

The Solitary Wave Solutions of Important Model in Particle Physics and Engineering According to Two Different Techniques

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Abstract

In this paper, we utilized the Jaulent-Miodek equation which is one of important models in particle physics and engineering. The exact traveling wave solutions for this equation “involving parameters” according to two different techniques are constructed. When these parameters are taken as special values, the solitary wave solutions are derived from it. A comparison between the obtained results using these two different methods with that obtained by previous authors is demonstrated.

Keywords

The Extended Jacobian Elliptic Function Expansion Method, The Ricatti-Bernolli Sub, ODE Method, The Jaulent-Miodek (JM)-Equation, Traveling Wave Solutions

1. Introduction

In various phenomena of physics, mathematics, and engineering the nonlinear partial differential equations are important to study them. In the fluid dynamics, the analytical solutions of the nonlinear evolution equations of shallow water waves and the commonly studied equations (the Korteweg-de Vries (KdV) equation [1], modified KdV equation, Boussinesq equation [2] [3], Green-Naghdi equation, Gardeners equation, and Whitham-Broer-Kaup and Jaulent-Miodek (JM) equations) are usually not available. Therefore, finding a new methods and techniques for dealing these types of equations is still an open area. The theory of solitons has contributed to understanding many experiments and complex phenomena in mathematical physics. Thus, it is of interest to evaluate new soli-

tary wave solutions of these equations. So that, during the past five decades, a lot of method was discovered by a diverse group of scientists to solve the nonlinear partial differential equations. For example, the modified simple equation-method [4] [5], the $\left(\frac{G'}{G}\right)$ -expansion method [6] [7] [8] [9], the modified $\left(\frac{G'}{G}, \frac{1}{G}\right)$ -expansion method [10], the extended Jacobi elliptic function method [11] [12] [13], the $\exp(-\phi(\zeta))$ -expansion method [14] [15] [16], the modified extended $\exp(-\phi(\zeta))$ -expansion method [17], the Riccati-Bernoulli Sub-ODE method [18] [19], the exp-function method [20], the extended tanh-function method [21] [22], the Modified extended tanh-function method [23] [24], Quadratic B-Spline Galer-kin Finite Element Method [25] and so on. The purpose of this paper is to find the exact solution of Jaulent-Miodek (JM)-equation [26] [27] [28] (contain some parameters) using the two different methods mention above. If these parameters take definite values the solaitry wave solutions can be derived from it. The objective of this article is to investigate more applications than obtained in [29] [30] to justify and demonstrate the advantages of extended Jacobian elliptic function expansion method and the Riccati-Bernoulli Sub-ODE method. Here, we apply these two methods to Jaulent-Miodek (JM)-equation; firstly we apply the extended Jacobian elliptic function expansion method.

2. Extended Jacobian Elliptic Function Expansion Method [11] [12] [13]

The solution according to the extended Jacobian elliptic functions takes the form:

$$w(\zeta) = a_0 + \sum_{j=1}^N f_i^{j-1}(\zeta) [a_j f_j(\zeta) + b_j g_j(\zeta)], \quad i = 1, 2, 3, \dots \quad (1.1)$$

With

$$\begin{aligned} f_1(\zeta) &= sn(\zeta), g_1(\zeta) = cn(\zeta), \\ f_2(\zeta) &= sn(\zeta), g_2(\zeta) = dn(\zeta), \\ f_3(\zeta) &= ns(\zeta), g_3(\zeta) = cs(\zeta), \\ f_4(\zeta) &= ns(\zeta), g_4(\zeta) = ds(\zeta), \\ f_5(\zeta) &= sc(\zeta), g_5(\zeta) = nc(\zeta), \\ f_6(\zeta) &= sd(\zeta), g_6(\zeta) = nd(\zeta), \end{aligned} \quad (1.2)$$

where $sn(\zeta), cn(\zeta)$ and $dn(\zeta)$ are respectively the Jacobian elliptic sine function, the Jacobian elliptic cosine function and the Jacobian elliptic function of the third kind and the other Jacobi functions which is denoted by Glaisher's symbols and are generated by these three kinds of functions, namely

$$\begin{aligned} ns(\zeta) &= 1/sn(\zeta), nc(\zeta) = 1/cn(\zeta), nd(\zeta) = 1/dn(\zeta), sc(\zeta) = cn(\zeta)/sn(\zeta) \\ cs(\zeta) &= sn(\zeta)/cn(\zeta), ds(\zeta) = dn(\zeta)/sn(\zeta), sd(\zeta) = sn(\zeta)/dn(\zeta) \end{aligned} \quad (1.3)$$

That has the relations

$$\begin{aligned} sn^2(\zeta) + cn^2(\zeta) &= 1, dn^2(\zeta) + m^2 sn^2(\zeta) = 1, ns^2(\zeta) = 1 + cs^2(\zeta), \\ ns^2(\zeta) &= m^2 + ds^2(\zeta), sc^2(\zeta) + 1 = nc^2(\zeta), m^2 sd^2(\zeta) + 1 = nd^2(\zeta) \end{aligned} \tag{1.4}$$

with modulus m where $0 < m < 1$.

In addition we know that

$$\frac{d}{d\zeta}(sn(\zeta)) = cndn, \frac{d}{d\zeta}(cn(\zeta)) = -sncn \text{ and } \frac{d}{d\zeta}(dn(\zeta)) = -m^2 sn \tag{1.5}$$

The derivatives of other Jacobi elliptic functions are obtained by using Equation (1.5).

According to the balance rules, we can balance the highest order derivative term and nonlinear term so that N in Equation (1.1) can be determined. In addition we see that when $m \Rightarrow 1$, $sn(\zeta)$, $cn(\zeta)$, and $dn(\zeta)$ degenerate as $\tanh \tanh \zeta$, $\text{sech} \zeta$ and $\text{cosech} \zeta$ respectively, while when $m = 0$, it will be $\tan \zeta$, $\sec \zeta$ and $\text{cosec} \zeta$, therefore Equation (1.1) degenerate as the following forms

$$\begin{aligned} w(\zeta) &= a_0 + \sum_{j=1}^N \tanh^{j-1}(\zeta) [a_j \tanh(\zeta) + b_j \text{sech}(\zeta)] \\ w(\zeta) &= a_0 + \sum_{j=1}^N \coth^{j-1}(\zeta) [a_j \coth(\zeta) + b_j \coth(\zeta)] \\ w(\zeta) &= a_0 + \sum_{j=1}^N \tan^{j-1}(\zeta) [a_j \tan(\zeta) + b_j \sec(\zeta)] \\ w(\zeta) &= a_0 + \sum_{j=1}^N \cot^{j-1}(\zeta) [a_j \cot(\zeta) + b_j \csc(\zeta)] \end{aligned} \tag{1.6}$$

Therefore the extended Jacobian elliptic function expansion method is more general than sine-cosine method, the tan function method and Jacobi elliptic function expansion method.

Application

Here, we will apply extended Jacobian elliptic function expansion method described in Section 2 to find the exact traveling wave solutions and then the solitary wave solutions for the Jaulent-Miodek (JM)-equation.

Consider the Jaulent-Miodek (JM)-equation [26] [27] [28] [29]

$$u_{,xt} + \frac{1}{4}u_{,xxx} - \frac{3}{2}u_x^2 u_{,xx} + \frac{3}{4}u_{,xx} u_y = 0 \tag{1.7}$$

Let us preceding the transformation $\zeta = x + y - ct$, substitute at Equation (1.7) we get

$$-cu'' + \frac{1}{4}u''' - \frac{3}{2}u'^2 u'' + \frac{3}{4}u'' u' = 0 \tag{1.8}$$

Let $w = u' \Rightarrow w' = u''$, $w'' = u'''$ and $w''' = u''''$, then Equation (1.8) became

$$-cw' + \frac{1}{4}w''' - \frac{3}{2}w^2 w' + \frac{3}{4}ww' = 0 \tag{1.9}$$

Integrating (1.9) once, we get

$$-cw + \frac{1}{4}w'' - \frac{1}{2}w^3 + \frac{3}{8}w^2 = 0 \tag{1.10}$$

Balancing the nonlinear term with the highest order derivative term, we find $m + 2 = 3m \Rightarrow m = 1$.

Consequently, according to the constructed method by taking the first pair of the elliptic function, the solution is

$$w(\zeta) = a_0 + a_1 sn(\zeta) + b_1 cn(\zeta), \tag{1.11}$$

$$\begin{aligned} w'(\zeta) &= a_1 cn(\zeta) dn(\zeta) - b_1 sn(\zeta) dn(\zeta) \\ w''(\zeta) &= -a_1 sn(\zeta) dn^2(\zeta) - a_1 m^2 sn(\zeta) cn^2(\zeta) \\ &\quad - b_1 cn(\zeta) dn^2(\zeta) + b_1 m^2 cn(\zeta) sn^2(\zeta) \end{aligned} \tag{1.12}$$

Substitute about w, w^2, w^3 and w'' at Equation (1.10) and equating all the coefficients of $sn^3, cnsn, sn^2, sn, cn, cnsn^2, sn^0$ to zero, we get this system of algebraic equations

$$\begin{aligned} \frac{1}{2}a_1 m^2 + \frac{3a_1 b_1^2}{2} - \frac{1}{2}a_1^3 &= 0 \\ \frac{1}{2}b_1 m^2 + -\frac{3b_1 a_1^2}{2} + \frac{1}{2}b_1^3 &= 0 \\ -\frac{3a_0 a_1^2}{2} + \frac{3a_0 b_1^2}{2} + \frac{3a_1^2}{8} - \frac{3b_1^2}{8} &= 0 \\ -ca_1 - \frac{1}{4}a_1 - \frac{1}{4}a_1 m^2 - \frac{3a_1 a_0^2}{2} - \frac{3a_1 b_1^2}{2} + \frac{3}{4}a_1 a_0 &= 0 \\ -3a_1 a_0 b_1 + \frac{3b_1 a_1}{4} &= 0 \\ -cb_1 - \frac{1}{4}b_1 - \frac{3b_1 a_0^2}{2} - \frac{b_1^3}{2} + \frac{3}{4}b_1 a_0 &= 0 \\ -ca_0 - \frac{3a_0 b_1^2}{2} - \frac{a_0^3}{2} + \frac{3a_0^2}{8} + \frac{3b_1^2}{8} &= 0 \end{aligned} \tag{1.13}$$

Solving this system of algebraic equations by Maple, we get the following results:

$$a_1 = \pm m, b_1 = 0, a_0 = \frac{1}{4}, c = \frac{1}{16} \tag{1.14}$$

So that the exact solution of Equation (1.10)

$$w(\zeta) = \frac{1}{4} \pm m sn(\zeta) \tag{1.15}$$

Now, if $m \rightarrow 1$ we can obtain the hyperbolic solution (**Figure 1**):

$$w(\zeta) = \frac{1}{4} \pm \tanh(\zeta) \tag{1.16}$$

Repeating, these previous work for the remaining pairs of the Jacobian elliptic functions, we can obtained the other six different new solutions.

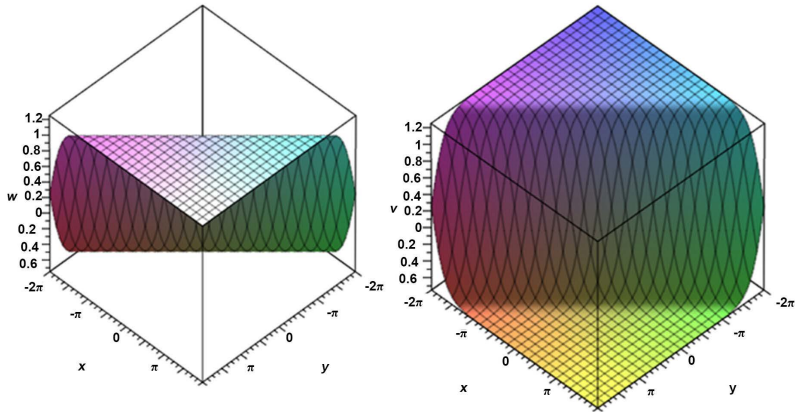


Figure 1. The plot of Equation (1.16) in three dimensions with values:

$$a_1 = 1, b_1 = 0, a_0 = \frac{1}{4}, c = \frac{1}{16}.$$

3. The Riccati-Bernoulli Sub-ODE Method [18] [19]

According to the Riccati-Bernoulli Sub-ODE method the suggested solution is

$$u' = au^{2-m} + bu + cu^m, \tag{2.1}$$

where a, b, c and m are constants to be determined later. It is important to know that when $ac \neq 0$ and $m = 0$, Equation (2.1) is a Riccati equation. When $a \neq 0, c = 0$ and $m \neq 1$, Equation (2.1) is a Bernoulli equation.

Differentiate (2.1) once we get

$$u'' = ab(3-m)u^{2-m} + a^2(2-m)u^{3-2m} + mc^2u^{2m-1} + bc(m+1)u^m + (2ac + b^2)u. \tag{2.2}$$

Substituting the derivatives of u into Equation (2.1) yields an algebraic equation of u , by considering the symmetry of the right-hand item of Equation (2.1) and setting equivalence for the highest power exponents of u we can determine m . Comparing the coefficients of u^i yields a set of algebraic equations for a, b, c and λ which solving to get a, b, c, λ .

According to the obtained values of these constants and using the transformation $\zeta = x + y - \lambda t$ the Riccati-Bernoulli Sub-ODE equation admits the following solutions:

1) When $m = 1$, the solution of Equation (2.1) is

$$u(\zeta) = C_1 e^{(a+b+c)\zeta} \tag{2.3}$$

2) When $m \neq 1, b = 0$ and $c = 0$, the solution of Equation (2.1) is

$$u(\zeta) = (a(m-1)(\zeta + C_1))^{1/(1-m)} \tag{2.4}$$

3) When $m \neq 1, b \neq 0$ and $c = 0$, the solution of Equation (2.1) is

$$u(\zeta) = \left(-\frac{a}{b} + C_1 e^{b(m-1)\zeta} \right)^{1/(m-1)} \tag{2.5}$$

4) When $m \neq 1, a \neq 0$ and $b^2 - 4ac < 0$, the solution of Equation (2.1) is

$$u(\zeta) = \left(\frac{-b}{2a} + \frac{\sqrt{4ac - b^2}}{2a} \tan \left(\frac{(1-m)\sqrt{4ac - b^2}}{2} (\zeta + C_1) \right) \right)^{\frac{1}{(1-m)}} \quad (2.6)$$

and

$$u(\zeta) = \left(\frac{-b}{2a} + \frac{\sqrt{4ac - b^2}}{2a} \cot \left(\frac{(1-m)\sqrt{4ac - b^2}}{2} (\zeta + C_1) \right) \right)^{\frac{1}{(1-m)}} \quad (2.7)$$

5) When $m \neq 1, a \neq 0$ and $b^2 - 4ac > 0$, the solution of Equation (15) is

$$u(\zeta) = \left(\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \coth \left(\frac{(1-m)\sqrt{b^2 - 4ac}}{2} (\zeta + C_1) \right) \right)^{\frac{1}{(1-m)}} \quad (2.8)$$

and

$$u(\zeta) = \left(\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \tanh \left(\frac{(1-m)\sqrt{b^2 - 4ac}}{2} (\zeta + C_1) \right) \right)^{\frac{1}{(1-m)}} \quad (2.9)$$

6) When $m \neq 1, a \neq 0$ and $b^2 - 4ac = 0$ the solution of Equation (15) is

$$u(\zeta) = \left(\frac{1}{a(m-1)(\zeta + C_1)} - \frac{b}{2a} \right)^{\frac{1}{(1-m)}} \quad (2.10)$$

where C_1 is an arbitrary constant.

Application

Now, we apply this method for solving the Jaulent-Miodek (JM)-equation (1.10) mention above,

$$-c_1 w + \frac{1}{4} w'' - \frac{1}{2} w^3 + \frac{3}{8} w^2 = 0 \quad (2.11)$$

According to the Riccati-Bernolli Sub-ODE method,

$$w' = aw^{2-m} + bw + cw^m \quad (2.12)$$

Differentiate once we get,

$$w'' = ab(3-m)w^{2-m} + a^2(2-m)w^{3-2m} + mc^2w^{2m-1} + bc(m+1)w^m + (2ac - b^2)w \quad (2.13)$$

Substitute about w'' at Equation (2.12) and by a suitable choose of m and equating the coefficients of different power of w to zero, we get this system of algebraic equations,

$$\begin{aligned} a^2 - 1 &= 0 \\ 2ab + 1 &= 0 \\ -4c_1 + 2ac - 2b^2 &= 0 \\ 2bc &= 0 \end{aligned} \quad (2.14)$$

Solving this system by Maple, we get $a = \pm 1, b = \frac{-1}{2}, c_1 = -\frac{1}{4}, c = 0$

According to this solution and the constructed method we take only the two cases (3) and (5).

Case (3): when $m \neq 1, b \neq 0$ and $c = 0$:

1) When $a > 0$, $\left(a = 1, b = \frac{-1}{2}, c_1 = -\frac{1}{4}, c = 0 \right)$ the solution is (**Figure 2**):

$$w(x, y, t) = \left(2 + e^{\frac{1}{2}(x+y-2t)} \right)^{-1} \tag{2.15}$$

2) When $a < 0$, $\left(a = -1, b = \frac{1}{2}, c_1 = -\frac{1}{4}, c = 0 \right)$ the solution is (**Figure 3**):

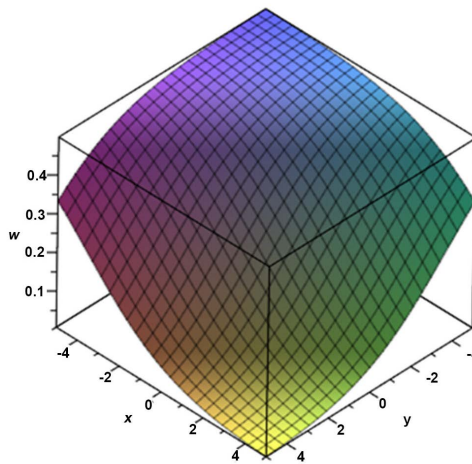


Figure 2. The plot of Equation (2.15) in three dimensions with values:

$$a = 1, b = \frac{-1}{2}, c_1 = -\frac{1}{4}, c = 0 .$$

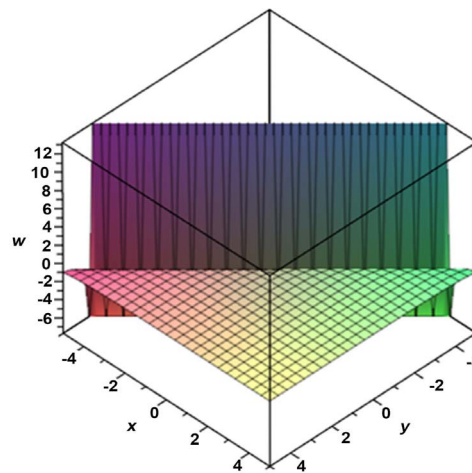


Figure 3. The plot of Equation (2.16) in three dimensions with values:

$$a = -1, b = \frac{1}{2}, c_1 = -\frac{1}{4}, c = 0 .$$

$$w(x, y, t) = \left(-2 + e^{-\frac{1}{2}(x+y-2t)} \right)^{-1} \tag{2.16}$$

Case (5): when $m \neq 1, a \neq 0$ and $b^2 - 4ac > 0$.

1) When $a = -1, b = \frac{1}{2}, c_1 = -\frac{1}{4}, c = 0, b^2 - 4ac > 0$ the solution is (**Figure 4**):

$$w(x, y, t) = \frac{1}{4} - \frac{1}{4} \coth \left[\frac{1}{4}(x + y - 2t) + 1 \right] \tag{2.17}$$

Or (**Figure 5**):

$$w(x, y, t) = \frac{1}{4} - \frac{1}{4} \tanh \left[\frac{1}{4}(x + y - 2t) + 1 \right] \tag{2.18}$$

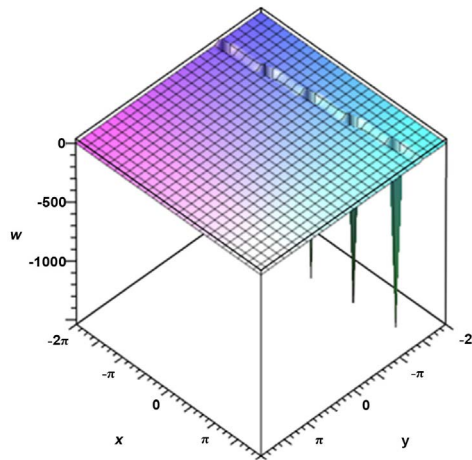


Figure 4. The plot of Equation (2.17) in three dimensions with values:

$$a = -1, b = \frac{1}{2}, c_1 = -\frac{1}{4}, c = 0.$$

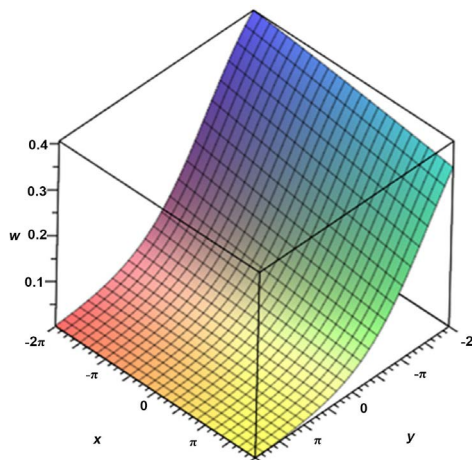


Figure 5. The plot of Equation (2.18) in three dimensions with values:

$$a = -1, b = \frac{1}{2}, c_1 = -\frac{1}{4}, c = 0.$$

4. Conclusion

The extended Jacobian elliptic function expansion method (which depends on the balance rule) has been used successfully to find the exact traveling wave solutions of the Jaulent-Miodek (JM) model. Also, the non-balanced Riccati-Bernoulli Sub-ODE method which is a powerful technique does not depend on the balance rule has been applied to get the solitary wave solution of this model. The comparison between our obtained results in this article with that obtained previously is included. It can be concluded that these two methods are reliable, effective, reduce the volume of calculations and can be applied to many other nonlinear evolution equations.

5. Captions, Citations and Physical Meaning of the Figures

We can briefly give the captions and citations for the obtained figures instead of the corresponding physical meaning. Let us start from **Figure 1** which represents the plot of the solitary solution Equation (1.16) in three dimensions according to the values of the obtained parameters realized for the first pairs of the extended Jacobi elliptic function which is hyperbolic one. Furthermore, six new different solitary solutions are realized using the other remaining pairs of the Jacobian elliptic functions. **Figure 2** and **Figure 3** denotes to the plot of the solitary solutions Equation (2.15), Equation (2.16) respectively in three dimensions according to the values of the obtained parameters realized by the Riccati-Bernoulli Sub-ODE method in the interval ($x = -4:4$, $t = -4:4$ and $y = -4:4$) which are periodic singular dark soliton, while **Figure 4** denotes to the plot of the solitary solution Equation (2.17) in the interval ($x = -2:2$, $t = -2:2$ and $y = -2:2$) which is kink shape. Furthermore, **Figure 5** denotes to the plot of the solitary solution Equation (2.18) in the interval ($x = -2:2$, $t = -2:2$ and $y = -2:2$) which is hyperbolic soliton solution.

Conflicts of Interest

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. The author did not have any competing interests in this research.

Authors' Contributions

All parts contained in the research carried out by the researcher through hard work and a review of the various references and contributions in the field of mathematics and the physical Applied.

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