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Multi-Port Resistance Networks and a Generalized Theory for Flow Preserving Clustered Equivalents for Market Analysis of Power Grids

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Abstract

The excessive computational burden encountered in power market analysis has necessitated the need for obtaining reduced equivalent networks that preserve flows along certain selected lines called tie lines in a larger power system. In this context, the concept of PTDF (Power Transfer Distribution Factors) matrix was introduced and studied using the DC flow model. On the other hand, the concept of modified circuit matrix of a multi-port resistance network was introduced by Thulasiraman and Murti. In this paper we draw attention to certain limitations of the approach by Cheng and Overbye to determine an equivalent that preserves a PTDF matrix. We then show the equivalence of the concept of modified circuit matrix of a multi-port resistance network and the concept of the PTDF matrix under the DC flow model. We then present a generalized theory of flow preserving equivalence that is not constrained by these limitations. We give a methodology to generate a flow preserving equivalent network and demonstrate its feasibility through simulations.

Keywords

PTDF Matrix, Modified Circuit Matrix, Power Network Equivalence, Graph Theory

1. Introduction

Power system networks are becoming complex with increasing size, and so associated analyses dealing with market or stability are becoming increasingly complex too [1]. To fulfill any analysis requirements, the complexity of the network needs to be addressed [2]. Early approaches for network equivalencing include [3]-[8]. These approaches usually eliminate less important elements on the basis of certain parameters. Transmission lines and generators connected to the boundary buses may be eliminated with minor impacts.

There has been a great deal of interest in recent years in reducing the computational burden in the analysis of power markets and hence certain equivalent network models have been proposed and used [9]-[12]. In particular, the PTDF-based system equivalents have received increasing attention [11] [12]. Basically the PTDF-based equivalent networks based on the DC power flow model preserve the flows across certain links (called tie-lines) of the original larger power network. In this paper, we are concerned with the PTDF-based equivalent network model.

Resistive electrical networks have found increasing importance in several applications to model random walks [13]. Our interest is in the context of the role of resistances in serving as a model for DC power flows in power systems.

We present certain graph theoretic concepts and circuit analysis techniques in Section 2. The concept of modified circuit matrix of a resistance multi-port network [14] is presented in Section 3. We present in Section 4 an inverse problem of designing a multi-port resistance network. We also present in this section, a methodology to solve this problem. Sections 5 - 6 deal with the basic notations in power matrix analysis and the equivalence of the PTDF to the modified circuit matrix of a multi-port resistance network. Section 7 deals with a generalized theory of flow preserving equivalents that is not constrained by the limitations of the approach in [11]. Simulation results are presented in Section 8.

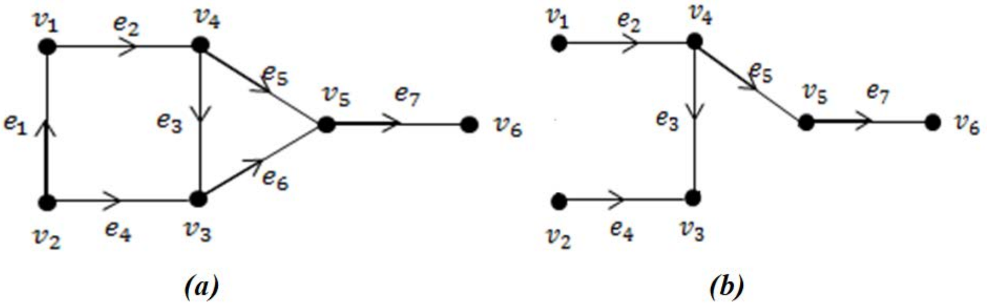


Figure 1. (a) Graph representation G; (b) Spanning tree T of G.

2. Basic Concepts

Consider a connected directed graph $G = (V, E)$ with no self-loops where $V = \{1, 2, \cdots, n\}$ is the set of n vertices and $E = \{e_1, e_2, \cdots, e_m\}$ is the set of m edges of the graph. Each edge directed from vertex i to j will be denoted by (i, j). A spanning tree T of G is a connected subgraph of G containing all the vertices of G and no circuits. For example, for the graph G in **Figure 1(a)** the edges $\{e_2, e_3, e_4, e_5, e_7\}$ form a spanning tree (see **Figure 1(b)**). The edges of

T are called branches of T and those that are not in T are called chords of T. For each spanning tree there are $n-1$ branches and $m-n+1$ chords. Three matrix representations of a graph used extensively in circuit theory literature are defined in what follows.

2.1. Incidence Matrix [15]

The all-vertex incidence matrix $A_c = \left[a_{ij}\right]$, of G is of dimension $n \times m$. The element a_{ij} of A_c is defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if the } j\text{th edge is incident on the } i\text{th vertex} \\ & \text{and oriented away from it;} \\ -1, & \text{if the } j\text{th edge is incident on the } i\text{th vertex} \\ & \text{and oriented towards it;} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The vertex which corresponds to the row of A_c which is not in A will be called the reference vertex of A.

Theorem 1: [15] *The determinant of any incidence matrix of a tree is equal to* ±1.

2.2. Fundamental Circuit Matrix [15]

Let the branches of a spanning tree T be denoted by $b_1, b_2, \cdots, b_{n-1}$, and let the chords of T be denoted by $c_1, c_2, \cdots, c_{m-n+1}$. While T has no circuits, the graph $T \cup \{c_i\}$ contains exactly one circuit C_i. The circuit C_i is called the fundamental circuit of G with respect to the chord c_i.

Note that each chord is present in exactly one fundamental circuit and every fundamental circuit contains exactly one chord. A circuit can be traversed in one of two directions, clockwise or anticlockwise. The direction we choose for traversing a circuit defines its orientation. The fundamental circuit matrix $B_f = \left[b_{ij}\right]$ with respect to T is of dimension $(m-n+1) \times m$ and is defined as follows.

1. The i th row corresponds to the fundamental circuit defined by c_i.
2. $$b_{ij} = \begin{cases} 1, & \text{if the } j\text{th edge is in the } i\text{th circuit and its orientation} \\ & \text{agrees with the circuit orientation of } c_i; \\ -1, & \text{if the } j\text{th edge is in the } i\text{th circuit and its orientation} \\ & \text{does not agree with the circuit orientation of } c_i; \\ 0, & \text{otherwise.} \end{cases}$$

If in addition, we assume that the orientation of a fundamental circuit is so chosen as to agree with that of the defining chord, then the matrix B_f can be displayed in a convenient form as follows:

$$B_f = \left[B_{ft} \quad | \quad U\right] \tag{2}$$

Where U is the unit matrix of order $m-n+1$ and its columns correspond to the chords of T; B_{ft} is the submatrix with its columns corresponding to the branches of T.

2.3. Fundamental Cutset Matrix [15]

Let V_1, V_2 be a partition of the vertex set V of a graph G. Then the set of edges with one vertex in V_1 and the other in V_2 is called a cut. This cut is denoted as (V_1, V_2).

Let b be a branch of T. The removal of b disconnects a spanning tree T into exactly two components T_1 and T_2. Let V_1 and V_2, respectively, denote the vertex sets of T_1 and T_2. The cut (V_1, V_2) is known as the fundamental cutset of G with respect to b of the spanning tree T of G. Note that every branch is present in exactly one fundamental cutset and each fundamental cutset has exactly one branch.

To define the fundamental cutset matrix of a directed graph we first assign an orientation to each cutset of the graph. Given a spanning tree T of an n-vertex connected graph G, let $b_1, b_2, \cdots, b_{(n-1)}$ denote the branches of T. The fundamental cutset matrix $Q_f = [q_{ij}]$ is of dimension $(n-1) \times m$ and is defined as follows:

$$q_{ij} = \begin{cases} 1, & \text{if the } j\text{th edge is in the } i\text{th cut and its orientation} \\ & \text{agrees with the cut orientation} \\ -1, & \text{if the jth edge is in the } i\text{th cut and its orientation} \\ & \text{does not agree with the cut orientation;} \\ 0, & \text{otherwise.} \end{cases}$$

If, in addition, we assume that the orientation of a fundamental cutset is so chosen as to agree with that of the defining branch, then the matrix Q_f can be displayed as follows:

$$Q_f = [U \mid Q_{fc}] \tag{3}$$

Where U is the unit matrix of order $n-1$ and its rows correspond to the branches of T. It is known that ([15])

$$Q_f B_f^t = 0. \tag{4}$$

Using this relation, we get

$$B_{ft} = -Q_{fc}^t. \tag{5}$$

See [15] for a more detailed discussion of graph-theoretic concepts.

2.4. Circuit Analysis

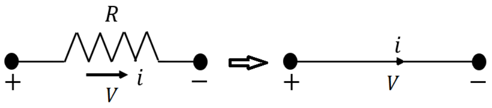


Figure 2. Convention for current direction and voltage polarities of resistance elements.

In the graph-theoretic representation of a resistance network, each resistance element is represented by an edge. We assign an arbitrary orientation to each such edge. The direction of current i in a resistance element will be the same as the direction of the corresponding edge. We follow the convention in **Figure 2** for the polarities of the voltage v in the resistance element.

Then by Ohm's law, we have $v = Ri$, where R is the value of the resistance in ohms (Ω) and i is the current flowing through the resistor. Let I_e denote the vector of currents in all resistance elements and V_e denote the vector of voltages across resistance elements. If B_f and Q_f are fundamental circuit and cutset matrices of the graph representation of the resistance network with respect to a spanning tree T, then we have

$$\text{KVL : Kirchoff's voltage law} \quad B_f V_e = 0, \tag{6}$$

$$\text{KCL : Kirchoff's current law} \quad Q_f I_e = 0. \tag{7}$$

Since the edges incident on a vertex form a cut, the incidence matrix is the representation of a special set of cutsets. So KCL can also be written as

$$AI_e = 0. \tag{8}$$

We know from (2) and (3), $Q_f = \left[U \mid Q_{fc}\right]$ and $B_f = \left[B_{ft} \mid U\right]$. Then in view of (4) and (5), we have

$$B_f = \left[-Q_{fc}^t \quad | \quad U\right], \tag{9}$$

$$Q_f = \left[U \quad | \quad -B_{ft}^t\right]. \tag{10}$$

Let V_b and I_b denote the vectors of branch voltages and currents, respectively. Also, let V_c and I_c denote the vectors of chord voltages and currents, respectively. Then we have the following relationship by KCL.

$$\left[U \quad | \quad Q_{fc}\right]\begin{bmatrix} I_b \\ I_c \end{bmatrix} = 0, \tag{11}$$

and

$$I_b = -Q_{fc} I_c. \tag{12}$$

Similarly, by KVL we have

$$\left[B_{ft} \quad | \quad U\right]\begin{bmatrix} V_b \\ V_c \end{bmatrix} = 0, \tag{13}$$

and

$$V_c = -B_{ft} V_b. \tag{14}$$

Using (9) and (10), we get

$$I_b = B_{ft}^t I_c, \tag{15}$$

$$V_c = Q_{fc}^t V_b. \tag{16}$$

From these equations, we can see that each branch current can be represented as a linear combination of chord currents. Similarly, each chord voltage can be represented as a linear function of the branch voltages.

3. The Modified Circuit Matrix of a Resistive Multi-Port Network

Consider a connected resistance network with vertex set V and edge set E. Let

P be a set of pairs of vertices of the network. For each pair (i, j), we connect a current source across the vertices i and j. These pairs of vertices are called ports of the network, as illustrated in **Figure 3**.

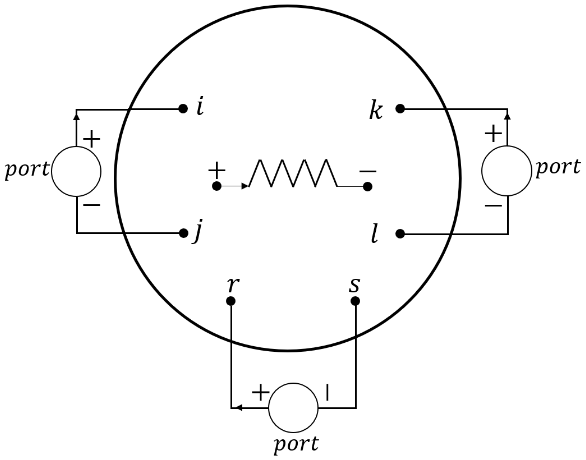


Figure 3. Resistive multi-port network.

In the graph representation of N, each port and each resistance element is assigned an orientation. Each resistance element is assigned an orientation as shown in **Figure 2** such that the voltage drop V is in the direction of the current. On the other hand, each port is assigned an orientation such that the port voltage drop is opposite to the direction of the port current, which is shown in **Figure 4**.

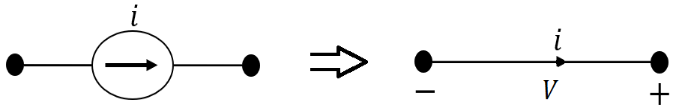


Figure 4. Convention for port current direction and voltage polarities.

In the following, we use the symbol N to denote the network as well as the corresponding graph. Let T be a spanning tree of N such that all its edges are resistance elements i.e. all the port edges are chords of T. Also, there could be some chords which are resistance elements; such chords are called non-port chords.

Let P be the set of port chords, NP the set of non-port chords and T the set of tree branches. Note $NP \cup T$ is the set of all resistive elements. Let V_p, I_p be the column vectors of port voltages and port currents respectively, V_{np}, I_{np} the column vectors of non-port chord voltages and currents in the set NP respectively and V_t, I_t the column vectors of tree branch (resistive) voltages and currents respectively. Then we can write B_f as

$$B_f = \begin{array}{c|c|c|c} & P & NP & T \\ P & U & 0 & B_1' \\ \hline NP & 0 & U & B_2' \end{array}. \tag{17}$$

By KVL we have

$$\begin{bmatrix} U & 0 & B_1' \\ 0 & U & B_2' \end{bmatrix} \begin{bmatrix} -V_p \\ V_{np} \\ V_t \end{bmatrix} = 0.$$

So

$$\begin{bmatrix} V_p \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & B_1' \\ U & B_2' \end{bmatrix} \begin{bmatrix} V_{np} \\ V_t \end{bmatrix} = 0. \tag{18}$$

Setting $B_1 = \begin{bmatrix} 0 & B_1' \end{bmatrix}$ and $B_2 = \begin{bmatrix} U & B_2' \end{bmatrix}$, we can rewrite (18) as

$$\begin{bmatrix} V_p \\ 0 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} V_{np} \\ V_t \end{bmatrix}. \tag{19}$$

By (15)

$$\begin{bmatrix} I_{np} \\ I_t \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^t \begin{bmatrix} I_p \\ I_{np} \end{bmatrix}. \tag{20}$$

Letting $V_R = \begin{bmatrix} V_{np} \\ V_t \end{bmatrix}$, $I_R = \begin{bmatrix} I_{np} \\ I_t \end{bmatrix}$ and Z_R be the diagonal matrix of resistances of all the resistive elements, we have

$$V_R = Z_R I_R. \tag{21}$$

Combining (19), (20) and (21),

$$\begin{aligned} \begin{bmatrix} V_p \\ 0 \end{bmatrix} &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} V_R \\ &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} Z_R \begin{bmatrix} B_1^t & B_2^t \end{bmatrix} I_R \\ &= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_p \\ I_{np} \end{bmatrix}, \end{aligned} \tag{22}$$

Where $Z_{ij} = B_i Z_R B_j^t$, $i = 1,2$, and $j = 1,2$.

Then

$$I_{np} = -Z_{22}^{-1} Z_{21} I_p. \tag{23}$$

Using (23) in (22)

$$\begin{aligned} V_p &= \left(Z_{11} - Z_{12} Z_{22}^{-1} Z_{21} \right) I_p \\ &= Z_p I_p. \end{aligned} \tag{24}$$

The matrix $Z_p = Z_{11} - Z_{12} Z_{22}^{-1} Z_{21}$ is called the open-circuit resistance matrix of the multi-port network N. The modified circuit matrix B of N defined in [14] is

$$B = B_1 - Z_{12} Z_{22}^{-1} B_2. \tag{25}$$

We can also verify (24) by showing that

$$Z_P = B Z_R B_1^t. \tag{26}$$

To illustrate the ideas developed thus far, consider the 3-port resistance network N in **Figure 5(a)**. The corresponding graph is in **Figure 5(b)**. The resistance elements are labelled as 1, 2, 3, 4, 5, 6 and the port edges are labelled as 7, 8, 9. We follow the conventions in **Figure 2** and **Figure 4** regarding the orientations of resistance elements and ports. Since the columns of B_1 and B_2 matrices are arranged as non-port chords $NP = \{2, 6\}$ and branches of $T = \{1, 3, 4, 5\}$, the resistance elements in Z_R matrix are arranged accordingly.

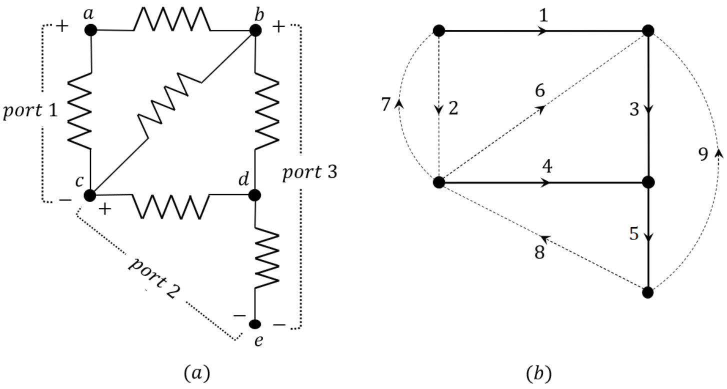


Figure 5. (a) Three port resistance network; (b) Corresponding graph.

Let r_j be the resistance of element labelled j with $j = \{2, 6, 1, 3, 4, 5\}$. Let us assume that

$$Z_R = \begin{bmatrix} r_2 & & & & & \\ & r_6 & & & & \\ & & r_1 & & & \\ & & & r_3 & & \\ & & & & r_4 & \\ & & & & & r_5 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 1 \end{bmatrix} \tag{27}$$

Consider the spanning tree T consisting of edges 1, 3, 4, 5. In the graph in **Figure 5(b)**, the branches of T are shown in solid lines and the port chords in dotted lines. Note that all port edges are chords of T. As before, let

$$P: \text{set of port chords} = \{7, 8, 9\},$$

$$NP: \text{set of non-port chords} = \{2, 6\},$$

$$T: \text{set of branches of } T = \{1, 3, 4, 5\}.$$

Then the fundamental circuit matrix B_f with respect to T is

$$B_f = \begin{array}{c} \\ P \\ \\ NP \end{array} \begin{array}{c|c|cccc} P & NP & & T & & \\ \hline U & 0 & 1 & 1 & -1 & 0 \\ & & 0 & 0 & 1 & 1 \\ & & 0 & 1 & 0 & 1 \\ \hline 0 & U & -1 & -1 & 1 & 0 \\ & & 0 & 1 & -1 & 0 \end{array} \tag{28}$$

$$= \begin{array}{c} \\ P \\ NP \end{array} \begin{array}{c|c|c} P & NP & T \\ \hline U & 0 & B_1' \\ \hline 0 & U & B_2' \end{array}. \tag{29}$$

Then

$$
\begin{aligned}
B_1 &= \begin{bmatrix} 0 & B_1' \end{bmatrix} \\
&= \begin{array}{c} \\ 7 \\ 8 \\ 9 \end{array}
\begin{array}{cc|cccc}
2 & 6 & 1 & 3 & 4 & 5 \\
\hline
0 & 0 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array}.
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
B_2 &= \begin{bmatrix} 0 & B_2' \end{bmatrix} \\
&= \begin{array}{c} \\ 2 \\ 6 \end{array}
\begin{array}{cc|cccc}
2 & 6 & 1 & 3 & 4 & 5 \\
\hline
1 & 0 & -1 & -1 & 1 & 0 \\
0 & 1 & 0 & 1 & -1 & 0
\end{array},
\end{aligned}
\tag{31}
$$

Then

$$
\begin{aligned}
Z_P &= Z_{11} - Z_{12} Z_{22}^{-1} Z_{21} \\
&= \begin{bmatrix} 0.8077 & -0.1538 & 0.0769 \\ -0.1538 & 2.0769 & 1.4615 \\ 0.0769 & 1.4615 & 1.7692 \end{bmatrix}
\end{aligned}
\tag{32}
$$

and the modified circuit B of the given resistance network is

$$B = B_1 - Z_{12} Z_{22}^{-1} B_2 \tag{33}$$

$$= \begin{bmatrix} 0.8077 & -0.1154 & 0.1923 & 0.0769 & -0.0769 & 0 \\ -0.1538 & 0.3077 & 0.1538 & 0.4615 & 0.5385 & 1 \\ 0.0769 & -0.1538 & -0.0769 & 0.0769 & 0.2308 & 1 \end{bmatrix}. \tag{34}$$

4. An Inverse Problem on Multi-Port Resistive Networks

In this section we present and develop a solution to an inverse problem on resistance networks. The inverse problem is: Given a multi-port resistance network N with unknown resistance values, determine the diagonal matrix Z_R of positive edge resistances that guarantees that N has a specified modified circuit matrix, B.

We first give a characterization of the modified circuit matrix $B = \left[b_{ij} \right]$ of a resistance multi-port network.

Theorem 2: *The* (i, j) *element,* b_{ij}*, in the modified circuit matrix of a multi-port resistance network* N *is equal to the current that flows through jth resistance, when port* i *is connected to an independent current source of unit value and all other ports are open-circuited.*

Proof. Let T be a spanning tree of the given multi-port network with no port edges i.e. the port edges are chords of T and all edges of T are resistive elements. As we have defined in Section 3, let I_p and I_{np} be the vectors of port chord currents and non-port chord currents respectively. Then the vector I_R of currents in all resistance elements is given by (20). That is,

$$I_R = B_1^t I_p + B_2^t I_{np}. \tag{35}$$

Consider now (23) repeated below

$$I_{np} = -Z_{22}^{-1} Z_{21} I_p. \tag{36}$$

Substituting the above expression for I_{np} in (35)

$$\begin{aligned} I_R &= B_1^t I_p - Z_{22}^{-1} Z_{21} I_p \\ &= \left(B_1^t - Z_{22}^{-1} Z_{21}\right) I_p \\ &= B^t I_p. \end{aligned} \quad (37)$$

From (37), we get

$$\begin{aligned} (i, j) \text{ element of } B &= (j, i) \text{ element of } B^t \\ &= \text{current in the } j^{\text{th}} \text{ resistance element,} \end{aligned}$$

When port i is connected to a source of unit value with all other port currents equal to zero.

Note that (37) is a generalization of (15). Whereas (15) expresses I_R in terms of the fundamental circuit matrix and all the chord currents, (37) expresses I_R in terms of the modified circuit matrix and port chord currents.

Theorem 3: *A specified matrix* B *is the modified circuit matrix of resistive n-port network* N *with a given port structure if and only if the real diagonal matrix* Z_R *of element instances satisfies* $Z_R > 0$ *and*

$$BZ_R B_2^t = 0. \quad (38)$$

(*Note: The elements of* B *are specified in Theorem* 2.)

Proof.

Sufficiency: Assume that $Z_R > 0$ and $BZ_R B_2^t = 0$. Then

$$BZ_R B_2^t = 0. \quad (39)$$

By definition of the modified circuit matrix, the matrix B is representable as

$$B = B_1 - MB_2 = 0.$$

(Note: B_1 and B_2 can be obtained from the topology of N.)

Then

$$\left(B_1 - MB_2\right) Z_R B_2^t = 0.$$

$$\Rightarrow B_1 Z_R B_2^t - MB_2 Z_R B_2^t = 0$$

$$\Rightarrow Z_{12} - MZ_{22} = 0$$

$$\Rightarrow M = Z_{12} Z_{22}^{-1}$$

$$\Rightarrow B = B_1 - Z_{12} Z_{22}^{-1} B_2,$$

as required by the definition of the modified matrix.

Necessity: Assume that the resistive n-port network N has B as its modified circuit. Clearly $Z_R > 0$. Also,

$$\begin{aligned} BZ_R B_2^t &= \left(B_1 - Z_{12} Z_{22}^{-1} B_2\right) Z_R B_2^t \\ &= B_1 Z_R B_2^t - Z_{12} Z_{22}^{-1} B_2 Z_R B_2^t \\ &= Z_{12} - Z_{12} Z_{22}^{-1} B_2 Z_R B_2^t \\ &= Z_{12} - Z_{12} Z_{22}^{-1} Z_{22} \\ &= 0, \end{aligned}$$

as required.

Though it is easy to write B_2 by inspection for small networks, it is rather involved in the case of large graphs. So an equation expressing B_2 in terms of the elements of the incidence matrix A will be derived below.

Let the columns of the incidence matrix of the resistive multi-port network be partitioned as follows:

A_1 - Columns corresponding to the (resistance) elements in the tree T,

A_2 - Columns corresponding to the port chords,

A_3 - Columns corresponding to the non-port chords.

Let $A = [A_1\ A_2\ A_3]$. Then fundamental cutset matrix Q_f with respect to T is

$$\begin{aligned} Q_f &= A_1^{-1}\left[A_1 \quad A_2 \quad A_3\right] \\ &= \left[U \quad A_1^{-1}\cdot A_2 \quad A_1^{-1}\cdot A_3\right], \end{aligned} \tag{40}$$

and from (9)

$$B_f = \begin{array}{c|c|c|c} & P & NP & T \\ P & U & 0 & -(A_1^{-1}.A_2)^t \\ \hline NP & 0 & U & -(A_1^{-1}.A_3)^t \end{array} \tag{41}$$

$$= \begin{bmatrix} U & B_1^t \\ 0 & B_2^t \end{bmatrix}. \tag{42}$$

So

$$B_1 = \left[0 \quad -\left(A_1^{-1}\cdot A_2\right)^t\right], \tag{43}$$

and

$$B_2 = \left[U \quad -\left(A_1^{-1}\cdot A_3\right)^t\right]. \tag{44}$$

5. Power System Market Analysis

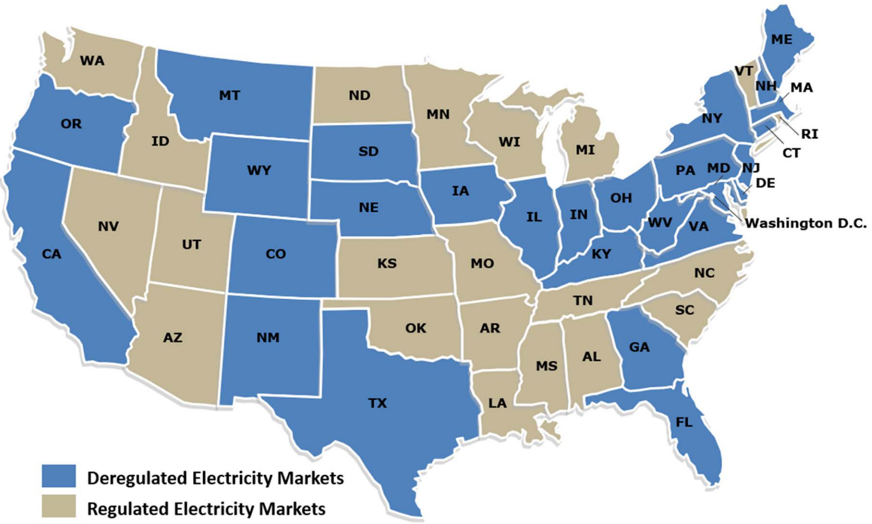


Figure 6. Deregulated electricity markets.

Power system market involves the sale of electricity from generators/producers to load serving entities (LSEs). Majority of interstate power transactions in the US are regulated by Federal Energy Regulatory Commission (FERC). The power grid in the US is divided into three main interconnections and within these interconnections lie regional entities which operate under FERC. They dispatch the electricity in accordance with their respective market rules [16].

Earlier most of the electricity regional markets in US were regulated; however due to its nature of monopoly and limitations on consumer choice there has been a recent deregulation of these markets which has increased consumer control over decisions. In addition to the increasing size of the grid, the impact of deregulation along with economic, political and environmental reasons has resulted in making power system market analysis in North America more complex [17]. The current deregulated electricity markets in the US are shown in **Figure 6**.

The power flow determination and following dispatching decisions as a part of analysis can become computationally challenging when based on full ac implementation approach [9]. Although full ac approach is accurate but making informed dispatch decisions on real-time requires the analysis to be quick even if the level of accuracy is reduced. The dc approach for power system analysis approximates the calculations and therefore allows network operators to make all the necessary decisions in a considerably shorter duration of time [17].

6. Equivalence of PTDF and Modified Circuit Matrix

Considering the enormity of scale of a power system network, analysis based on dc approach can still be complex and time-consuming. Power system network equivalencing is targeted at reducing the network in order to make the analysis computationally feasible. However, in the process of equivalencing, it must be ensured that the power flows across the regions or interconnections must be preserved along with the inter-ties. Conventional approaches for network equivalencing followed the process of elimination of unnecessary elements based on critical geographical and electrical parameters [4]-[8]. Some of the recent approaches are REI equivalent [3] and PTDF-based equivalents [11] [12]. Recently, the PTDF-based approach has been widely used along with approximation using dc flow analysis.

Introduced in [11], PTDF is defined as the fraction of the amount of transaction flowing through a transmission line for every injection and a respective withdrawal at different buses in the system. If the sink is the slack bus in all (injection, withdrawal) pairs, the PTDF is called ISF [1]. In the following mathematical description of PTDF, it is assumed that all sinks are the slack bus. In an effort to further reduce the computational time a dc power flow approach has been employed. In this model a power system is treated as a single element-kind circuit consisting of only inductive elements (reactances). Thus all the concepts and results on resistance network discussed in the previous sections are applicable to the study of the dc flow model of a power system where each resistance is replaced by the reactance of the same magnitude.

For a system having $N+1$ buses including the slack bus and L transmission lines, let

P_{in}: $(N\times 1)$ vector of bus active power injections.

Y_L: $(L\times L)$ diagonal matrix of line reactances.

A: $(N\times L)$ reduced incidence matrix (slack bus excluded).

Θ: vector of bus voltage angles. (Note: In the dc model all voltages are assumed to be of unit value.) $\vartheta_{bus}=\left(AY_LA^t\right)^{-1}$.

The power flow in the dc model is then given by

$$P_{in}=\vartheta_{bus}^{-1}\Theta. \tag{45}$$

Let Φ = PTDF matrix of dimension $L\times N$.

P_{flow} = $(L\times 1)$ vector of line flows.

$$Y_{BR}=Y_LA^t.$$

P_{in} = $(N\times 1)$ vector of the bus power injections.

Then

$$\begin{aligned} P_{flow} &= Y_LA^t\Theta \\ &= Y_{BR}\vartheta_{bus}^{-1}P_{in}. \end{aligned} \tag{46}$$

In a system with $N+1$ buses and L transmission lines where all transactions are between pair of nodes given by a set of injection nodes, $\mathcal{I}\in\{i_1,i_2,\cdots,i_N\}$ and a set of withdrawal nodes, $\mathcal{W}\in\{w_1,w_2,\cdots,w_N\}$, the PTDF matrix is defined as

$$\Phi=\begin{bmatrix} \phi_1^{i_1w_1} & \phi_1^{i_2w_2} & \cdots & \phi_1^{i_Nw_N} \\ \phi_2^{i_1w_1} & \phi_2^{i_2w_2} & \cdots & \phi_2^{i_Nw_N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_L^{i_1w_1} & \phi_L^{i_2w_2} & \cdots & \phi_L^{i_Nw_N} \end{bmatrix}, \tag{47}$$

Where $\phi_x^{i_jw_k}$ is the power flow on the line x when unit power is injected at bus i_j and withdrawn at bus w_k. Thus from (46) the PTDF matrix is defined as

$$\Phi=Y_{BR}\vartheta_{bus}^{-1}. \tag{48}$$

Noting the correspondence between link flows and line currents and the correspondence between bus voltage magnitudes and bus voltage angles, it follows from (48) that $\phi_x^{i_jw_k}$ is the current on line x when unit current is injected at bus i_j and withdrawn at bus w_k. In view of Theorem 2 we have

$$\Phi=B^t$$

Where B is the modified circuit of the dc model when each (injection, withdrawal) pair is treated as a port. We summarize this in the following theorem.

Theorem 4: *Given the PTDF matrix of the dc model of a power system, the modified circuit matrix* B *of the corresponding multi-port network* Φ *is given by*

$$B=\Phi^t.$$

7. A Generalized Theory of Flow Preserving Equivalents for Power Grids

Consider a power grid partitioned into different regions. Let each region be called a cluster. Let the transmission lines connecting the different clusters be called tie-lines. For a given set of (injection, withdrawal) pairs, let Φ be the sub-matrix of the PTDF matrix of the original power system corresponding to the flows in tie-lines. Let us construct a smaller network in which each cluster is a single node and all tie-lines between any two clusters are represented by a single line of unknown reactance value. We assume that a cluster does not contain both the buses in any (injection, withdrawal) pair. Now we wish to consider the problem of determining the unknown reactances so that the resulting equivalent network has the PTDF matrix Φ.

As we pointed out in Section 6, the problem is the same as the inverse problem we defined in Theorem 3. That is, we wish to determine Z_L such that

$$BZ_L B_2^t = 0, \tag{49}$$

Where $B = \Phi^t$ and B_2 is defined in Section 4 (Equation (19)). Note that Z_L is the absolute values of the impedances of the desired equivalent network.

In [11], the authors proposed a solution to the flow preserving equivalence problem assuming that the PTDF matrix is an ISF matrix. This solution has certain limitations:

1. The method in [11] assumes that the ports of the corresponding multi-port resistance network form a star. It further assumes that the port contains all the nodes of the required network. In other words, the method is applicable only when all the ports corresponding to (injection, withdrawal) pairs form a spanning tree of the required network.

2. For a PTDF matrix that does not satisfy the requirement in [11] the corresponding ISF cannot be uniquely determined. In other words, the ISF matrix is uniquely determined from the PTDF matrix if and only if the port structure defining the PTDF matrix and the port structure defining the ISF matrix are both spanning trees of the required equivalent structure.

In view of these limitations, the method in [11] for determining a flow preserving equivalent is not general. For the sake of completeness we next make certain remarks about the theory developed in [11]. We first show that the formula in [11] for the ISF matrix indeed satisfies the condition in our Theorem 3, thereby verifiying the correctness of this expression for the ISF matrix satisfying the limitation mentioned earlier.

In [11], the authors developed the following expression for Φ (ISF) where all (injection, withdrawal) pairs include the slack bus.

$$\Phi = Y_L A^t \vartheta_{bus}. \tag{50}$$

So in this case

$$B = \Phi^t \tag{51}$$

$$= \vartheta_{bus} A Y_L. \tag{52}$$

We now verify that $B = \Phi^t$ satisfies condition in Theorem 3. Note that in the definition of the ISF matrix there are no columns corresponding to the (injection, withdrawal) pairs (x, y) with neither x nor y is a slack. Hence the A matrix will have no columns corresponding to these pairs. Following the notation used in Section 4, let

$$A = \begin{bmatrix} A_3 & A_1 \end{bmatrix}.$$

Note that we have rearranged columns of A to conform to the columns of B_2. Then from (44)

$$B_2 = \begin{bmatrix} U & -\left(A_1^{-1} A_3\right)^t \end{bmatrix}.$$

So

$$\begin{aligned} B Z_L B_2^t &= \left(\vartheta_{bus} A Y_L\right) Z_L B_2^t \\ &= \vartheta_{bus} \begin{bmatrix} A_3 & A_1 \end{bmatrix} \begin{bmatrix} U \\ -A_1^{-1} A_3 \end{bmatrix} \\ &= 0. \end{aligned}$$

Thus we have the following theorem.

Theorem 5: *The ISF matrix* Φ *defined by*

$$\Phi = \vartheta_{bus} A Y_L$$

Satisfies the condition in Theorem 3, verifying the correctness of the formula in [11].

8. Simulations

In this section we experimentally evaluate the effectiveness of the methodology described in this paper The optimization problem is solved by cvx convex optimization toolbox in MATLAB. The instruction to install the cvx toolbox in MATLAB is provided in the link below and the tutorial are explained in the video links below. The simulations are performed on MATLAB R2018a.

http://cvxr.com/cvx/download/

https://www.youtube.com/watch?v=N2b_B4TNfUM

https://www.youtube.com/watch?v=h31bP5yw1gw

For the simulations we have used the synthetic transmission grid of 200 buses and 245 transmission lines, from Texas A&M University, Electric Grid Test Case Repository:

https://electricgrids.engr.tamu.edu/electric-grid-test-cases/

Two cases are presented to reduce the actual network to their equivalent networks Two folders named as "Case 1" and "Case 2" are provided. Each folder contains the Simulink (.slx) files for actual and reduced networks (CaseX. slx and VerifyX. slx, where X = 1, 2), and a MATLAB script file (CodeX. m). In the equivalent network each cluster is represented by a node. If there are lines connecting two clusters an edge is added between the corresponding nodes in the equivalent network. Each edge in the equivalent network is given an orientation. The MATLAB script file performs the following tasks:

1. Determine clusters for each power grid. Select a tree for the graph of the equivalent network. The generators will form the chords of the tree. Without loss of generality it is assumed that there are no generators inside a cluster. Matrix B_2 is then obtained.

2. Determine the tie line currents in the actual circuits for each generator ($G_1, G_2, \cdots$). Only one Generator remains ON for a particular instance and others remain OFF.

3. The algebraic sum of the tie-line currents in the lines connecting two clusters in the original network is the value of the required line current in the required equivalent. The value of this current is the corresponding element in the modified circuit matrix B of the equivalent network.

4. The equation $BZ_RB_2^t = 0$, with $Z_R > 0$ may not have a feasible solution. So we solve the following optimization problem to determine Z_R.

$$\min\left(\left\|BZ_RB_2^t = 0\right\|\right)$$

Subject to

$$BZ_RB_2^t = 0 \,\&\, Z_R > 0$$

The values of matrix Z_R are then assigned to the resistors in the VerifyX.slx file.

5. For the purpose of verification of $BZ_RB_2^t = 0$, all the generators are turned ON both in the actual network and reduced network, and the results are compared in the tie-line branches for both the actual and reduced network.

Inputs and results for Case 1 are presented in **Tables 1-4**, and for Case 2 in **Tables 5-8**. The reduced networks and the corresponding graphs are shown in **Figure 7** and **Figure 8**. The MATLAB script files in the folders also contain the comments for all cases to guide the user about the optimization and verification sections.

Table 1. Value of the B matrix.

Modified Circuit Matrix, B =								
	I_{12}	I_{23}	I_{24}	I_{25}	I_{35}	I_{45}	I_{46}	I_{56}
G1	1	0.2169	0.2236	0.5595	0.2169	0	0.2236	0.7764
G2	1	0.2436	0.1079	0.6486	0.2436	0	0.1079	−0.1079
G3	0	−0.0363	0.0198	0.0165	0.9637	−1	0.0198	−0.0198
G4	0	0.2165	0.2231	0.5604	0.2165	0	0.2231	0.7769
G5	0	0.2432	0.1073	0.6495	0.2432	0	0.1073	−0.1073
G6	0	−0.2711	−0.0741	−0.6548	0.7289	0	−0.0741	0.0741
G7	0	0.2348	0.0939	0.6713	0.2348	−1	0.0939	−0.0939
G8	0	−0.0183	0.1291	−0.1109	−0.0183	1	−0.1291	0.8709

Table 2. Values of B_2.

B_2 =

	R_4	R_5	R_8	R_1	R_2	R_3	R_6	R_7
R_4	1	0	0	0	0	−1	−1	0
R_5	0	1	0	0	1	−1	−1	0
R_8	0	0	1	0	0	0	1	−1

Table 3. Resistance values.

Resistance values after performing $\min\left(\left\|B \times R \times B_2^t\right\|_\infty\right)$ optimization:

R_4	R_5	R_8	R_1	R_2	R_3	R_6	R_7
0.0331	0.3343	0.1000	0.1000	7.4889	7.5878	0.1000	0.1000

Table 4. Tie-line currents comparison.

Tie-Line Currents Comparison:

Tie-Line Current	Actual Network	Reduced Network	Percentage Error
I_{12}	2.0000	2.0000	0
I_{23}	0.8292	0.7887	−3.7960
I_{24}	0.8306	0.9244	11.2928
I_{25}	2.3402	2.2779	−2.6631
I_{35}	2.8292	2.7977	−1.1125
I_{45}	−1.0000	−1.0504	5.0404
I_{46}	0.8306	0.9748	17.3611
I_{56}	2.1694	2.0252	−6.6470

Table 5. Value of the B matrix.

Modified Circuit Matrix, B =

	I_{12}	I_{13}	I_{14}	I_{24}	I_{34}	I_{35}	I_{45}
G1	0.2169	0.2236	0.5595	0.2169	0.0000	0.2236	0.7764
G2	0.2436	0.1079	0.6486	0.2436	0.0000	0.1079	−0.1079
G3	−0.0363	0.0198	0.0165	0.9637	−1.0000	0.0198	−0.0198
G4	−0.2715	−0.0747	−0.6538	0.7285	−0.0000	−0.0747	0.0747
G5	0.2352	0.0945	0.6704	0.2352	−1.0000	0.0945	−0.0945
G6	−0.0183	0.1291	−0.1109	−0.0183	1.0000	0.1291	0.8709

Table 6. B_2 values.

$B_2 =$							
	R_3	R_4	R_7	R_1	R_2	R_5	R_6
R_3	1	0	0	0	−1	−1	0
R_4	0	1	0	1	−1	−1	0
R_7	0	0	1	0	0	1	−1

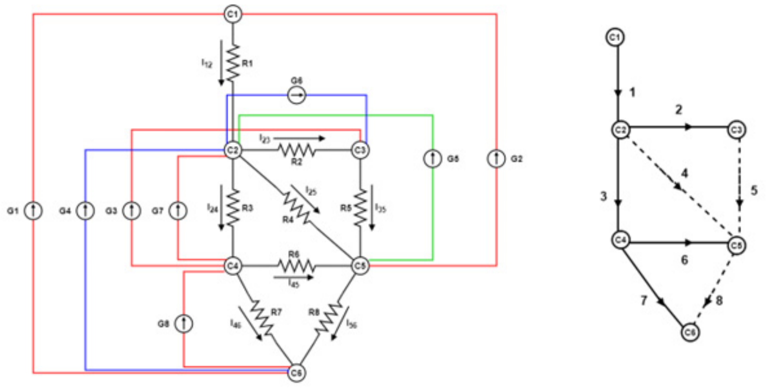
Table 7. Resistance values.

Resistance Values after performing $\min\left(\left\|B \times R \times B_2^t\right\|_\infty\right)$ optimization:						
R_3	R_4	R_7	R_1	R_2	R_5	R_6
0.0331	0.3343	0.1000	7.4889	7.5878	0.1000	0.1000

Table 8. Tie-line currents comparison.

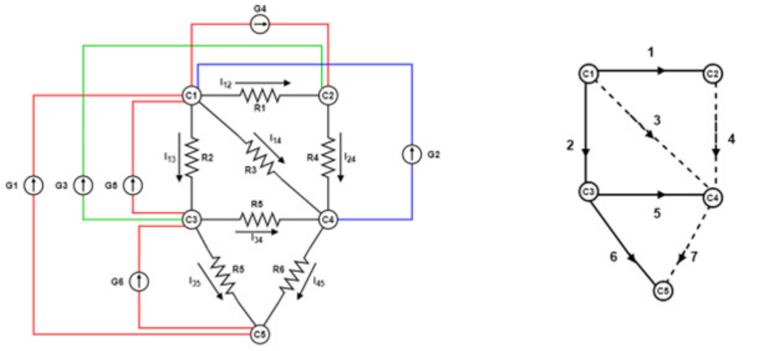
Tie-Line Currents Comparison:			
Tie-Line Current	Actual Network	Reduced Network	Percentage Error
I_{12}	0.3695	0.3640	−1.4955
I_{13}	0.5002	0.4768	−4.6820
I_{14}	1.1303	1.1593	2.5607
I_{24}	2.3695	2.3640	−0.2332
I_{34}	−1.0000	−1.0155	1.5487
I_{35}	0.5002	0.4923	−1.5858
I_{45}	1.4998	1.5077	0.5289

The above results demonstrate that the methodologies to generate Z_R to achieve a given modified circuit matrix are effective.



Reduced Network **Graph Representation**

Figure 7. Case 1: Simulation results (6 Clusters and 8 Generators).



Reduced Network

Graph Representation

Figure 8. Case 2: Simulation results (5 Clusters and 6 Generators).

9. Summary and Remarks

Despite its limitations, the work reported in [11] provided the motivation for the research presented in this paper. To make the paper self-contained, we first presented relevant graph theoretic concepts and techniques for analysis of multiport resistance networks. We then introduced the concept of modified circuit matrix first defined in [14]. We showed the PTDF matrix defined [11] is the transpose of the modified circuit matrix. We presented a necessary and sufficient condition to determine the element resistances that would generate a given modified circuit matrix (Theorem 3). We then gave a physical interpretation of the condition in Theorem 3. We also showed that the condition given in Theorem 3 is a generalization of the condition in [11]. Finally, we showed the relevance of Theorem 3 in generating clustered equivalents for market analysis of large scale power grids.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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