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Explicit Equimodular Curves for Prism-Graph Chromatic Polynomials and the Beraha-Kahane-Weiss Limit Set

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Abstract

For the prism (cyclic ladder) graphs $G_n = C_n \square P_2$, the chromatic polynomial admits a four-branch transfer-matrix expansion $P(G_n, z) = \sum_{j=0}^{3} \alpha_j(z)\lambda_j(z)^n$, $\lambda_0(z) = z^2 - 3z + 3$, $\lambda_1(z) = 1 - z$, $\lambda_2(z) = 3 - z$, $\lambda_3(z) = 1$, with explicit polynomial amplitudes $\alpha_j(z)$. By the Beraha-Kahane-Weiss mechanism, accumulation of chromatic roots as $n \to \infty$ is confined to loci where two or more dominant eigenvalues tie in modulus, together with isolated points arising from vanishing dominant amplitudes. We give a complete real-algebraic description of these modulus-tie sets for the prism family, including closed-form Cartesian quartic equations for the quadratic-linear balances $|z^2 - 3z + 3| = |z - 1|$ and $|z^2 - 3z + 3| = |z - 3|$. These identities replace plot-based equimodular boundaries with verifiable equations and allow direct symbolic certification of the dominance inequalities governing the BKW accumulation arcs. For comparison, we also recall the cycle family C_n, whose nontrivial chromatic roots lie on the circle $|z - 1| = 1$ and are uniformly distributed in angle.

Keywords

Chromatic Polynomial, Chromatic Roots, Prism Graphs, Cyclic Ladders, Beraha-Kahane-Weiss Theorem, Equimodular Curves, Transfer Matrix, Graph Families, Real-Algebraic Curves

1. Introduction

For a finite simple graph $G = (V, E)$, the chromatic polynomial $P(G, z)$ counts proper vertex colorings when z is a nonnegative integer and extends uniquely to a monic polynomial of degree $|V|$ with integer coefficients [1]-[3]. Its complex zeros, the *chromatic roots*, encode subtle structural information about G and have been studied extensively from combinatorial, analytic, and statistical-mechanical perspectives. A central theme is to describe the geometry and asymptotic distribution of these roots for natural recursive families of graphs.

Many such families $\{G_n\}$ admit a finite transfer-matrix representation

$$P(G_n, z) = \sum_{j=1}^{m} \alpha_j(z) \lambda_j(z)^n. \quad (1)$$

In this setting, the Beraha-Kahane-Weiss (BKW) theorem [4] governs the limiting behavior of chromatic roots as $n \to \infty$: accumulation is confined to loci where two or more eigenvalues tie in modulus while dominating all others, together with isolated points arising from the vanishing of a dominant amplitude. Thus, the limiting root set is determined by explicit modulus-tie equations supplemented by dominance inequalities. While such boundaries are often explored numerically, fully explicit real-algebraic descriptions remain rare.

In this paper, we provide an explicit real-algebraic verification of the equimodular geometry for the prism (cyclic ladder) graphs $G_n = C_n \square P_2$. Beyond the standard transfer-matrix eigen-branch expansion, we derive closed Cartesian equations for the nontrivial modulus ties involving the quadratic branch $\lambda_0(z) = z^2 - 3z + 3$, including the quartic loci $|\lambda_0(z)| = |z-1|$ and $|\lambda_0(z)| = |z-3|$, and we formulate the limiting set as a union of explicitly defined dominant subsets of these ties, together with isolated amplitude-zero points.

The prism family $G_n = C_n \square P_2$, $n \geq 3$, is the smallest natural example in which multibranch competition occurs. Using the transfer-matrix framework together with the symmetric-group reduction introduced by Biggs [5] [6], we obtain an explicit four-branch expansion of the form (1) and determine all associated modulus-tie sets. The distinguishing feature is a closed Cartesian description of the quadratic-linear equimodular loci, which appear as real-algebraic curves of degree four alongside the remaining elementary ties. These formulas replace plot-based boundaries with exact algebraic conditions and permit direct, symbolic verification of the dominance relations required by the BKW mechanism.

Viewed in parallel, cycles and prisms form complementary test cases. Cycles exhibit a single dominant branch, forcing all nontrivial chromatic roots onto the circle $|z-1| = 1$, while prisms mark the transition to higher-degree real-algebraic limit sets generated by competing eigenvalues. Making this transition explicit clarifies the geometric content of the BKW framework and provides a concrete template for the analysis of more complex recursive families.

The paper is organized as follows. Section 2 briefly revisits cycles, fixing notation and recalling the explicit root parametrization and limiting distribution. Section 3 develops the transfer-matrix formulation for prisms, derives the four-branch eigenvalue expansion, and gives a concrete real-algebraic description of the associated BKW limit set, including the quartic modulus-tie curves.

2. Cycles: Chromatic Polynomial and Explicit Roots

Cycles provide a canonical family in which the chromatic polynomial, its full root set, and the limiting root geometry admit closed forms.

They serve here as a reference case: a single dominant eigen-branch controls the asymptotic behavior, yielding an unambiguous accumulation set.

This sharply contrasts with the multibranch behavior encountered later for prisms.

Theorem 2.1 (Chromatic polynomial of cycles). *For* $n \geq 3$,

$$P(C_n, z) = (z-1)^n + (-1)^n (z-1) = (z-1)\left((z-1)^{n-1} + (-1)^n\right). \tag{2}$$

Proof. The identity follows by any standard method.

For instance, deletion-contraction on an edge $e \in E(C_n)$ gives

$$P(C_n, z) = z(z-1)^{n-1} - P(C_{n-1}, z),$$

and since $P(C_3, z) = z(z-1)(z-2)$ satisfies (2), induction completes the argument.

Equivalent derivations via endpoint constraints on paths or adjacency-matrix traces lead to the same polynomial identity and are omitted.

□

With the closed form fixed, the root structure follows immediately.

Theorem 2.2 (Explicit chromatic roots of cycles). *Let* $n \geq 3$ *and set* $m = n-1$. *The roots of* $P(C_n, z)$ *consist of* $z = 1$ *together with the* m *solutions of* $(z-1)^m = (-1)^{n+1}$, *given explicitly by*

$$z_k = 1 + \eta \omega_m^k, \; k = 0, 1, \cdots, m-1, \tag{3}$$

where $\omega_m = e^{2\pi i/m}$ and

$$\eta = \begin{cases} 1, & n \text{ odd}, \\ e^{i\pi/m}, & n \text{ even}. \end{cases}$$

Proof. This is immediate from the factorization in Theorem 2.1. □

Corollary 2.3 (Root locus). *All nontrivial chromatic roots of* C_n *lie on the circle*

$$\Gamma = \{ z \in \mathbb{C} : |z-1| = 1 \}.$$

Proof. Equation (3) gives $|z_k - 1| = 1$. □

Figure 1 illustrates this configuration for representative values of n. As n increases, the nontrivial roots form a rotated regular $(n-1)$-gon on Γ and converge to a uniform angular distribution.

The resulting geometry is rigid and fully determined by a single modulus constraint.

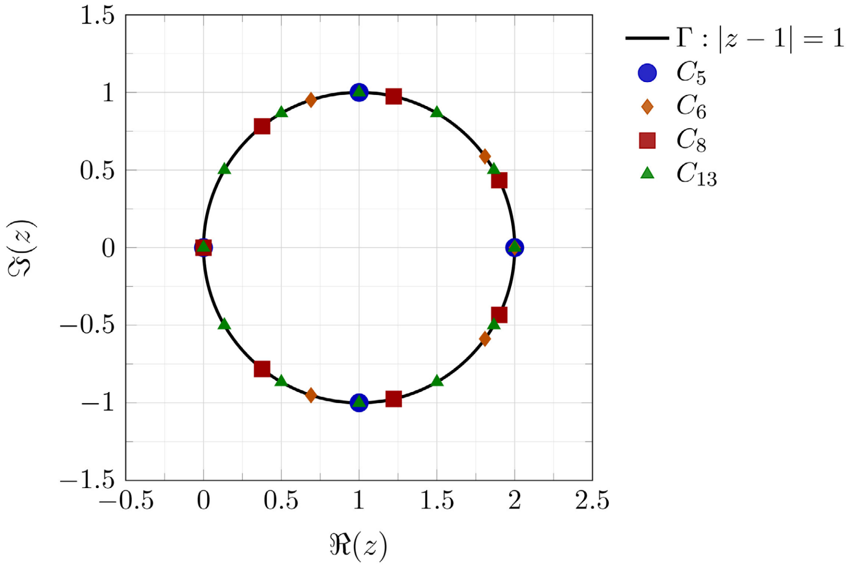

Figure 1. Chromatic roots of C_n for $n \in \{5,6,8,13\}$. All nontrivial roots lie on the circle $|z-1|=1$ and become uniformly distributed as n grows.

2.1. Cycle: Geometry of the Roots

We retain the indexing from Theorem 2.2.

Corollary 2.4 (Real and imaginary parts). *Let* $n \ge 3$ *and set* $m = n-1$. *For each nontrivial chromatic root* $z_k = a_k + \mathrm{i}b_k$,

$$a_k = 1+\cos\left(\frac{\pi(2k+\delta)}{m}\right),\quad b_k = \sin\left(\frac{\pi(2k+\delta)}{m}\right), \tag{4}$$

where $\delta = 1$ for even n and $\delta = 0$ for odd n.

Proof. Writing $z_k - 1 = \mathrm{e}^{\mathrm{i}\theta_k}$ with $\theta_k = \pi(2k+\delta)/m$, Euler's formula yields the result. □

Thus, the translated roots $z_k - 1$ are equally spaced on the unit circle, with constant angular increment $2\pi/m$.

Corollary 2.5 (Real roots and symmetry). *The nontrivial chromatic roots of* C_n *occur in complex conjugate pairs. They include* 0 *for all* $n \ge 3$, *include* 2 *if and only if* n *is odd, and admit no other real values.*

Proof. Real roots correspond to $\mathrm{e}^{\mathrm{i}\theta_k} \in \{\pm 1\}$. The value −1 occurs for all m, while +1 occurs precisely when n is odd. □

Remark 2.6 *All nontrivial chromatic roots of* C_n *lie on* Γ *and hence in the rectangle* $0 \le \Re(z) \le 2$, $|\Im(z)| \le 1$, *with extremal points* 0, 2, *and* $1 \pm i$.

This rigid geometry reflects the presence of a single dominant eigenvalue branch.

Beyond exact location, the roots of C_n admit a simple asymptotic description: as $n \to \infty$ they become uniformly distributed on Γ, and the associated logarithmic potential is elementary. Since these facts are classical and not used later, we record them briefly for context.

2.2. Cycles: Equidistribution and Logarithmic Potential

The nontrivial roots of $P(C_n, z)$ form an equally spaced set on Γ, up to a parity-dependent rotation. Consequently, the empirical root measures converge weakly to the uniform probability measure on Γ.

Theorem 2.7 (Equidistribution). *The measures* $\mu_n = \frac{1}{n}\sum_{z\in\mathcal{R}_n}\delta_z$ *converge weakly to the uniform measure* σ *on* Γ.

Proof. This follows directly from the explicit parametrization $z_k = 1+\eta\omega_{n-1}^k$ and standard Riemann-sum convergence. □

Lemma 2.8 (Logarithmic potential). *For all* $z \in \mathbb{C}$,

$$\int_\Gamma \log|z-w|\,\mathrm{d}\sigma(w) = \log^+|z-1|.$$

Proof. By rotational symmetry, the integral reduces to a classical mean-value computation on the unit circle. □

Theorem 2.9 (Potential convergence). *For every* $z \in \mathbb{C}\setminus\{1\}$,

$$\lim_{n\to\infty}\frac{1}{n}\log|P(C_n,z)| = \log^+|z-1|.$$

Proof. The closed form $P(C_n,z) = (z-1)^n + (-1)^n(z-1)$ yields the result by a direct comparison of exponential rates in the regimes $|z-1|<1$, $|z-1|>1$, and $|z-1|=1$. □

For cycles, the BKW limit set reduces to a single circle. In more complex recursive families, multiple eigenvalue branches compete, and the limiting geometry is dictated by nontrivial modulus ties. This transition motivates the prism analysis that follows.

3. Prisms: Transfer Matrices and the Beraha-Kahane-Weiss Mechanism

3.1. Transfer-Matrix Formulation

Let $G_n = C_n \square P_2$ denote the prism (cyclic ladder) graph, viewed as n rungs joining two horizontal n-cycles. A transfer matrix arises by propagating a proper coloring rung-by-rung while maintaining distinct colors on each rung.

Definition 3.1 (Rung state space). *Fix* $q \ge 0$ *and write* $[q] = \{1,2,\cdots,q\}$. *Set*

$$\Omega_q := \{(a,b)\in[q]\times[q] : a\ne b\},\quad |\Omega_q| = q(q-1),$$

so that rung i is encoded by the ordered pair $(c(u_i), c(v_i)) \in \Omega_q$.

The horizontal edges enforce $c(u_{i+1}) \ne c(u_i)$ and $c(v_{i+1}) \ne c(v_i)$, while each rung enforces $c(u_{i+1}) \ne c(v_{i+1})$, *i.e.*, membership in Ω_q.

Definition 3.2 (Transfer matrix). *For* $q \ge 0$, *define the matrix* $M(q)$ *indexed by* Ω_q *by*

$$M(q)_{(a,b),(a',b')} = \begin{cases} 1, & a'\ne a,\ b'\ne b,\ a'\ne b', \\ 0, & \text{otherwise.}\end{cases}$$

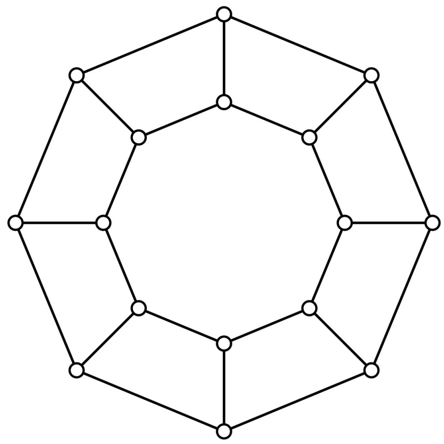

Figure 2. The prism graph $G_8 = C_8 \square P_2$ with 16 vertices and 24 edges. The graph consists of two concentric n-cycles (inner and outer) connected by n vertical edges (rungs). The Cartesian product structure $C_n \square P_2$ reflects the natural decomposition into horizontal cycles and vertical paths.

Proposition 3.3 (Trace formula). *For every* $n \ge 3$ *and every integer* $q \ge 0$,

$$P(G_n, q) = \operatorname{tr}\left(M(q)^n\right).$$

In particular, for fixed n the right-hand side is a polynomial in q agreeing with $P(G_n, q)$ on $q \in \mathbb{N}$, hence the identity extends to all $z \in \mathbb{C}$.

Proof. A proper q-coloring of G_n corresponds to a cyclic sequence of states

$$(a_1, b_1) \to (a_2, b_2) \to \cdots \to (a_n, b_n) \to (a_1, b_1),$$

where each transition satisfies $a_{i+1} \ne a_i$, $b_{i+1} \ne b_i$, and $a_{i+1} \ne b_{i+1}$. These are precisely the closed walks of length n in the directed transition graph on Ω_q with adjacency matrix $M(q)$, counted by $\operatorname{tr}\left(M(q)^n\right)$. □

The eigen-branch expansion follows from the S_q-symmetry given by relabeling colors.

Lemma 3.4 (Row-column subspace). *Fix* $q \ge 0$ *and write* $T_q := M(q)$. *Let* $\mathcal{W}_q$ *be the subspace of functions* $f : \Omega_q \to \mathbb{C}$ *of the form* $f(a,b) = g(a) + h(b)$ *with*

$$\sum_{i=1}^{q} g(i) = \sum_{i=1}^{q} h(i) = 0.$$

Then $\mathcal{W}_q$ is T_q-invariant and, for every $(a,b) \in \Omega_q$,

$$(T_q f)(a,b) = \left(-(q-2)g(a) + h(a)\right) + \left(g(b) + (2-q)h(b)\right).$$

Equivalently, for each $i \in [q]$ the pair $(g(i), h(i))$ is updated by

$$A_q = \begin{pmatrix} -(q-2) & 1 \\ 1 & 2-q \end{pmatrix},$$

whose eigenvalues are $1-q$ and $3-q$, each yielding a $(q-1)$-dimensional mean-zero eigenspace in $\mathcal{W}_q$.

Proof. Fix $(a,b)\in\Omega_q$ and write $f(a',b')=g(a')+h(b')$. Summing over admissible successors (a',b') with $a'\neq a$, $b'\neq b$, $a'\neq b'$ gives

$$(T_q f)(a,b)=\sum_{\substack{a'\neq a\\ b'\neq b\\ a'\neq b'}} g(a')+\sum_{\substack{a'\neq a\\ b'\neq b\\ a'\neq b'}} h(b').$$

For the g-sum, fix a' and count admissible b'. If $a'=b$ then b' may be any element of $[q]\setminus\{b\}$, giving $q-1$ choices; if $a'\notin\{a,b\}$ then b' must avoid $\{b,a'\}$, giving $q-2$ choices. Hence

$$\sum_{\substack{a'\neq a\\ b'\neq b\\ a'\neq b'}} g(a')=(q-1)g(b)+(q-2)\sum_{a'\neq a,b} g(a')=-(q-2)g(a)+g(b),$$

using $\sum g=0$. By symmetry, the h-sum equals $h(a)+(2-q)h(b)$, yielding the stated formula.

Finally,

$$\det(A_q-\lambda I)=(\lambda+q-2)^2-1,$$

so the eigenvalues are $1-q$ and $3-q$. Each eigendirection in $\mathbb{C}^2$, together with the mean-zero condition, contributes a $(q-1)$-dimensional eigenspace in $\mathcal{W}_q$. □

We now identify the spectrum relevant to the trace.

Theorem 3.5 (Eigenvalue expansion). *There exist four eigenvalue branches*

$$\lambda_0(z)=z^2-3z+3,\ \lambda_1(z)=1-z,\ \lambda_2(z)=3-z,\ \lambda_3(z)=1, \tag{5}$$

with polynomial amplitudes

$$\alpha_0(z)=1,\ \alpha_1(z)=z-1,\ \alpha_2(z)=z-1,\ \alpha_3(z)=z^2-3z+1, \tag{6}$$

such that for all $n\geq 3$ and all $z\in\mathbb{C}$,

$$P(G_n,z)=\sum_{j=0}^{3}\alpha_j(z)\lambda_j(z)^n. \tag{7}$$

Proof. Work first at integer inputs $z=q$ with $q\geq 4$, then extend by polynomial identity.

Fix $q\geq 4$ and set $\mathcal{V}_q=\mathbb{C}^{\Omega_q}$. Let T_q be the operator with matrix $M(q)$, so $\operatorname{tr}\left(M(q)^n\right)=\operatorname{tr}\left(T_q^n\right)$ by Proposition 3.3. The symmetric group S_q acts on Ω_q by relabeling colors and commutes with T_q, hence $\mathcal{V}_q$ decomposes into S_q-isotypic components on which T_q acts scalarly.

On the fixed line $\mathbb{C}\mathbf{1}$, each state (a,b) has $(q-1)^2-(q-2)=q^2-3q+3$ admissible successors, so $T_q\mathbf{1}=(q^2-3q+3)\mathbf{1}$ and $\lambda_0(q)=q^2-3q+3$.

The S_q-stable subspace $\mathcal{W}_q$ from Lemma 3.4 has dimension $2(q-1)$ and splits into eigenspaces with eigenvalues $\lambda_1(q)=1-q$ and $\lambda_2(q)=3-q$, each of multiplicity $q-1$.

Let $\mathcal{U}_q$ be an S_q-stable complement of $\operatorname{span}\{\mathbf{1}\}\oplus\mathcal{W}_q$. Then $\dim(\mathcal{U}_q)=q(q-1)-1-2(q-1)=q^2-3q+1$. Choosing distinct $a,b,c,d\in[q]$ and

$$\phi := e_{(a,b)} - e_{(a,c)} - e_{(d,b)} + e_{(d,c)},$$

one has $\phi \perp \mathbf{1}$ and $\phi \perp \mathcal{W}_q$, hence $\phi \in \mathcal{U}_q$. A direct transition check gives $T_q \phi = \phi$, so $\lambda_3(q) = 1$ on $\mathcal{U}_q$.

Further, the ordered-pair permutation representation of S_q on Ω_q has exactly these four isotypic constituents [6]. Therefore

$$\operatorname{tr}\left(M(q)^n\right) = \left(q^2 - 3q + 3\right)^n + (q-1)(1-q)^n + (q-1)(3-q)^n + \left(q^2 - 3q + 1\right),$$

which matches (7) upon rewriting in terms of $\lambda_j(q)$ and $\alpha_j(q)$. For fixed n, both sides are polynomials in q that agree for all integers $q \geq 4$, hence are identical; evaluating at $q = z$ gives (7) for all $z \in \mathbb{C}$. □

Remark 3.6 (Closed form and consistency checks). *Substituting* (5) *and* (6) *into* (7) *yields*

$$P(G_n, z) = \left(z^2 - 3z + 3\right)^n + (z-1)\left((1-z)^n + (3-z)^n\right) + \left(z^2 - 3z + 1\right). \tag{8}$$

In particular, $P(G_n, 0) = P(G_n, 1) = 0$ for all $n \geq 3$. Small-n expansions provide checks only; the identity (7) is forced by the trace formula together with the S_q-reduction producing the four eigen-branches.

3.2. The Beraha-Kahane-Weiss Mechanism

The representation (7) is a finite exponential sum in n. Consequently, root accumulation can occur only where at least two branches tie in modulus at the dominant scale, or at isolated points where a uniquely dominant branch has vanishing amplitude. The Beraha-Kahane-Weiss theorem makes this localization precise.

Theorem 3.7 (BKW accumulation set for prism chromatic roots). *Let* $\mathcal{L}$ *be the set of accumulation points in* $\mathbb{C}$ *of chromatic roots of the prism family* $\{G_n\}_{n \geq 3}$. *With*

$$P(G_n, z) = \sum_{j=0}^{3} \alpha_j(z) \lambda_j(z)^n$$

as in Theorem 3.5, *define*

$$\mathcal{E} := \bigcup_{0 \leq i < j \leq 3} \left\{ z \in \mathbb{C} : \left|\lambda_i(z)\right| = \left|\lambda_j(z)\right| = \max_k \left|\lambda_k(z)\right| \right\}, \tag{9}$$

$$\mathcal{A} := \left\{ z \in \mathbb{C} : \exists j \text{ with } \left|\lambda_j(z)\right| > \max_{i \neq j} \left|\lambda_i(z)\right| \text{ and } \alpha_j(z) = 0 \right\}. \tag{10}$$

Then:

1) $\mathcal{L} \subseteq \mathcal{E} \cup \mathcal{A}$.

2) If $z_0 \in \mathcal{E}$ satisfies

$$\left|\lambda_i(z_0)\right| = \left|\lambda_j(z_0)\right| > \max_{k \notin \{i,j\}} \left|\lambda_k(z_0)\right|,$$

$\alpha_i(z_0)\alpha_j(z_0) \neq 0$, and λ_i / λ_j is not locally constant near z_0, then $z_0 \in \mathcal{L}$.

For the prism branches (5), the elementary tie loci are

$$\mathcal{C}_{12} = \{z : |1-z| = |3-z|\} = \{z : \Re(z) = 2\}, \tag{11}$$

$$\mathcal{C}_{13} = \{z : |1-z| = 1\} = \{z : |z-1| = 1\}, \tag{12}$$

$$\mathcal{C}_{23} = \{z : |3-z| = 1\} = \{z : |z-3| = 1\}, \tag{13}$$

$$\mathcal{C}_{03} = \{z : |z^2 - 3z + 3| = 1\}. \tag{14}$$

The amplitude zeros are

$$\{z : \alpha_1(z) = 0\} = \{z : \alpha_2(z) = 0\} = \{1\},\ \{z : \alpha_3(z) = 0\} = \left\{\frac{3 \pm \sqrt{5}}{2}\right\}. \tag{15}$$

Thus, on any subset of a tie locus where exactly two branches dominate and the tied amplitudes are nonzero, the points belong to $\mathcal{L}$ by (ii).

Proof. (i) Fix $z_0 \in \mathbb{C}$ and assume a unique index j_0 satisfies

$$|\lambda_{j_0}(z_0)| > \max_{i \neq j_0} |\lambda_i(z_0)| \text{ and } \alpha_{j_0}(z_0) \neq 0.$$

By continuity, there exists a neighborhood U of z_0 and constants $c > 0$ and $\rho \in (0,1)$ such that $|\alpha_{j_0}(z)| \geq c$ and $|\lambda_i(z)| \leq \rho |\lambda_{j_0}(z)|$ for all $z \in U$ and $i \neq j_0$. Factoring the dominant branch gives

$$P(G_n, z) = \alpha_{j_0}(z) \lambda_{j_0}(z)^n (1 + R_n(z)),\ R_n(z) := \sum_{i \neq j_0} \frac{\alpha_i(z)}{\alpha_{j_0}(z)} \left(\frac{\lambda_i(z)}{\lambda_{j_0}(z)}\right)^n.$$

On U one has $|R_n(z)| \leq C\rho^n$ uniformly for some $C > 0$, hence $1 + R_n(z) \neq 0$ on U for all sufficiently large n. Thus z_0 cannot be an accumulation point of zeros, proving $\mathcal{L} \subseteq \mathcal{E} \cup \mathcal{A}$.

(ii) Let $z_0 \in \mathcal{E}$ with exactly two dominant branches $i \neq j$ and $\alpha_i(z_0)\alpha_j(z_0) \neq 0$. Shrinking to a neighborhood where all remaining branches are uniformly subdominant, write

$$P(G_n, z) = \lambda_j(z)^n \left(\alpha_j(z) + \alpha_i(z)\left(\frac{\lambda_i(z)}{\lambda_j(z)}\right)^n + \tilde{R}_n(z)\right),\ |\tilde{R}_n(z)| \leq C\rho^n,$$

uniformly for some $\rho \in (0,1)$. Since λ_i/λ_j is analytic and not locally constant, its argument varies in every neighborhood of z_0, and the two leading terms attain near-opposition for infinitely many n. The standard BKW argument for exponential sums then yields zeros of $P(G_n, \cdot)$ arbitrarily close to z_0 for arbitrarily large n, hence $z_0 \in \mathcal{L}$.

Finally, (11)-(14) and (15) follow by direct substitution from (5) and (6). □

We now specialize the Beraha-Kahane-Weiss description to the explicit prism eigen-branches, isolating the subsets that can contribute to the limiting chromatic root set.

Corollary 3.8 (Prism limiting set). *Let* $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ *and* $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ *be as in Theorem 3.5. For* $0 \leq i < j \leq 3$ *define*

$$\mathcal{E}_{ij} := \left\{z \in \mathbb{C} : |\lambda_i(z)| = |\lambda_j(z)| \geq \max_{k \notin \{i,j\}} |\lambda_k(z)|, \alpha_i(z)\alpha_j(z) \neq 0\right\}$$

and define

$$\mathcal{A} := \left\{ z \in \mathbb{C} : \exists j \text{ with } \left|\lambda_j(z)\right| > \max_{i \neq j} \left|\lambda_i(z)\right| \text{ and } \alpha_j(z) = 0 \right\}.$$

Then

$$\mathcal{L} \subseteq \left(\bigcup_{0 \le i < j \le 3} \mathcal{E}_{ij} \right) \cup \mathcal{A}.$$

Moreover, if $z_0 \in \mathcal{E}_{ij}$ is a point at which exactly the two branches i, j are dominant and λ_i/λ_j is not locally constant near z_0, then $z_0 \in \mathcal{L}$.

This is an immediate specialization of Theorem 3.7 to the eigenvalues (5) and amplitudes (6), with the dominance conditions made explicit.

3.3. Quartic Ties Involving the Quadratic Branch λ_0

For $G_n = C_n \square P_2$ the eigen-branch expansion in Theorem 3.5 reads

$$P(G_n, z) = \sum_{j=0}^{3} \alpha_j(z) \lambda_j(z)^n, \ \lambda_0(z) = z^2 - 3z + 3, \lambda_1(z) = 1 - z, \lambda_2(z) = 3 - z, \lambda_3(z) = 1.$$

The ties among $\lambda_1, \lambda_2, \lambda_3$ are elementary:

$$\mathcal{C}_{13} = \{|z-1| = 1\}, \ \mathcal{C}_{23} = \{|z-3| = 1\}, \ \mathcal{C}_{12} = \{\Re(z) = 2\},$$

and together with $\mathcal{C}_{03} = \{|\lambda_0(z)| = 1\}$ they account for all ties involving λ_3. The remaining balances tie λ_0 with λ_1 or λ_2 and yield quartic real-algebraic curves.

Theorem 3.9 (Quartic modulus ties for λ_0). *Let* $z = x + \mathrm{i}y$ *with* $x, y \in \mathbb{R}$, *and set*

$$\lambda_0(z) = z^2 - 3z + 3, \ \lambda_1(z) = 1 - z, \ \lambda_2(z) = 3 - z.$$

The modulus-tie sets

$$\mathcal{C}_{01} := \{z : |\lambda_0(z)| = |\lambda_1(z)|\}, \ \mathcal{C}_{02} := \{z : |\lambda_0(z)| = |\lambda_2(z)|\}$$

are quartic real-algebraic curves given by

$$\mathcal{C}_{01} : \left(x^2 - y^2 - 3x + 3\right)^2 + y^2 (2x-3)^2 = (x-1)^2 + y^2, \tag{16}$$

$$\mathcal{C}_{02} : \left(x^2 - y^2 - 3x + 3\right)^2 + y^2 (2x-3)^2 = (x-3)^2 + y^2. \tag{17}$$

Proof. Writing $z = x + \mathrm{i}y$ gives

$$\lambda_0(z) = \left(x^2 - y^2 - 3x + 3\right) + \mathrm{i}y(2x-3),$$

so

$$|\lambda_0(z)|^2 = \left(x^2 - y^2 - 3x + 3\right)^2 + y^2 (2x-3)^2.$$

Since $|\lambda_1(z)|^2 = (x-1)^2 + y^2$ and $|\lambda_2(z)|^2 = (x-3)^2 + y^2$, the equalities $|\lambda_0| = |\lambda_1|$ and $|\lambda_0| = |\lambda_2|$ are equivalent to (16) and (17). Each is polynomial in (x, y) and has total degree four. □

Corollary 3.10 (Basic properties). *The quartic curves* $\mathcal{C}_{01}$ *and* $\mathcal{C}_{02}$ *satisfy*:

1) $\mathcal{C}_{01} \cap \mathbb{R} = \{2\}$, with even intersection multiplicity;

2) $\mathcal{C}_{02} \cap \mathbb{R} = \{0, 2\}$;

3) both curves are invariant under complex conjugation.

Proof. Setting $y = 0$ in (16) gives $\left(x^2 - 3x + 3\right)^2 = \left(x - 1\right)^2$, hence $x^2 - 3x + 3 = \pm\left(x - 1\right)$. The + sign yields $\left(x - 2\right)^2 = 0$; the − sign yields $x^2 - 2x + 2 = 0$ with no real solutions. Thus $\mathcal{C}_{01} \cap \mathbb{R} = \{2\}$ with even multiplicity.

Setting $y = 0$ in (17) gives $\left(x^2 - 3x + 3\right)^2 = \left(x - 3\right)^2$, hence $x^2 - 3x + 3 = \pm\left(x - 3\right)$. The − sign yields $x\left(x - 2\right) = 0$, giving $\{0, 2\}$; the + sign has no real solutions. Both equations depend on y only via y^2, so they are invariant under conjugation. □

Remark 3.11 (Dominance on quartic ties). *On* $\mathcal{C}_{01}$ *we have* $|\lambda_0| = |\lambda_1| = |z - 1|$. *The linear comparison*

$$|z-3|^2 - |z-1|^2 = 4\left(1 - \Re\left(z\right)\right)$$

implies that for every $z \in \mathcal{C}_{01}$ with $\Re\left(z\right) \le 1$, $|z-3| \ge |z-1| = |\lambda_0|$, so λ_0 cannot be strictly dominant there. On $\mathcal{C}_{02}$ we have $|\lambda_0| = |\lambda_2| = |z-3|$, and

$$|z-1|^2 - |z-3|^2 = 4\left(\Re\left(z\right) - 2\right)$$

implies that for every $z \in \mathcal{C}_{02}$ with $\Re\left(z\right) \ge 2$, $|z-1| \ge |z-3| = |\lambda_0|$, again excluding strict dominance of λ_0 on that portion. In these regions the quartic ties cannot contribute to accumulation arcs.

Equations (16) (17) also sharpen the tie geometry beyond what contour plots alone resolve: tangencies and near-intersections can be checked algebraically, and the dominance inequalities can be verified directly without relying on numerical resolution.

4. Conclusion

Cycles and prisms illustrate a sharp transition in chromatic-root geometry, from a single dominant eigen-branch to multibranch competition governed by the Beraha-Kahane-Weiss mechanism. For cycles C_n, the explicit factorization of $P\left(C_n, z\right)$ confines all nontrivial roots to the circle $|z-1| = 1$ and yields a uniform angular limiting distribution. For prisms $G_n = C_n \square P_2$, the transfer-matrix formulation produces a four-branch eigenvalue expansion, placing the limiting root set under precise BKW control.

For the prism family, we give an explicit real-algebraic description of all equimodular boundaries. In addition to the elementary ties among the linear and constant branches, we derive closed Cartesian quartic equations for the quadratic-linear balances.

$$|z^2 - 3z + 3| = |z - 1| \text{ and } |z^2 - 3z + 3| = |z - 3|.$$

These identities replace plot-based boundaries with exact algebraic conditions and allow direct verification of the dominance inequalities that determine which portions of the tie loci contribute to the BKW accumulation set.

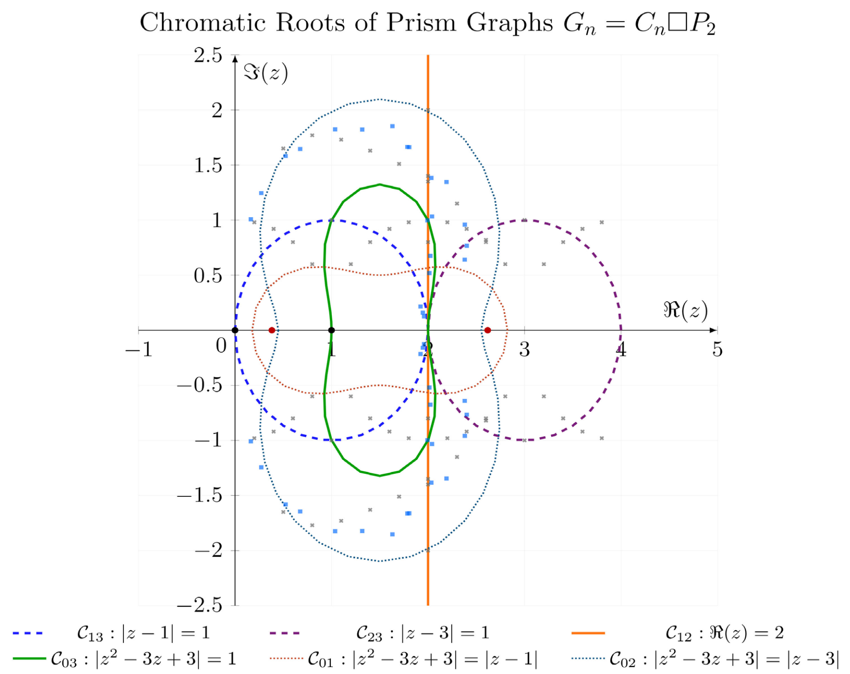

Figure 3. Modulus-tie curves for $G_n = C_n \square P_2$. Dashed curves indicate dominant BKW accumulation arcs; dotted curves are subdominant. Squares show roots for n = 6, 8, 10.

More broadly, prism graphs provide a compact setting in which the full equimodular geometry can be made explicit. The same combination of transfer-matrix reduction, algebraic elimination of modulus ties, and dominance analysis applies to wider cylindrical ladders $C_n \square P_m$ and to other recursive families, where additional eigen-branches are expected to generate richer real-algebraic limit sets [7] [8]. Extending these techniques offers a systematic route toward rigorous, computation-assisted descriptions of chromatic-root accumulation geometry beyond the cases accessible by numerical exploration alone.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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