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Density Functions and Descriptive Statistics for Distance Measures in Cartesian Space

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Abstract

This research provides density functions and descriptive statistics for the distance between points for basic shapes in Cartesian space. Both Euclidean and Rectilinear Distances are determined for these shapes. The shapes are in the unit line, the unit square, unit circle, unit sphere and unit triangle. In all instances, an exact, symbolic solution is sought, but there are four occasions where that is not possible—for these cases, a reasonable approximate solution is obtained. For each distance measure, mean, median and standard deviation measures are provided as well.

Keywords

Density, Probability, Stochastic

1. Introduction

Density functions are an important part of probability and statistics. They illustrate continuous probability distributions, giving us a visual image of possible outcomes and their associated likelihoods of occurring. These density functions can also be exploited so that we can also learn the expected value, standard deviation and other statistics associated with continuous random variables [1] [2].

The research presented here explores the density functions associated with distances between two points in Cartesian Space. The space that is of interest to us is confined to unit shapes: the unit line, the unit square, the unit cube, the unit circle, the unit sphere and the unit triangle. We are interested in two types of distance: Euclidean, or straight-line distance, and Rectilinear distance—a distance that is broken into its horizontal and vertical components.

The two points which are used as endpoints for the line lengths we measure are always independent and uniformly distributed in the interval spanning the minimum and maximum possible values for the interval of interest. For example, in the unit square, both points have minimum and maximum values of 0 and 1 because 0 is the minimum possible value in the unit square, and 1 is the maximum possible value in the unit square. All values on this (0, 1) interval are equally likely to occur. Additionally, whatever locations are chosen for one point are completely independent of the locations for the other point.

In all scenarios, the exact distances are sought in symbolic form. This, however, is not always feasible for the distance measures we seek due to the complexities associated with symbolic integration of distance measures. In such cases, we seek a numerical approximation that is considered reasonable in terms of both accuracy and computational effort.

The general intent of this paper is essentially twofold: first, it is intended to provide investigators with some information about important density functions. Second, while some of these density functions have already been determined, some have not, and it is intended to provide a convenient aggregate source of information for those interested.

The following sections discuss the probability density function and show how the distance and density functions are determined for each distance measure. The numerical estimation process for density functions is presented, along with descriptive statistics. Conclusions are also offered.

2. Background

The continuous random variables of interest here are essentially simplistic measures of distance in the unit line, unit square, unit cube, unit circle, unit cube, and the unit triangle. These length measures are Euclidean (straight-line) distances and Rectilinear distances—distances as a sum of their vertical and horizontal components.

Consider a continuous random variable X. This random variable has the density function, $f(x)$:

$$P(a \le X \le b) = \int_a^b f(x)\,\mathrm{d}x \tag{1}$$

Consider then, $F(x)$ to be the cumulative distribution of X:

$$F(x) = \int_{-\infty}^{x} f(u)\,\mathrm{d}u \tag{2}$$

The density function is the first derivative of the cumulative density function. The density function is related to the cumulative distribution function via the following:

$$f(x) = \mathrm{d}\left[F(x)\right]/\mathrm{d}x \tag{3}$$

It should also be noted that the total area under the density function is unity. Mathematically, this is as follows:

$$F(x) = \int_a^b f(x)\,\mathrm{d}x = 1 \tag{4}$$

For this research, all stochastic input entities are uniformly distributed on the

(a,b) interval. This means that the lower boundary is no less than a and the upper boundary is no more than b. All values between a and b are equally likely to occur, as shown in Figure 1.

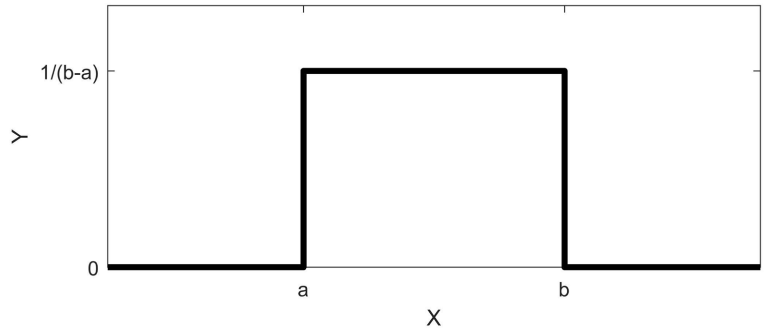


Figure 1. Density function.

3. Methodology

Each subsection below addresses a specific distance distribution and is broken into Euclidean and Rectilinear Distance measures.

3.1. Distance in the Unit Line

We are interested in finding the density function for the difference between X_1 and X_2 ($|X_1 - X_2|$). If X_1 and X_2 are independent and uniformly distributed on (0,1) in the unit square, we then have the following probabilistic statement:

$$P\left(|X_1 - X_2| > d\right) = (1-d)^2 \quad (5)$$

Here, $(1-d)^2$ is the total area of the two shaded right triangles, as shown in the unit square in Figure 2 [3]-[5].

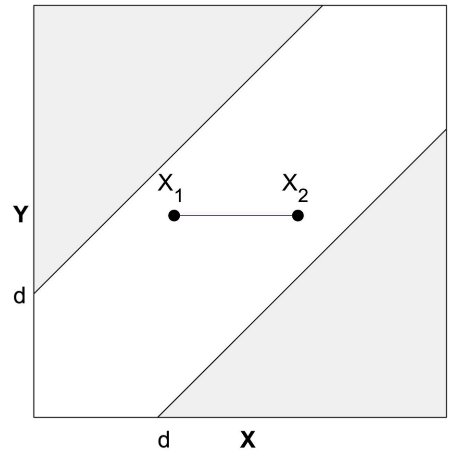


Figure 2. Diagonal strip2.

The total area that is non-shaded is $1-(1-d)^2$, which can be related to $P\left(|X_1 - X_2| < d\right)$ via the following:

$$P\left(\left|X_1 - X_2\right| < d\right) = 1 - \left(1 - d\right)^2 \quad (6)$$

The right-hand side is a cumulative density function, since it accounts for the totality of the non-shaded area. Taking the first derivative of the right-hand side with respect to " d ", we have the following:

$$P\left(\left|X_1 - X_2\right| = d\right) = D_d\left[1 - \left(1 - d\right)^2\right] = 2\left(1 - d\right) \quad (7)$$

Thus, the density function for $\left|X_1 - X_2\right|$ on the unit square appears as follows in **Figure 3**.

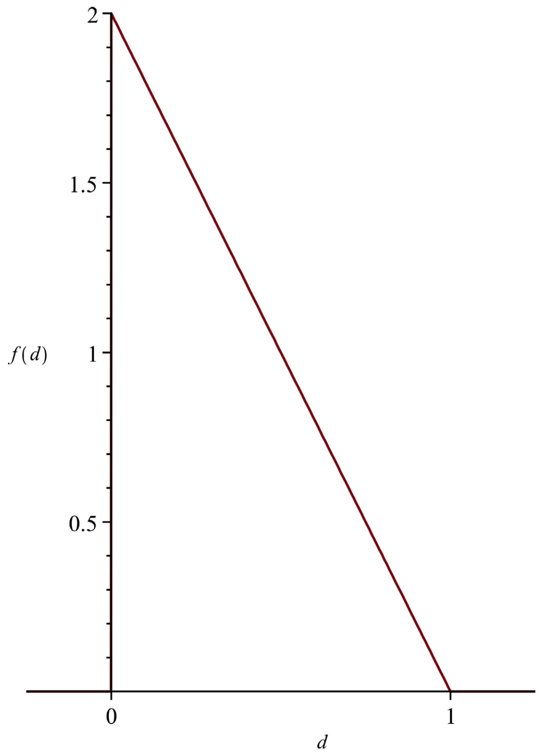


Figure 3. Line density.

3.2. Distance in the Unit Square

There are two different distance measures in the unit square that we are interested in: the Euclidean distance and the Rectilinear distance. For each, both points are uniformly distributed and in the (0, 1) interval, and shown in **Figure 4(a)** and **Figure 4(b)**.

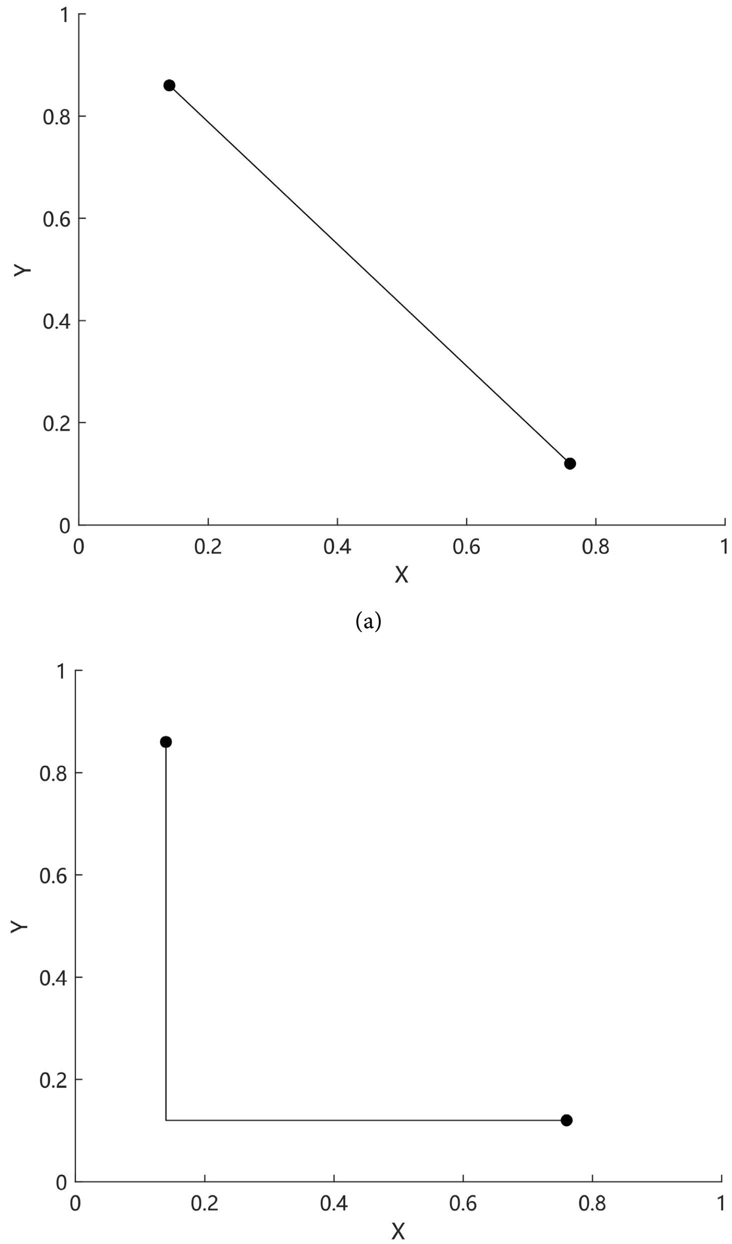
3.2.1. Euclidean Distance

For Euclidean distance, our distance measure is the following [3]:

$$d = \sqrt{\left(X_1 - X_2\right)^2 + \left(Y_1 - Y_2\right)^2} \quad (8)$$

This distance is on the (0, $\sqrt{2}$) interval. This equation can also be shown as follows:

$$d^2 = \left(X_1 - X_2\right)^2 + \left(Y_1 - Y_2\right)^2 \quad (9)$$



(a)

(b)

Figure 4. (a) Euclidean 2d; (b) Rectilinear 2d.

The reasoning for deriving the density function for the unit line in Equation (9) and **Figure 2** can be extrapolated as follows:

$$P\left(\left(X_1 - X_2\right)^2 + \left(Y_1 - Y_2\right)^2 > d\right) = \left(1 - d\right)^2 \quad (10)$$

From this, we can see that two components are added, permitting us to use the convolution to find the density function for d [6]:

$$\left(f * g\right)\left(d\right) = \int_{-\infty}^{\infty} f\left(t\right) * g\left(d - t\right) \mathrm{d}t \quad (11)$$

This provides us with the density function:

$$d = \begin{cases} 0, & d \le 0 \\ 2d\left(d^2 + \pi - 4d\right), & d \le 1 \\ -2d\left(d^2 + 2\sin^{-1}\left(\frac{d^2-2}{d^2}\right) + 2 + \frac{4-4d^2}{\sqrt{d^2-1}}\right), & d \le \sqrt{2} \\ 0, & d > \sqrt{2} \end{cases} \quad (12)$$

3.2.2. Rectilinear Distance

For Rectilinear Distance, our distance measure is on the (0, 2) interval and is as follows:

$$d = \left|X_1 - X_2\right| + \left|Y_1 - Y_2\right| \quad (13)$$

Because two components are added, we can once again convolve the two components directly above to obtain the density function for the rectilinear distance in the unit square, which is as follows:

$$f\left(d\right) = \begin{cases} 0, & d \le 0 \\ \frac{2}{3}\left(d^3 - 6d^2 + 6d\right), & d \le 1 \\ \frac{-2}{3}\left(d-2\right)^3, & d \le 2 \\ 0, & d > 2 \end{cases} \quad (14)$$

A comparison of these two density functions is shown in **Figure 5** below.

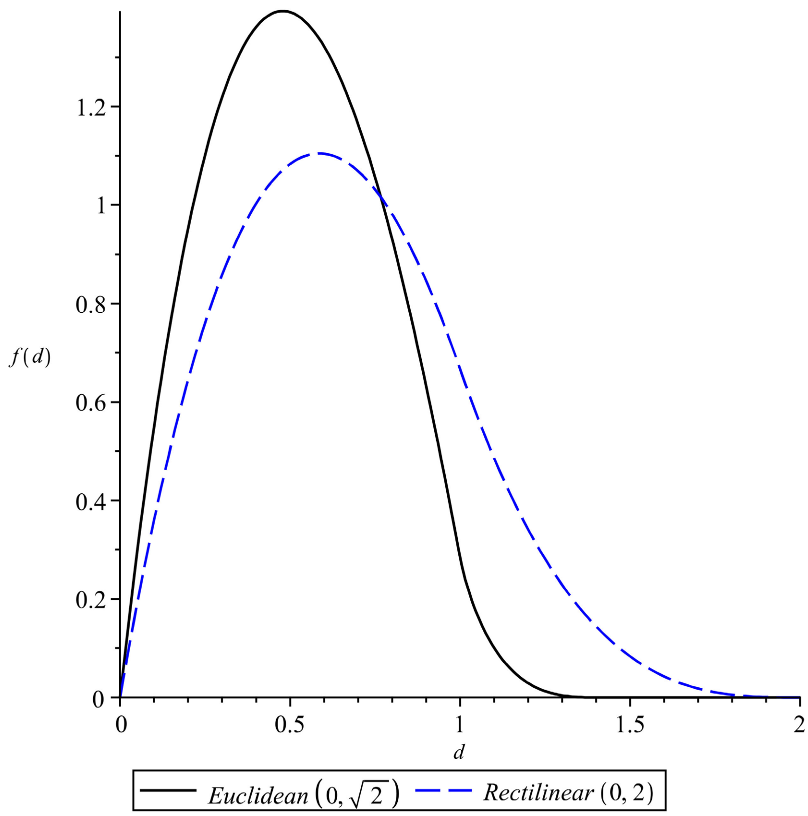
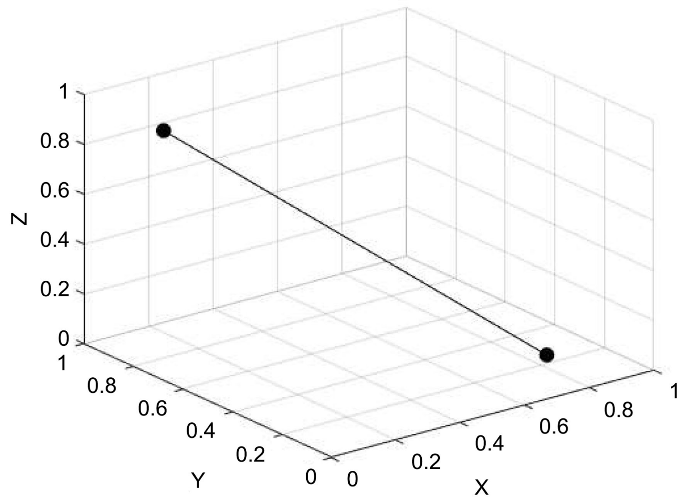


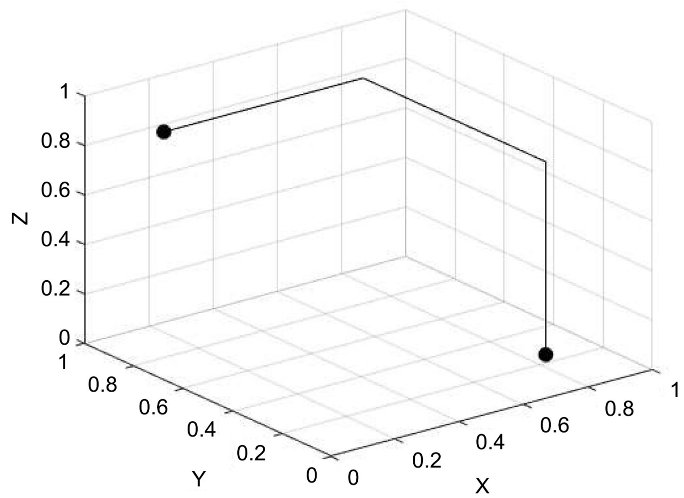
Figure 5. Density unit square.

3.3. Distance in the Unit Cube

In the unit cube, we have two points each with X, Y and Z coordinates, all uniformly distributed in the (0, 1) interval. This is shown in **Figure 6(a)** and **Figure 6(b)**.



(a)



(b)

Figure 6. (a) Euclidean 3D; (b) Rectilinear 3D.

3.3.1. Euclidean Distance

The Euclidean Distance between two points, each point having three coordinates, is as follows

$$d = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2} \tag{15}$$

This can also be written as follows:

$$d^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 \tag{16}$$

One will note that the above now has three components and has been added.

As such, the convolution can be applied twice here to obtain the density function for the distance measure above. The first convolution involves the distance density involving X and Y from the unit square. The second convolution involves the convolution obtained by employing (X and Y) with Z. This density function is as follows:

$$f(d)=\begin{cases}0, & d\le 0\\ d\left(d^3+6d\pi-8d^2-4\pi\right), & d\le 1\\ -2d\left(a_1+a_2+a_3+a_4\right), & d\le\sqrt{2}\\ -d\left(b_1-b_2+b_3+b_4+b_5-b_6+b_7\right), & d\le\sqrt{3}\\ 0, & d>\sqrt{3}\end{cases} \quad (17)$$

Where:

$$a_1=4d^2\sin^{-1}\left(\frac{d^2-2}{d^2}\right) \quad (18)$$

$$a_2=4d^2\tan^{-1}\left(\sqrt{d^2-1}\right) \quad (19)$$

$$a_3=d^4+(2\pi+3)d^2-4d\pi+3\pi-\frac{1}{2} \quad (20)$$

$$a_4=\frac{-8d^4+4d^2+4}{\sqrt{d^2-1}} \quad (21)$$

$$b_1=8d\tan^{-1}\left(\frac{d^3+d^2-3d-1}{\sqrt{d^2-2}\cdot\left(d^3-d^2-d-1\right)}\right) \quad (22)$$

$$b_2=-4d^2\cot^{-1}\left(\frac{d^2-d-1}{\sqrt{d^2-2}}\right) \quad (23)$$

$$b_3=4d(d+2)\tan^{-1}\left(\frac{d^2+d-1}{\sqrt{d^2-2}}\right) \quad (24)$$

$$b_4=\left(8d^2-4\right)\sin^{-1}\left(\frac{d^2-3}{d^2-1}\right) \quad (25)$$

$$b_5=16\sin^{-1}\left(\frac{\sqrt{d^2-2}}{\sqrt{d^2-1}}\right) \quad (26)$$

$$b_6=16\sin^{-1}\left(\frac{1}{\sqrt{d^2-1}}\right) \quad (27)$$

$$b_7=\frac{(d+5)\left(\sqrt{d^2-2}-8d^2+16\right)(d-1)}{\sqrt{d^2-2}} \quad (28)$$

3.3.2. Rectilinear Distance

As is the case for rectilinear distance in the unit square, we can continue the logic for the three-dimensional case. The rectilinear distance between two points in the unit cube is as follows:

$$d=\left|X_1-X_2\right|+\left|Y_1-Y_2\right|+\left|Z_1-Z_2\right| \quad (29)$$

The density function is as follows:

$$f(d)=\begin{cases}0, & d\le 0\\ \frac{1}{15}d^2\cdot\left(d^3-15d^2+60d-60\right), & d\le 1\\ \frac{1}{15}\left(-2d^5+30d^4-150d^3+330d^2-315d+93\right), & d\le 2\\ \frac{1}{15}(d-3)^2, & d\le 3\\ 0, & d>3\end{cases} \tag{30}$$

Figure 7 shows this density function.

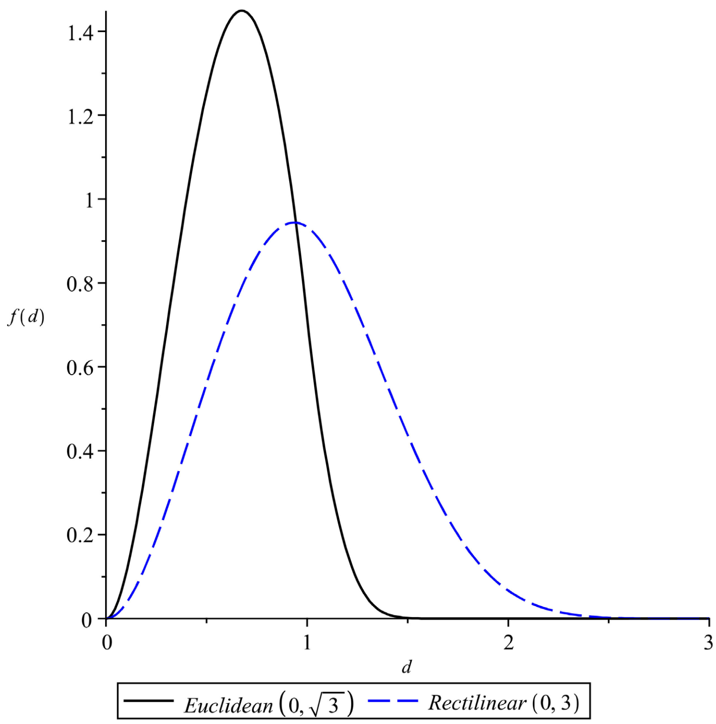


Figure 7. Density unit cube.

3.4. Distance in the Unit Circle

In the unit circle, we have the center at the origin (0, 0) and a radius limited to 1. We are not guaranteed a feasible point by random points on the (0, 1) interval. We must generate random points such that coordinates do not exceed a radius of 1. For any point, the x and y coordinates are limited to the maximum radius of 1:

$$x^2+y^2\le 1 \tag{31}$$

Because of this, we employ polar coordinates, such that the random angle is shown as θ on the (0, 2π) interval, and r, the random radius, is on the (0, 1) interval. Our X and Y coordinates are as follows:

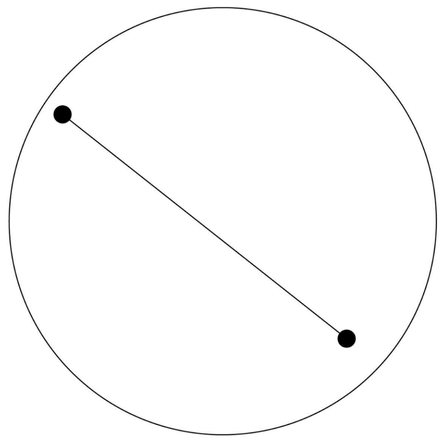
$$X_1=\sqrt{r_1}\cos\theta_1, Y_1=\sqrt{r_1}\sin\theta_1 \tag{32}$$

$$X_2=\sqrt{r_2}\cos\theta_2, Y_2=\sqrt{r_2}\sin\theta_2 \tag{33}$$

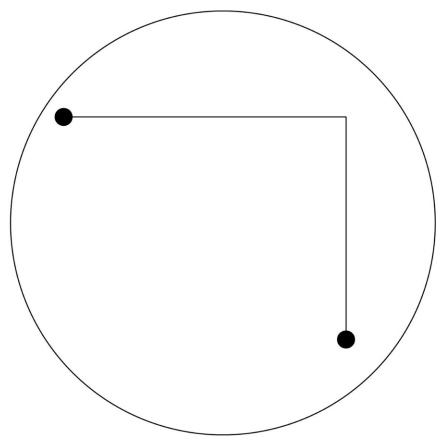
Take note of the radius component in the square root form. This must be done to guarantee a random distribution of a radius on the (0, 1) interval. If the square root were omitted, the random radius would be skewed toward the center of the circle. We would like a radius that is proportional to the circle's area. Since the area (A) of a circle is proportional to πr^2, we desire a radius that takes on the following form:

$$r = \sqrt{\frac{A}{\pi}} \tag{34}$$

This should justify taking the square root of the random number in the (0, 1) interval for the radius. These distances are shown in Figure 8(a) and Figure 8(b).



(a)



(b)

Figure 8. (a) Circle euclidean; (b) Circle rectilinear.

3.4.1. Euclidean Distance

Using Equations (32) and (33) above, we have the Euclidean Distance between two points in the unit circle:

$$d = \sqrt{\left(\sqrt{r_1} \cdot \cos(\theta_1) - \sqrt{r_2} \cdot \cos(\theta_2)\right)^2 + \left(\sqrt{r_1} \cdot \sin(\theta_1) - \sqrt{r_2} \cdot \sin(\theta_2)\right)^2} \tag{35}$$

Unfortunately, we are unable to obtain a density function via the distance function above. The required integration of Equation (35) above does not permit a symbolic solution. Fortunately, there is a clever proof available for the density function of the Euclidian distance between two points in the unit circle, which is as follows [7]:

$$f(d) = \frac{4d}{\pi}\left(\cos^{-1}\left(\frac{d}{2}\right) - \frac{d}{2}\sqrt{1-\left(\frac{d}{2}\right)^2}\right) \tag{36}$$

3.4.2. Rectilinear Distance

Using Equations (32) and (33) above, we have the Rectilinear Distance between two points in the unit circle.

$$d = \left|\sqrt{r_1}\cdot\cos(\theta_1) - \sqrt{r_2}\cdot\cos(\theta_2)\right| + \left|\sqrt{r_1}\cdot\sin(\theta_1) - \sqrt{r_2}\cdot\sin(\theta_2)\right| \tag{37}$$

As is the case with the Euclidean distance between two points in the unit circle, the Rectilinear Distance is also difficult to obtain, and there are no exact solutions known to the authors. As such, a numerical approximation is employed. The Pearson Distribution is exploited to obtain an approximation to density function of a non-symmetric nature. This approximation is detailed in a subsequent section of this paper. The interval for possible rectilinear distances in the unit circle is (0, $2\sqrt{2}$). For this problem, we approximate the distance density via the following:

$$f(d) = 1.3940\left(-0.2912\cdot d^2 + 0.8089\cdot d + 0.0316\right)^{1.7168} \mathrm{e}^{-0.8992\cdot\tanh^{-1}(0.7006\cdot d - 0.9730)} \tag{38}$$

The density functions for the unit circle measurements are as shown in **Figure 9** below:

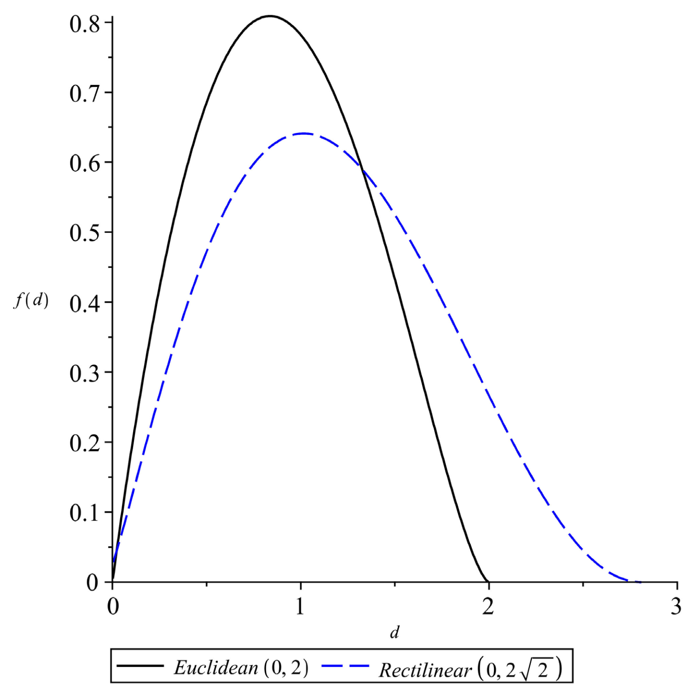


Figure 9. Circle density.

3.5. Distance in the Unit Sphere

In the unit sphere, we have the center at the origin (0, 0) and a radius limited to 1. We are not guaranteed a feasible point by random points on the (0, 1) interval. We must generate random points such that coordinates do not exceed a radius of 1. For any point, the x, y and z coordinates are limited to the maximum radius of 1:

$$x^2 + y^2 + z^2 \le 1 \quad (39)$$

Because of this, we again employ polar coordinates, such that the random alt-azimuth angle is shown as θ on the (0, 2π) interval. The declination angle, φ, is in the (0, π) interval, and r, the random radius, is on the (0, 1) interval. Our X, Y and Z coordinates are as follows:

$$X_1 = \sqrt{r_1}\cos\theta_1 \sin\varphi_1, Y_1 = \sqrt{r_1}\sin\theta_1 \sin\varphi_1, Z_1 = \sqrt{r_1}\cos\varphi_1 \quad (40)$$

$$X_2 = \sqrt{r_2}\cos\theta_2 \sin\varphi_2, Y_2 = \sqrt{r_2}\sin\theta_2 \sin\varphi_2, Z_2 = \sqrt{r_2}\cos\varphi_2 \quad (41)$$

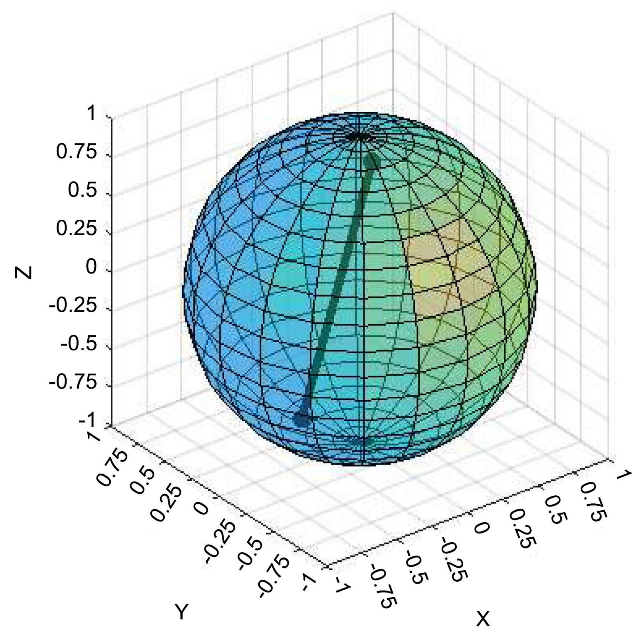
As was the case for the unit circle, we must take the square root of the radius so that the distribution of the points is uniform throughout the unit sphere. **Figure 10(a)** and **Figure 10(b)** are shown in **Figure 10(a)** and **Figure 10(b)**.

3.5.1. Euclidean Distance

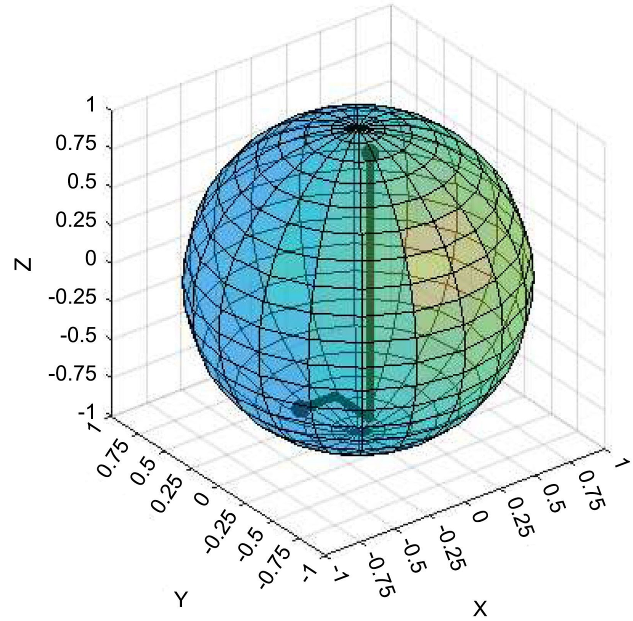
The Euclidean Distance between two points in the unit sphere is as follows:

$$d = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2} \quad (42)$$

where the X, Y and Z coordinates are determined by Equations (40 and 41) above. A proof by Lellouche and Souris shows that the density function of a unit sphere has an interval of (0, 2) and is as follows [8] [9]:



(a)



(b)

Figure 10. (a) Sphere euclidean; (b) Sphere rectilinear.

$$f(d) = \frac{3d^2}{2}\left(\left(\frac{d}{2}\right)^3 - 3\left(\frac{d}{2}\right) + 2\right) \tag{43}$$

3.5.2. Rectilinear Distance

The Rectilinear Distance between two points in the unit sphere is as follows:

$$d = |X_1 - X_2| + |Y_1 - Y_2| + |Z_1 - Z_2| \tag{44}$$

where X, Y and Z coordinates are determined by equations above. As was the case with the Rectilinear Distance in the unit circle, the author is not aware of an exact mathematical solution. As such, the Pearson distribution was again used to obtain the following density function approximation:

$$f(d) = 3.628\left(-0.1748 \cdot d^2 + 0.5828 \cdot d + 0.0476\right)^{2.8599} \mathrm{e}^{-0.5473 \cdot \tanh^{-1}(0.5726 \cdot d - 0.9543)} \tag{45}$$

This density function spans the (0, $2\sqrt{3}$) interval, and is shown in **Figure 11** below.

3.6. Distance in the Unit Triangle

The Unit triangle has vertices at (0, 0), $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and (1, 0). The X coordinate must be on the (0, 1) interval, and the Y coordinate must be on the $\left(0, \frac{\sqrt{3}}{2}\right)$ interval. Additionally, to guarantee the random point is in the unit triangle, the following two conditions must be met:

$$y \le \sqrt{3}x \tag{46}$$

$$y \le \sqrt{3} - \sqrt{3}x \tag{47}$$

Figure 12(a) and Figure 12(b) show an example of Euclidean and Rectilinear distances in the unit triangle.

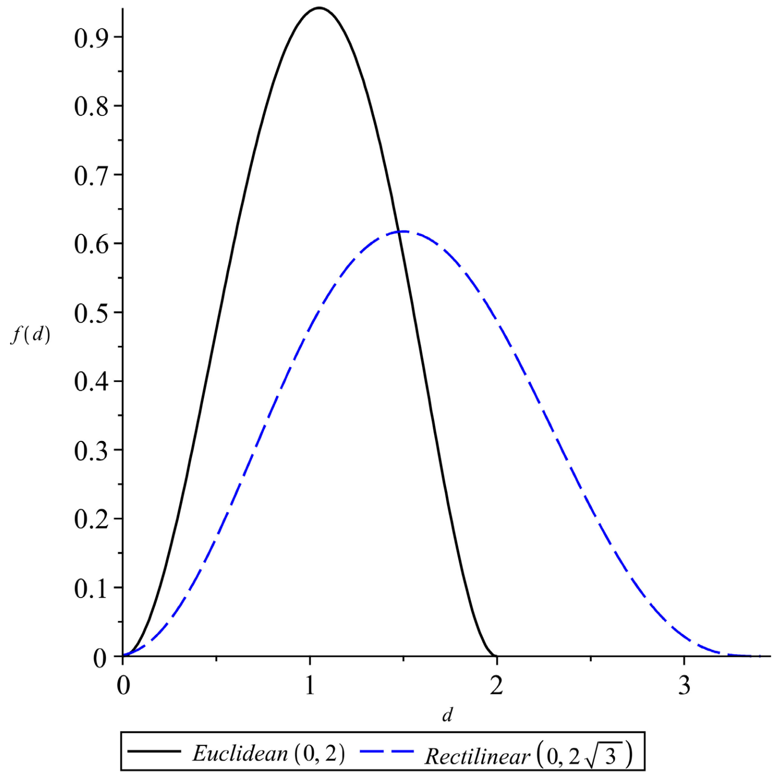
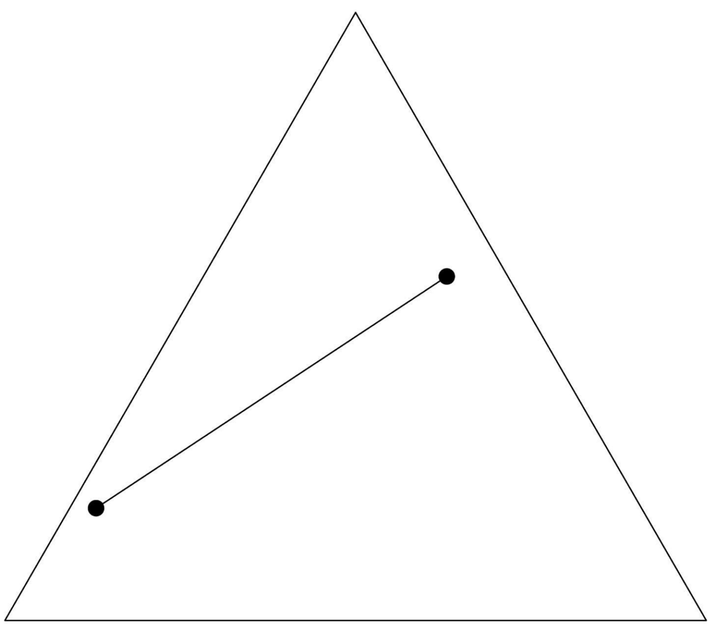
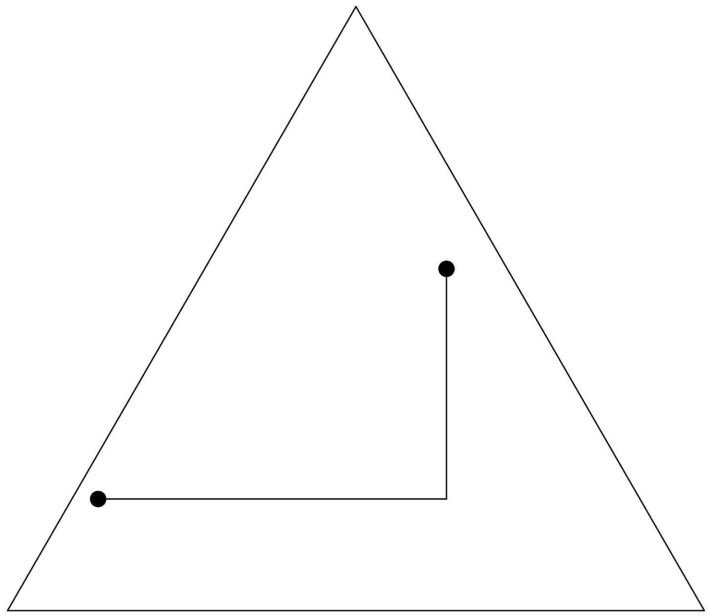


Figure 11. Sphere density.



(a)



(b)

Figure 12. (a) Triangle euclidean; (b) Triangle rectilinear.

3.6.1. Euclidean Distance

After feasible points are generated in the unit triangle, the Euclidean distance is as follows:

$$d = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2} \quad (48)$$

The density function is:

$$f(d) = 132.8\left(-0.3047d^2 + 0.2844d + 0.0012\right)^{1.6407} \mathrm{e}^{-1.144\tanh^{-1}(2.124d - 0.9912)} \quad (49)$$

3.6.2. Rectilinear Distance

For the points generated above, the Rectilinear distance is as follows:

$$d = |X_1 - X_2| + |Y_1 - Y_2| \quad (50)$$

The density function is:

$$f(d) = 140.8\left(-0.2295d^2 + 0.3033d + 0.0048\right)^{2.1791} \mathrm{e}^{-1.851\tanh^{-1}(1.478d - 0.9771)} \quad (51)$$

Figure 13 shows the density function.

4. Pearson Distribution for Estimation of Density Functions

For this research effort, some density functions cannot be determined symbolically. As such, an estimate is used to find the density function. The Pearson Distribution is employed, as it is able to find density functions, specifically of a non-symmetric nature. Specifically, the Pearson Distribution, Type 4 is employed here, which considers the skewed behavior of the resultant distance data. The distance measures exploiting the Pearson Distribution are the rectilinear distance for the unit circle, the rectilinear distance for the unit sphere, and both the Euclidean and rectilinear distance for the unit triangle.

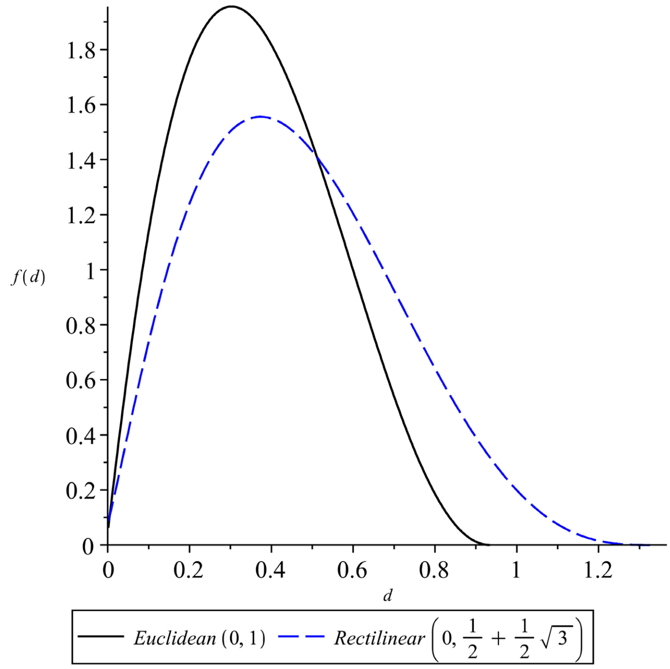


Figure 13. Triangle density.

For each of these four scenarios where a symbolic solution is not possible, 1,000,000 measures of distance (d) were obtained via simulation and recorded. These simulated distances were essentially employed as data sets to be "fitted". The intent is to find a density function that best "fits" the simulated data [10] [11].

The Pearson Distribution function starts as a differential equation:

$$\frac{\mathrm{d}f(d)}{\mathrm{d}d}=\frac{-(d+a)\cdot f(d)}{c_2d^2+c_1d+c_0} \tag{52}$$

The solution to this differential equation is as follows:

$$f(d)=\left(c_2d^2+c_1d+c_0\right)^{\frac{-1}{2c_2}}\cdot c_3\cdot\exp\left(\frac{\tan^{-1}\left(\frac{2c_2d+c_1}{\sqrt{4c_0c_2-c_1^2}}\right)\cdot\left(2c_2a-c_1\right)}{c_2\sqrt{4c_0c_2-c_1^2}}\right) \tag{53}$$

After numerically solving for the constants, and simplifying, we have the following general solution:

$$f(d)=b_7\left(b_2d^2+b_1d+b_0\right)^{b_3}\mathrm{e}^{b_4\tanh^{-1}(b_5d-b_6)} \tag{54}$$

Here, the constant values (b_i) vary according to the density function pursued. The values of b_i are shown in the Section 3, where the specific distance measure of interest is presented.

For purposes of comparison, The Pearson Distribution function estimated for the Euclidean Distance in a unit sphere is compared to the actual density function for Euclidean Distance in a unit sphere. **Figure 14** shows the result [12] [13].

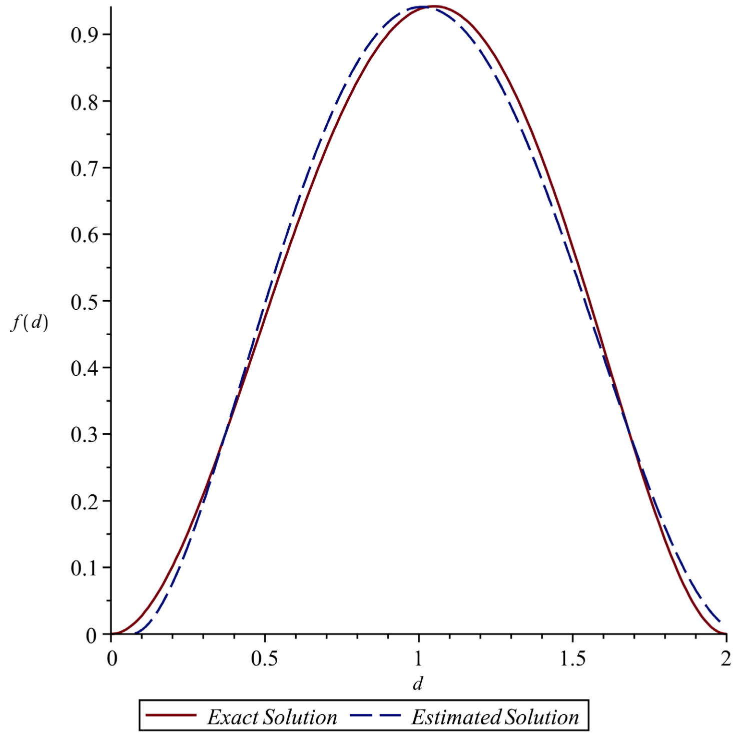


Figure 14. Sphere comparison.

From close inspection, one should notice that the difference between the actual density function and the estimated density function is small. The authors consider this small difference a strong argument as to the viability of estimating density functions that cannot be determined to exactness.

5. Statistics Associated with Density Functions

In addition to the density functions obtained, it is desired to capture descriptive statistics of these random variables. Of specific interest are the mean, standard deviation and median values. For the mean or expected value (μ), we exploit the density function ($f(d)$) as follows:

$$\mu = \int_0^b d \cdot f(d)\,\mathrm{d}d \tag{55}$$

where b is the highest possible value of the random variable of interest. For the standard deviation of the random variable of interest (σ), we exploit the following via the density function of interest:

$$\sigma = \sqrt{\int_0^b (d-\mu)^2 f(d)\,\mathrm{d}d} \tag{56}$$

For the median, we find the value of the density function at the 50th percentile. With the density function for the random variable of interest, we take the following and solve for the unknown (x):

$$\frac{1}{2} = \int_0^x f(d)\,\mathrm{d}d \tag{57}$$

Tables 1-3 below summarize the descriptive statistics associated with each density function pursued [14].

Table 1. Expected values for density functions.

Distance Measurement	Expected Value: Symbolic Value (Fixed Point Value)
Line	$\frac{1}{3}$, (0.333333)
Unit Square (Euclidean)	$\frac{2}{15}+\frac{\sqrt{2}}{15}-\ln\left(1+\sqrt{2}\right)+\frac{4\coth^{-1}\sqrt{2}}{3}$, (0.5214054335)
Unit Square (Rectilinear)	$\frac{2}{3}$ (0.66666667)
Unit Cube (Euclidean)	$\frac{4}{105}+\frac{17}{105}\sqrt{2}-\frac{2}{35}\sqrt{3}+\frac{1}{5}\ln\left(1+\sqrt{2}\right)+\frac{2}{5}\ln\left(2+\sqrt{3}\right)-\frac{1}{15}\pi$, (0.6617071838)
Unit Cube (Rectilinear)	1 (1.000000)
Unit Circle (Euclidean)	$\frac{128}{45\pi}$ (0.9054147874)
Unit Circle (Rectilinear)	Not Available (1.15258320967201)
Unit Sphere (Euclidean)	$\frac{36}{35}$ (1.0285714286)
Unit Sphere (Rectilinear)	Not Available (1.5428441)
Unit Triangle (Euclidean)	Not Available (0.3647171)
Unit Triangle (Rectilinear)	Not Available (0.4639936)

Table 2. Standard deviations for density functions.

Distance Measurement	Standard Deviation: Symbolic Value (Fixed Point Value)
Line	$\frac{\sqrt{2}}{6}$ (0.235702264)
Unit Square (Euclidean)	$\frac{1}{15}\left(69-225\ln\left(1+\sqrt{2}\right)^2+30\left(2+\sqrt{2}+20\coth^{-1}\left(\sqrt{2}\right)\right)\ln\left(1+\sqrt{2}\right)-400\coth^{-1}\left(\sqrt{2}\right)^2 40\left(-2-\sqrt{2}\right)\coth^{-1}\left(\sqrt{2}\right)-4\sqrt{2}\right)^{1/2}$, (0.6617071838)
Unit Square (Rectilinear)	$\frac{1}{3}$ (0.33333333)
Unit Cube (Euclidean)	Available from Authors* (0.2492861908)
Unit Cube (Rectilinear)	$\frac{\sqrt{6}}{6}$ (0.4082482906)
Unit Circle (Euclidean)	$\frac{\sqrt{2025\pi^2-16384}}{45\pi}$ (0.4245280475)

Continued

Unit Circle (Rectilinear)	Not Available (0.554961333335191)
Unit Sphere (Euclidean)	$\frac{\sqrt{174}}{35}$ (0.3768830274)
Unit Sphere (Rectilinear)	Not Available (0.5899576)
Unit Triangle (Euclidean)	Not Available (0.1834006)
Unit Triangle (Rectilinear)	Not Available (0.2385581)

*This symbolic result is very lengthy, and in the interest of convenience, is not shown here.

Table 3. Median values for density functions.

Distance Measurement	Median: Symbolic Value (Fixed Point Value)
Line	$1-\frac{\sqrt{2}}{2}$ (0.2928932190)
Unit Square (Euclidean)	Not Available (0.512003269083295)
Unit Square (Rectilinear)	Not Available (0.642156234507211)
Unit Cube (Euclidean)	Not Available (0.662283184809551)
Unit Cube (Rectilinear)	Not Available (0.976278306309760)
Unit Circle (Euclidean)	Not Available (0.8912907750)
Unit Circle (Rectilinear)	Not Available (1.12469353312852)
Unit Sphere (Euclidean)	Not Available (1.033247409)
Unit Sphere (Rectilinear)	Not Available (1.5339)
Unit Triangle (Euclidean)	Not Available (0.3513)
Unit Triangle (Rectilinear)	Not Available (0.44305)

6. Concluding Comments

For common distance measures in unit space, density functions and descriptive statistics were obtained to provide a better description of these entities. In all instances, exact, symbolic solutions were desired, but not always possible. For all but four density functions, exact, symbolic solutions were obtained. For the four density functions for which exact, symbolic solutions were not available, reasonably accurate numerical approximations via the Pearson distribution were obtained. These approximations are believed to be reasonable estimates of the actual density functions, as a comparison was made between the actual density function for the distance between two points on a unit sphere and an approximation of that same density. The comparison shows the approximation is, in fact, a reasonable fit.

For the descriptive statistics measures of mean, median and standard deviation, exact, symbolic solutions were more difficult to obtain. Symbolic solutions were obtained for the simpler distance measures. As such, numerical values were usually presented.

There are opportunities for subsequent research in this area. With the emphasis on wireless networks, microprocessors and the like, this has become an important problem [15]. For example, consider the electronics industry. Integrated circuits often require precious medals, such as silver, to permit an electrical signal to flow. Precious medals, such as silver, are expensive, and they need to be used parsimoniously. We can employ the methodology presented here, so that we have a better idea of how much silver is needed so that the proper amount of silver is procured, preventing needless waste.

Exact, symbolic solutions were obtained via Maple version 2023.2. It is always possible for the mathematical software to gain functionality over time, and enable the user to see more exact, symbolic solutions. Additionally, there are many other distance measures and subsequent density functions and descriptive statistics to investigate.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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