.5267/j.jfs.2023.2.002

[25] Pervin, M. and Weber, G.W. (2023) Sustainable Inventory Model with Environmental Impact for Non-Instantaneous Deteriorating Items with Composite Demand. *Rairo Operations Research*, **57**, 237-261. https://doi.org/10.1051/ro/2023005

[26] Sobia, S., Pardeepa, M. and Valliathal, M. (2024) A Deterministic Inventory Model for Constant Deteriorating Items with A Generalized Exponential Diminishing Demand, and Stable Holding Cost. *The International Journal of Interdisciplinary Organizational Studies*, **19**, 2286-2294.

[27] Kumar, S., Kumar, M., Singh, S. and Jaiswal, D. (2024) Sustainable Inventory System for Decaying Products with Expiration Date, Carbon Emission, and Price-Sensitive Exponentially Decreasing Demand. *Tuijin Jishu/Journal of Propulsion Technology*, **45**, 330-343.

[28] Guo, Z. and Zhang, Y. (2025) Solving the Optimal Order Quantity with Unknown Parameters for Products with Stock-Dependent Demand and Variable Holding Cost Rate. *Journal of Combinatorial Optimization*, **49**, 1-22. https://doi.org/10.1007/s10878-025-01260-z

[29] Rangarajan, R.S.K., Dey, B.K., Alrasheedi, A.F., Ivkovic, N. and Jana, C. (2025) A Sustainable Production Inventory Model for Power-Pattern Demand with Carbon Emissions and Self Life Considerations. *International Journal of Computational Intelligence Systems*, **18**, 1-32. https://doi.org/10.1007/s44196-025-00861-0

[30] Alharbi, M. (2025) Investigating a Sustainable Inventory System with Controlled Non-Instantaneous Deterioration for Green Products via the Dragonfly Algorithm. *Sustainability*, **17**, Article 1156. https://doi.org/10.3390/su17031156

Appendix 1

For, the sufficient condition of total cost function, the second order derivatives are as follows,

$$\begin{aligned}\frac{\partial^2 TC}{\partial T_1^2} = \frac{1}{T}\Big[& h_c\left\{-3c(1+m)+c(1+m)(1-m)^2+A_C(a-bs)(1-m)\right\} \\ & -A_C(a-bs)(s_c-p_c-2\delta p_c)+p_c c\left(1-m^2\right)+ls_c\delta A_C(a-bs) \\ & +sh_c(1+\delta)A_C(a-bs)+sh_c c\left(1-m^2\right)+c_T\gamma_2\{-3c(1+m) \\ & +c(1+m)(1-m)^2+A_C(a-bs)(2-m)\}+c_T\gamma_3\left\{c\left(1-m^2\right)\right. \\ & \left.+A_C(a-bs)(1+\delta)\right\}+\Big[5c+c\left(1-m^2\right) \\ & +2A_C(a-bs)\left(1+m-2\delta s_c+2\delta^2 ls_c\right) \\ & +\frac{c_T\gamma_2}{2}\left\{5c+c\left(1-m^2\right)+A_C(a-bs)(1+m+2\delta)\right\}\Big]T_1 \\ & +2\delta A_C(a-bs)(s_c-ls_c\delta-c_T\gamma_2)T \\ & -\frac{3h_c}{2}\left\{A_C(a-bs)+c(1+m)\right\}T_1^2\Big]\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 TC}{\partial T_1\partial s} = \frac{1}{T}\Big[& -A_C b(2-m)(p_c+sh_c+c_T\gamma_3)+A_C b\big[-h_c(1-m) \\ & +(s_c-p_c-2\delta p_c)-\delta ls_c-sh_c(1+\delta)-c_T\gamma_2(2-m) \\ & -c_T\gamma_3(1+\delta)\big]T_1-A_C b(s_c-\delta ls_c-\delta sh_c-c_T\gamma_2-c_T\gamma_3\delta)T \\ & -\frac{A_C b}{2}\left\{\left(1+m-2\delta s_c+2\delta^2 ls_c\right)+c_T\gamma_2(1+m+2\delta)\right\}T_1^2 \\ & -A_C b\delta(-s_c+\delta ls_c+c_T\gamma_2)T-2A_C b\delta(s_c-\delta ls_c-c_T\gamma_2)TT_1+\frac{h_c A_C b}{2}T_1^3\Big]\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 TC}{\partial T_1\partial T} = \frac{1}{T}\big[& A_C(a-bs)(s_c-\delta ls_c-\delta sh_c-c_T\gamma_2-c_T\gamma_3\delta)+2\delta p_c+2A_C(a \\ & -bs)\delta(-s_c+\delta ls_c+c_T\gamma_2)T+2\delta A_C(a-bs)(s_c-\delta ls_c-c_T\gamma_2)T_1\big] \\ & -\frac{1}{T^2}\Big[(p_c+sh_c+c_T\gamma_3)\left\{-cm(1+m)(2-m)+A_C(a-bs)(2-m)\right\} \\ & +h_c\left\{-3c(1+m)+c(1+m)(1-m)^2+A_C(a-bs)(1-m)\right\}T_1 \\ & +\left\{-A_C(a-bs)(s_c-p_c-2\delta p_c)+(p_c+sh_c)c\left(1-m^2\right)\right\}T_1 \\ & +A_C(a-bs)\left\{ls_c\delta+sh_c(1+\delta)\right\}T_1 \\ & +c_T\gamma_3\left\{c\left(1-m^2\right)+A_C(a-bs)(1+\delta)\right\}T_1 \\ & +c_T\gamma_2\left\{-3c(1+m)+c(1+m)(1-m)^2+A_C(a-bs)(2-m)\right\}T_1 \\ & +\left\{A_C(a-bs)(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta)+2p_c\delta\right\}T \\ & +\frac{1}{2}\left\{5c+c\left(1-m^2\right)+A_C(a-bs)\left(1+m-2s_c\delta+2ls_c\delta^2\right)\right\}T_1^2\end{aligned}$$

$$+\frac{c_T\gamma_2}{2}\left\{5c+c\left(1-m^2\right)+A_C\left(a-bs\right)\left(1+m+2\delta\right)\right\}T_1^2$$
$$+A_C\left(a-bs\right)\delta\left(-s_c+ls_c\delta+c_T\gamma_2\right)T^2+2A_C\left(a-bs\right)\delta\left(s_c\right.$$
$$\left.\left.-ls_c\delta-c_T\gamma_2\right)TT_1-\frac{h_c}{2}\left\{A_C\left(a-bs\right)+c\left(1+m\right)\right\}T_1^3\right]$$

$$\frac{\partial^2 TC}{\partial s^2}=0$$

$$\frac{\partial^2 TC}{\partial s\partial T_1}=\frac{1}{T}\left[-A_Cb\left(2-m\right)\left(p_c+sh_c+c_T\gamma_3\right)+A_Cb\left\{-h_c\left(1-m\right)+s_c-p_c-2\delta p_c\right.\right.$$
$$-ls_c\delta-sh_c\left(1+\delta\right)-c_T\gamma_2\left(2-m\right)-c_T\gamma_3\left(1+\delta\right)\big\}T_1-A_Cb\left(s_c-\delta ls_c\right.$$
$$\left.-\delta sh_c-c_T\gamma_2-c_T\gamma_3\delta\right)T-\frac{A_Cb}{2}\left\{1+m-2\delta s_c+2\delta^2 ls_c\right.$$
$$\left.+c_T\gamma_2\left(1+m+2\delta\right)\right\}T_1^2-A_Cb\delta\left(-s_c+\delta ls_c+c_T\gamma_2\right)T^2$$
$$\left.-2A_Cb\delta\left(s_c-\delta ls_c-c_T\gamma_2\right)TT_1+\frac{h_cA_Cb}{2}T_1^3\right]$$

$$\frac{\partial^2 TC}{\partial s\partial T}=\frac{1}{T}\Big[A_Cb\left(p_c+sh_c+c_T\gamma_3\right)-A_Cb\big\{-sh_c+p_c+\delta\left(ls_c+sh_c\right)$$
$$+c_T\left(\gamma_2+\gamma_3\delta\right)\big\}T-A_Cb\left\{s_c-\delta ls_c-\delta sh_c-c_T\gamma_2-c_T\gamma_3\right\}T_1$$
$$-A_Cb\left(s_c-\delta ls_c-c_T\gamma_2\right)T^2-2A_Cb\delta\left(-s_c+\delta ls_c+c_T\gamma_2\right)TT_1$$
$$-A_Cb\delta\left(s_c-\delta ls_c-c_T\gamma_2\right)T_1^2\Big]$$
$$-\frac{1}{T^2}\Big[-A_Cb\left(2-m\right)\left(p_c+sh_c+c_T\gamma_3\right)T_1+A_Cb\left(p_c+sh_c+c_T\gamma_3\right)T$$
$$+\frac{A_Cb}{2}\left\{-h_c\left(1-m\right)+s_c-p_c-2p_c\delta-ls_c\delta-sh_c\left(1+\delta\right)-c_T\gamma_2\left(2-m\right)\right\}T_1^2$$
$$-\frac{A_Cb}{2}c_T\gamma_3\left(1+\delta\right)T_1^2-\frac{A_Cb}{2}\left(-s_c+p_c+ls_c\delta+sh_c\delta+c_T\gamma_2+c_T\gamma_3\delta\right)T^2$$
$$-A_Cb\left(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta\right)TT_1-\frac{A_Cb\delta}{3}\left(s_c-ls_c\delta-c_T\gamma_2\right)T^3$$
$$-\frac{A_Cb}{6}\left\{1+m-2s_c\delta+2ls_c\delta^2+c_T\gamma_2\left(1+m+2\delta\right)\right\}T_1^3$$
$$-A_Cb\delta\left(-s_c+ls_c\delta+c_T\gamma_2\right)T^2T_1-A_Cb\delta\left(s_c-ls_c\delta-c_T\gamma_2\right)TT_1^2$$
$$+\frac{A_Cbh_c}{8}T_1^4\Big]$$

$$\frac{\partial^2 TC}{\partial T\partial T_1}=\frac{1}{T}\Big[A_C\left(a-bs\right)\left(s_c-\delta ls_c-\delta sh_c-c_T\gamma_2-c_T\gamma_3\delta\right)+2\delta p_c$$
$$+2A_C\left(a-bs\right)\delta\left(-s_c+\delta ls_c+c_T\gamma_2\right)T$$
$$+2\delta A_C\left(a-bs\right)\left(s_c-\delta ls_c-c_T\gamma_2\right)T_1\Big]$$
$$-\frac{1}{T^2}\Big[\left(p_c+sh_c+c_T\gamma_3\right)\left\{-cm\left(1+m\right)\left(2-m\right)+A_C\left(a-bs\right)\left(2-m\right)\right\}$$
$$+\Big[h_c\left\{-3c\left(1+m\right)+c\left(1+m\right)\left(1-m\right)^2+A_C\left(a-bs\right)\left(1-m\right)\right\}$$

$$
\begin{aligned}
&-A_C(a-bs)(s_c-p_c-2\delta p_c)+p_c c(1-m^2)+\delta ls_c A_C(a-bs)\\
&+sh_c(1+\delta)A_C(a-bs)+sh_c c(1-m^2)\\
&+c_T\gamma_2\left\{-3c(1+m)+c(1+m)(1-m)^2+A_C(a-bs)(2-m)\right\}\\
&+c_T\gamma_3\left\{c(1-m^2)+A_C(a-bs)(1+\delta)\right\}\Big]T_1\\
&+\left\{A_C(a-bs)(s_c-\delta ls_c-\delta sh_c-c_T\gamma_2-c_T\gamma_3\delta)+2\delta p_c\right\}T\\
&+\frac{1}{2}\Big[5c+c(1-m^2)+A_C(a-bs)(1+m-2\delta s_c+2\delta^2 ls_c)\\
&+c_T\gamma_2\left\{5c+c(1-m^2)+A_C(a-bs)(1+m+2\delta)\right\}\Big]T_1^2\\
&+A_C(a-bs)\delta(-s_c+\delta ls_c+c_T\gamma_2)T^2+2A_C(a-bs)\delta(s_c-\delta ls_c\\
&-c_T\gamma_2)TT_1-\frac{h_c}{2}\left\{A_C(a-bs)+c(1+m)\right\}T_1^3\Big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 TC}{\partial T\partial s}=\frac{1}{T}\Big[&A_C b(c_T\gamma_3+s_c-\delta ls_c-c_T\gamma_2-c_T\gamma_3\delta)T\\
&-A_C b(s_c-\delta ls_c-\delta sh_c-c_T\gamma_2-c_T\gamma_3\delta)T_1\\
&-A_C b\delta(s_c-\delta ls_c-c_T\gamma_2)T^2-2A_C b\delta(-s_c+\delta ls_c+c_T\gamma_2)TT_1\\
&-A_C b\delta(s_c-\delta ls_c-c_T\gamma_2)T_1^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 TC}{\partial T^2}=&\Big[A_C(a-bs)(-s_c+p_c+\delta ls_c+\delta sh_c+c_T\gamma_2+c_T\gamma_3\delta)+2A_C(a-bs)\delta(s_c\\
&-\delta ls_c-c_T\gamma_2)T+2A_C(a-bs)\delta(-s_c+\delta ls_c+c_T\gamma_2)T_1\Big]\\
&-\frac{1}{T^2}\Big[-A_C(a-bs)(p_c+sh_c+c_T\gamma_3)\\
&+A_C(a-bs)(-s_c+p_c+ls_c\delta+sh_c\delta+c_T\gamma_2+c_T\gamma_3\delta)T\\
&+\left\{A_C(a-bs)(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta)+2p_c\delta\right\}T_1\\
&+A_C(a-bs)\delta(s_c-ls_c\delta-c_T\gamma_2)T^2+2A_C(a-bs)\delta(-s_c\\
&+ls_c\delta+c_T\gamma_2)TT_1+A_C(a-bs)\delta(s_c-ls_c\delta-c_T\gamma_2)T_1^2\Big]\\
&-\frac{1}{T^2}\Big[-A_C(a-bs)(p_c+sh_c+c_T\gamma_3)+A_C(a-bs)(-s_c+p_c+ls_c\delta\\
&+sh_c\delta+c_T\gamma_2+c_T\gamma_3\delta)T+\left\{A_C(a-bs)(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta)\right.\\
&\left.+2p_c\delta\right\}T_1+A_C(a-bs)\delta(s_c-ls_c\delta-c_T\gamma_2)T^2+2A_C(a-bs)\delta(-s_c\\
&+ls_c\delta+c_T\gamma_2)TT_1+A_C(a-bs)\delta(s_c-ls_c\delta-c_T\gamma_2)T_1^2\Big]\\
&-\frac{1}{T^2}\Big[(o_c+c_T\gamma_1)+(p_c+sh_c+c_T\gamma_3)\left\{-c(1+m)+c(1-m)^2(1+m)\right.\\
&\left.+A_C(a-bs)(2-m)\right\}T_1-A_C(a-bs)(p_c+sh_c+c_T\gamma_3)T\\
&+\frac{1}{2}\Big[h_c\left\{-3c(1+m)+c(1+m)(1-m)^2+A_C(a-bs)(1-m)\right\}\\
&-A_C(a-bs)(s_c-p_c-2\delta p_c)+p_c c(1-m^2)+ls_c\delta A_C(a-bs)\\
&+sh_c A_C(a-bs)(1+\delta)+sh_c c(1-m^2)+c_T\gamma_2\left\{-3c(1+m)\right.
\end{aligned}
$$

$$+c\left(1-m\right)^2\left(1+m\right)+A_C\left(a-bs\right)\left(2-m\right)\Big\}+c_T\gamma_3\left\{c\left(1-m^2\right)\right.$$
$$\left.+A_C\left(a-bs\right)\left(1+\delta\right)\right\}\Big]T_1^2+\frac{A_C\left(a-bs\right)}{2}\left(-s_c+p_c+ls_c\delta+sh_c\delta\right.$$
$$\left.+c_T\gamma_2+c_T\gamma_3\delta\right)T^2+\left\{A_C\left(a-bs\right)\left(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta\right)\right.$$
$$\left.+2\delta p_c\right\}TT_1+\frac{1}{6}\Big[\Big\{5c+c\left(1-m^2\right)+A_C\left(a-bs\right)\left(1+m-2\delta s_c\right.$$
$$\left.\left.+2\delta^2 ls_c\right)\right\}+c_T\gamma_2\left\{5c+c\left(1-m^2\right)+A_C\left(a-bs\right)\left(1+m+2\delta\right)\right\}\Big]T_1^3$$
$$+\frac{A_C\left(a-bs\right)\delta}{3}\left(s_c-ls_c\delta-c_T\gamma_2\right)T^3+A_C\left(a-bs\right)\delta\left(-s_c+ls_c\delta\right.$$
$$\left.+c_T\gamma_2\right)T^2T_1+A_C\left(a-bs\right)\delta\left(s_c-ls_c\delta-c_T\gamma_2\right)TT_1^2$$
$$\left.-\frac{h_c}{8}\left\{A_C\left(a-bs\right)+c\left(1+m\right)\right\}T_1^4\right]$$
$$+\frac{2}{T^3}\Big[\left(o_c+c_T\gamma_1\right)+\left(p_c+sh_c+c_T\gamma_3\right)\Big\{-c\left(1+m\right)+c\left(1-m\right)^2\left(1+m\right)$$
$$\left.+A_C\left(a-bs\right)\left(2-m\right)\right\}T_1-A_C\left(a-bs\right)\left(p_c+sh_c+c_T\gamma_3\right)T$$
$$+\frac{1}{2}\Big[h_c\left\{-3c\left(1+m\right)+c\left(1+m\right)\left(1-m\right)^2+A_C\left(a-bs\right)\left(1-m\right)\right\}$$
$$-A_C\left(a-bs\right)\left(s_c-p_c-2\delta p_c\right)+p_cc\left(1-m^2\right)+ls_c\delta A_C\left(a-bs\right)$$
$$+sh_cA_C\left(a-bs\right)\left(1+\delta\right)+sh_cc\left(1-m^2\right)+c_T\gamma_2\Big\{-3c\left(1+m\right)$$
$$+c\left(1-m\right)^2\left(1+m\right)+A_C\left(a-bs\right)\left(2-m\right)\Big\}+c_T\gamma_3\left\{c\left(1-m^2\right)\right.$$
$$\left.+A_C\left(a-bs\right)\left(1+\delta\right)\right\}\Big]T_1^2+\frac{A_C\left(a-bs\right)}{2}\left(-s_c+p_c+ls_c\delta+sh_c\delta\right.$$
$$\left.+c_T\gamma_2+c_T\gamma_3\delta\right)T^2+\left\{A_C\left(a-bs\right)\left(s_c-ls_c\delta-sh_c\delta-c_T\gamma_2-c_T\gamma_3\delta\right)\right.$$
$$\left.+2\delta p_c\right\}TT_1+\frac{1}{6}\Big[\Big\{5c+c\left(1-m^2\right)+A_C\left(a-bs\right)\left(1+m-2\delta s_c\right.$$
$$\left.\left.+2\delta^2 ls_c\right)\right\}+c_T\gamma_2\left\{5c+c\left(1-m^2\right)+A_C\left(a-bs\right)\left(1+m+2\delta\right)\right\}]]T_1^3$$
$$+\frac{A_C\left(a-bs\right)\delta}{3}\left(s_c-ls_c\delta-c_T\gamma_2\right)T^3+A_C\left(a-bs\right)\delta\left(-s_c+ls_c\delta\right.$$
$$\left.+c_T\gamma_2\right)T^2T_1+A_C\left(a-bs\right)\delta\left(s_c-ls_c\delta-c_T\gamma_2\right)TT_1^2$$
$$\left.-\frac{h_c}{8}\left\{A_C\left(a-bs\right)+c\left(1+m\right)\right\}T_1^4\right]$$

Appendix 2

The sufficiency for total cost function $TC\left(s,T_1,T\right)$ is that the Hessian matrix H is positive semi-definite. And $\frac{\partial^2 TC\left(s,T_1,T\right)}{\partial s^2}>0$ and

$$H_{12}=\begin{pmatrix}\dfrac{\partial^2 TC\left(s,T_1,T\right)}{\partial s^2} & \dfrac{\partial^2 TC\left(s,T_1,T\right)}{\partial s\partial T_1}\\ \dfrac{\partial^2 TC\left(s,T_1,T\right)}{\partial T_1\partial s} & \dfrac{\partial^2 TC\left(s,T_1,T\right)}{\partial T_1^2}\end{pmatrix}>0$$

The Hessian matrix is defined as follows,

$$H=\begin{pmatrix} \frac{\partial^2 TC(s,T_1,T)}{\partial s^2} & \frac{\partial^2 TC(s,T_1,T)}{\partial s\partial T_1} & \frac{\partial^2 TC(s,T_1,T)}{\partial s\partial T} \\ \frac{\partial^2 TC(s,T_1,T)}{\partial T_1\partial s} & \frac{\partial^2 TC(s,T_1,T)}{\partial T_1^2} & \frac{\partial^2 TC(s,T_1,T)}{\partial T_1\partial T} \\ \frac{\partial^2 TC(s,T_1,T)}{\partial T\partial s} & \frac{\partial^2 TC(s,T_1,T)}{\partial T\partial T_1} & \frac{\partial^2 TC(s,T_1,T)}{\partial T^2} \end{pmatrix}$$

Numerically, the Hessian matrix H is given by,

$$H=\begin{pmatrix} 0 & 2.84039 & -0.09179 \\ 2.58245 & -124.05427 & 3.87998 \\ -1.31291 & 6.41626 & 50.32647 \end{pmatrix}$$