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One-Dimensional *Forced* Standing Waves

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Abstract

This research-oriented report analyzes the impact of a class of time-independent forces and their time-dependent counterparts on the characteristics of solutions to the wave equation on a one-dimensional string. The solution of these equations, subject to Initial and Boundary Value (IBV) conditions, is investigated, yielding modified standing waves. Analyses are heavily Computer Algebra oriented. The popular Computer Algebra System (CAS), specifically *Mathematica*, has been pivotal in uncovering the numerical, algebraic, and graphical aspects of the investigation. The time-dependent force functions not only generalize the scope of the analysis but also possess 1) a character that their functionality by transitioning to the coordinate-dependent only is conducive to expected output, and 2) the absence of these forces frees the imposed IBV, conducive to the characteristics of the standard standing waves. This comprehensive report embodies an atlas of graphs for both mentioned forces. It is shown that the proposed one-dimensional time-independent forces give rise to previously unseen standing waves with a peculiar character. The report includes the essential *Mathematica* code for most calculations and all the graphs depicted. The report allows reproduction by individuals familiar with *Mathematica*. It is crafted and developed so that modification and generalization of the applied IBVs are readily possible. The Conclusion segment embodies a suggestion to expand the scope of future investigations.

Keywords

Forced Standing Waves, Time-Independent and Dependent Forces, Inhomogeneous Wave Equations, Computer Algebra System, *Mathematica*

1. Introduction

The standard one-dimensional elliptic homogeneous partial differential wave equation with constant coefficients is:

$$\frac{\partial^2}{\partial x^2}u(x,t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(x,t)=0; -\infty<x<\infty \text{ and } 0<t<\infty \tag{1}$$

with initial conditions.

$$\begin{cases} u(x,0)=f(x) \\ \frac{\partial}{\partial t}u(x,0)=g(x) \end{cases} \tag{2}$$

Equation (1), which has no boundaries, describes the motion of an infinitely long string with given initial conditions. Its solution bears the name of the excavator, D'Alambert [1] given in (3),

$$u(x,t)=\frac{1}{2}\left[f(x-ct)+f(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\xi)\,\mathrm{d}\xi \tag{3}$$

The non-dispersive solution given in (3) is interpreted as a pair of moving waves traveling in opposite directions. The c is the speed of the non-dispersive waves.

The solution (3) and its applications are analyzed in physics and math courses.

One of the objectives of this report is to generalize (1), propose examples with solutions in the domain $0\le x\le \ell$, where ℓ is the finite length of the string.

The foremost generalization of (1) entails adding a variable term to convert the homogeneous equation into an inhomogeneous equation. The secondary aim stems from the fact that, due to the restricted coordinate domain, traveling waves are reflected at the boundaries, potentially interfering to form so-called "tationary standing" waves. Addressing the impact of the non-homogeneous term on the characteristics of the standing waves concludes the report.

The generalized (1) is (4). The non-homogeneous term, $F(x,t)$, is thereafter called "Force" and bears the title of the article.

$$\frac{\partial^2}{\partial x^2}u(x,t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(x,t)=F(x,t), \text{there stricted domain is } 0\le x\le \ell. \tag{4}$$

Although different types of boundary conditions may be considered, the interest is in the one where the string is of length ℓ. It is pinned at both ends. This restricts the solutions of (4) satisfying,

$$u(0,t)=u(\ell,t)=0, \text{ for } t>0 \tag{5}$$

Compatible with (5), the string's initial shape can be controlled via the initial conditions,

$$u(x,0)=f(x) \tag{6}$$

And for the sake of argument and transparency of the calculation, we consider the case that initially the string starts off freely, with no initial speed, *i.e.*,

$$\frac{\partial}{\partial t}u(x,0)=0 \tag{7}$$

In addition, two general cases are considered. 1a) $F(x,t)$ is time-independent *i.e.* $F(x,t)=f(x)$, and 1b) $F(x,t)$ is coordinate and time-dependent separable function, *i.e.* $F(x,t) = f(x)g(t)$.

With these assumptions in Section 2, the solution of (4) is presented.

This report is composed of three sections. In addition to Section 1, Introduction, Section 2, Formulation and codes, embodies a variety of graphic atlases. Conclusions and remarks are in Section 3.

2. Formulation

By applying the "modified" Fourier series with a finite number of terms that controls the accuracy of the computation, the formal solution of (4) is,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left[\frac{n\pi}{\ell} x\right] \tag{8}$$

The Fourier series with sine bases compatible with the boundary conditions is called "modified" because it has time-dependent coefficients.

In aforementioned case Ia, $F(x,t) = f(x)$; the Fourier series expansion of the latter function is,

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left[\frac{n\pi}{\ell} x\right] \tag{9}$$

Comparing (8) and (9) shows the difference between the two mentioned series.

The orthogonality of the bases is,

$$\int_0^{\ell} \sin\left[\frac{n\pi}{\ell} x\right] \sin\left[\frac{m\pi}{\ell} x\right] \mathrm{d}x = \begin{cases} 0, & n \neq m \\ \dfrac{\ell}{2}, & n = m \end{cases} \tag{10}$$

Utilizing (10) yields the constant coefficients of (9),

$$f_n = \frac{2}{\ell} \int_0^{\ell} f(\xi) \sin\left[\frac{n\pi}{\ell} \xi\right] \mathrm{d}\xi \tag{11}$$

Inserting (8) and (9) in (4) gives,

$$\sum_{n=1}^{n=\infty} \left\{ \ddot{u}_n(t) + \left(\frac{n\pi c}{\ell}\right)^2 u_n(t) - f_n \right\} \sin\left[\frac{n\pi}{\ell} x\right] = 0 \tag{12}$$

Double super dot is the derivative with respect to time, concluding, each coefficient of $\sin\left[\frac{n\pi}{\ell} x\right]$ ought to satisfy the ODE (13).

Equation (12) is satisfied if all the coefficients of the series are zero, conducive (13),

$$\ddot{u}_n(t) + \left(\frac{n\pi c}{\ell}\right)^2 u_n(t) = f_n \tag{13}$$

Equation (13) with a constant inhomogeneous term is a standard ODE; its solution is available, e.g., [2]-[4],

$$u_n(t) = f_n \frac{\ell}{n\pi c} \int_0^t \sin\left[\frac{n\pi c}{\ell}(t-\tau)\right] \mathrm{d}\tau \tag{14}$$

Inserting (14) in (8) formally concludes the formulation segment of the report.

Because we are interested in the "forced" issue, f_n (11) calls for some calculation. Quantity $f(\xi)$ in (11) addresses the force, and it can be any physically meaningful function. In our previously published article [5], we consider a class of polynomials. For the sake of consistency, here we utilize the same, *i.e.*, $f(\xi)=\xi^m$ for m = 1, 2, 3, and 4. These functions may be replaced with functions of choice.

From this point on, we use Mathematica [6] as a CAS to perform symbolic calculations. Codes are given for reproduction and potential generalization.

Descriptive Codes

Case I. $F(x,t)=f(x)=x^m$, m = 1, 2, 3 and 4. For the sake of simplicity, the string length and the wave speed are set to unity. In the follow-up cases, the time-dependent functions are parametrized via a variable quantity α. All these quantities are stored in the **values (1,2)** lists,

```
values:={ℓ->1.,c->1.,α->0.}
values1:={ℓ->1.,c->1.,α->0.5}
values2:={ℓ->1.,c->1.,α->10.}
```

There is a one-to-one connection between the numbered equations and the codes. For instance, the corresponding code of Equation (11) is,

$$\mathbf{f1[n\_,m\_]:=\frac{2}{\ell}\int_0^{\ell} x^m\ Sin[\frac{n\pi}{\ell}x]dx\ //Simplify}$$

The m is the power of the polynomial. For m = 1, 2, 3 and 4 the f1 is tabulated,

$$\mathbf{tab1m=Table[\{m,x^m,f1[n,m]\},\{m,1,4,1\}]}$$

$$\left\{\left\{1,x,\frac{2\ell\left(-n\pi\,\mathrm{Cos}[n\pi]+\mathrm{Sin}[n\pi]\right)}{n^2\pi^2}\right\},\right.$$

$$\left\{2,x^2,\frac{2\ell^2\left(-2+\left(2-n^2\pi^2\right)\mathrm{Cos}[n\pi]+2n\pi\,\mathrm{Sin}[n\pi]\right)}{n^3\pi^3}\right\},$$

$$\left\{3,x^3,-\frac{2\ell^3\left(n\pi\left(-6+n^2\pi^2\right)\mathrm{Cos}[n\pi]-3\left(-2+n^2\pi^2\right)\mathrm{Sin}[n\pi]\right)}{n^4\pi^4}\right\},$$

$$\left.\left\{4,x^4,\frac{2\ell^4\left(24-\left(24-12n^2\pi^2+n^4\pi^4\right)\mathrm{Cos}[n\pi]+4n\pi\left(-6+n^2\pi^2\right)\mathrm{Sin}[n\pi]\right)}{n^5\pi^5}\right\}\right\}$$

Then (14) is coded,

$$\mathbf{u1[t\_,n\_]=Table[tab1m[\![m,3]\!](\frac{\ell}{n\pi c}\int_0^{t} Sin[\frac{n\pi c(t-\tau)}{\ell}]d\tau\ ),\{m,1,4\}];}$$

For instance, its 3rd element is,

```
u1[t,n]⟦3⟧
```

$$-\frac{2\ell^4\left(\ell-\ell\,\mathrm{Cos}\left[\frac{cn\pi t}{\ell}\right]\right)\left(n\pi\left(-6+n^2\pi^2\right)\mathrm{Cos}[n\pi]-3\left(-2+n^2\pi^2\right)\mathrm{Sin}[n\pi]\right)}{c^2n^6\pi^6}$$

Equation (8) for the sum of the first 5 terms is,

```
u1Sum[x_,t_]=Table[{m,Sum[(ℓ^2/nπc)u1[t,n]⟦m⟧Sin[(nπc/ℓ)x]/.values,
{n,1,5}]},{m,1,4}];
```

Its first term corresponding to x^1 is,

```
u1Sum[x,t]⟦1,2⟧;(*the output is suppressed*)
```

The coefficients of the first five terms are.

```
{(u1Sum[x,t]⟦1,2,1,1⟧,u1Sum[x,t]⟦1,2,2,1⟧,u1Sum[x,t]⟦1,2,3,1⟧,u1Sum[x,t]
⟦1,2,4,1⟧,u1Sum[x,t]⟦1,2,5,1⟧)}
```

```
{0.020531964509368672,–0.001283247781835542,0.00025348104332553913,
–0.00008020298636472138,0.000032851143214989876}
```

This shows the coefficients alternate in sign and decrease drastically. The latter justifies that the series converges and that five terms in (8) are adequate.

Indexing the terms yields additional information. The outputs are suppressed but one

```
{u1Sum[x,t]⟦1,2,2⟧,u1Sum[x,t]⟦1,2,2,1⟧};
{u1Sum[x,t]⟦1,2,3⟧,u1Sum[x,t]⟦1,2,3,1⟧};
{u1Sum[x,t]⟦1,2,4⟧,u1Sum[x,t]⟦1,2,4,1⟧}
```

```
    {–0.00008020298636472138(1.–1.Cos[12.566370614359172t])
    Sin[12.566370614359172x],–0.00008020298636472138}
```

```
{u1Sum[x,t]⟦1,2,5⟧,u1Sum[x,t]⟦1,2,5,1⟧};
```

For instance, within the desired time span, the graph of each mode can be animated, e.g., the mode associated with the 4th polynomial is

```
Manipulate[Plot[Evaluate[u1Sum[x,t]⟦1,2,4⟧],{x,0,1},PlotRange→{-0.0002,
0.0002},GridLines→Automatic,AxesLabel→{"t","4th
term"},PlotLabel→α=0],{t,0.1,4.,0.01}];
(*output is suporessed*)
cc={Blue,Red,Green,Black}
{🟦,🟥,🟩,⬛}
manip1=Manipulate[Table[Plot[Evaluate[u1Sum[x,t]⟦m,2⟧/.t->τ],{x,0,1},
PlotRange->{-0.05,0.05},PlotStyle->cc⟦m⟧(*,Filling→Axis,FillingStyle→Ma-
genta*),GridLines->Automatic,AxesLabel->{"x","u1(x,t)"},GridLines->Auto-
matic,PlotLabel->"α=0",PlotLegends->{f[x]==x^m,"Expressions"}],{m,1,4}],{τ,
0.01,10.,0.001}];
```

Finally, the animation of the collective modes corresponding to the individual polynomial terms is shown in Figure 1.

In Case 1, with the imposed initial condition, the initial force functions are x^m, all with positive amplitudes, preventing the interfering pulses from acquiring negative amplitudes, so that the stationary configuration always sustains its positivity. On the contrary, had the initial shape been sin(x), not $\sin(x)^2$, it would have resulted in positive-negative standing waves, like the traditional standing waves.

Functional behavior of u1[x,t] for four cases of interest at t=5.0s is shown in Figure 2.

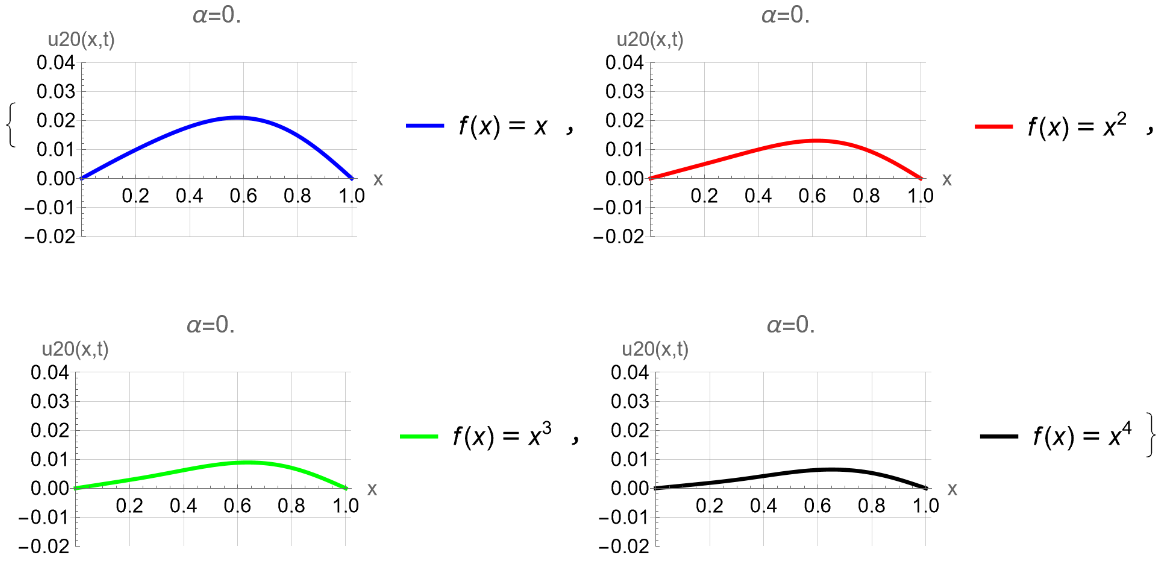


Figure 1. Profile of the vibrating string associated with the individual polynomial terms.

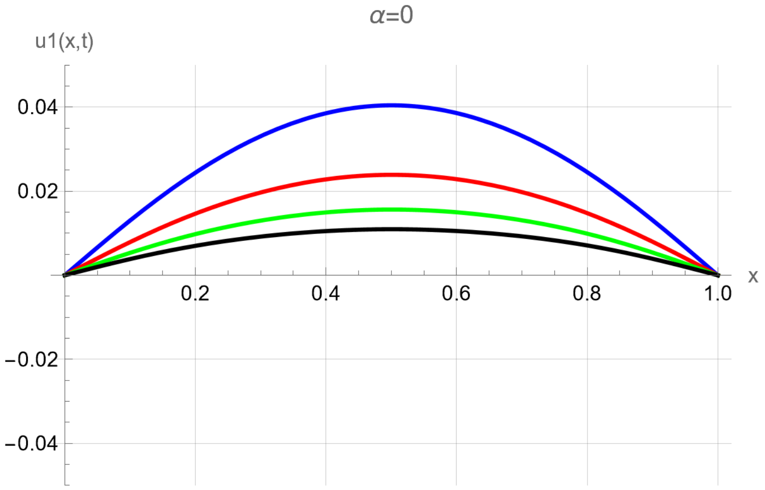


Figure 2. Collective amplitudes associated with $u_1(x,t)$. The color codes are the same as Figure 1.

An animation of the profiled vibrations shows that the string always maintains a positive amplitude; it never crosses the horizontal axis. This feature distinguishes itself from the classic standing waves.

Additional useful detailed codes for interested individuals are. Output is suppressed.

```
plot11:=Plot[Evaluate[u1Sum[x,t][[1,2]]],{x,0,1},Filling ->Axis,FillingStyle ->cc[[1]]];
plot12:=Plot[Evaluate[u1Sum[x,t][[2,2]]],{x,0,1},Filling ->Axis,FillingStyle ->cc[[2]]];
```

```
Overlay[Manipulate[{plot12,plot11},{t,(*0.01*)0.8,2.2(*,0.1*),0.01}]];
Manipulate[plot11,{t,(*0.01*)0.8,2.2(*,0.1*),0.01}];
Manipulate[Plot[{Evaluate[u1Sum[x,t][[1,2]]],Evaluate[u1Sum[x,t][[2,2]]],
Evaluate[u1Sum[x,t][[3,2]]],Evaluate[u1Sum[x,t][[4,2]]]},{x,0,1},PlotRange-> {-
0.05,0.05},PlotStyle->cc,{Filling->Axis,FillingStyle->{cc[[1]],cc[[3]]}},GridLines
->Automatic,AxesLabel->{"x","u1 (x,t)"},GridLines->Automatic],{t,(*0.01*) 0.
8,2.2(*,0.1*),0.01}];
```

Case II. The second case addresses coordinate-time dependent separable forces. For separable functions, *i.e.*, $F(x,t)=f(x)T(t)$, among a variety of choices for the time-dependent term, we consider $T(t)=\mathrm{e}^{-\alpha t}$, with $\alpha \in$ real. This allows us to control its impact. E.g., $\alpha = 0$ yields the previously studied case 1a. $\alpha = 0.5$ corresponds to a mild dependency, and $\alpha = 10$ drops it off, which is conducive to the classic free-standing waves. With these objectives, the codes are,

$$\mathbf{T[\alpha\_,t\_]:=e^{-\alpha t}(*\alpha\ should\ have\ a\ dimension\ of\ t^{-1}*)}$$

The code for (11) is,

$$\mathbf{f2[n\_,m\_,t\_]:=T[\alpha,t]\ \frac{\ell}{2}\int_0^{\ell} x^m\ Sin[\frac{n\pi}{\ell}x]dx\ //Simplify}$$

Its elements are,

```
tab2m=Table[{m,x^m,f2[n,m,t]},{m,1,4,1}]
```

$$\left\{\left\{1,x,\frac{2\mathrm{e}^{-t\alpha}\ell\left(-n\pi\,\mathrm{Cos}[n\pi]+\mathrm{Sin}[n\pi]\right)}{n^2\pi^2}\right\},\right.$$

$$\left\{2,x^2,\frac{2\mathrm{e}^{-t\alpha}\ell^2\left(-2+\left(2-n^2\pi^2\right)\mathrm{Cos}[n\pi]+2n\pi\,\mathrm{Sin}[n\pi]\right)}{n^3\pi^3}\right\},$$

$$\left\{3,x^3,-\frac{2\mathrm{e}^{-t\alpha}\ell^3\left(n\pi\left(-6+n^2\pi^2\right)\mathrm{Cos}[n\pi]-3\left(-2+n^2\pi^2\right)\mathrm{Sin}[n\pi]\right)}{n^4\pi^4}\right\},$$

$$\left.\left\{4,x^4,\frac{2\mathrm{e}^{-t\alpha}\ell^4\left(24-\left(24-12n^2\pi^2+n^4\pi^4\right)\mathrm{Cos}[n\pi]+4n\pi\left(-6+n^2\pi^2\right)\mathrm{Sin}[n\pi]\right)}{n^5\pi^5}\right\}\right\}$$

The code associated with (14) is,

$$\mathbf{u20[t\_,n\_]=Table[\ \frac{\ell}{n\pi c}\int_0^{t} Sin[\frac{n\pi c(t-\xi)}{\ell}]\ (tab2m[\![m,3]\!]/.\{t\text{->}\xi,\alpha\text{->}0.\})d\xi),\{m,1,4\}];}$$

For instance, its 4th element is,

```
u20[t,n]⟦4⟧
```

$$\frac{0.0006621873603551337\ell^6\left(-1+\mathrm{Cos}\left[\frac{cn\pi t}{\ell}\right]\right)\left(-24+\left(24-118.4352528130723n^2+n^4\pi^4\right)\mathrm{Cos}[n\pi]+\left(75.39822368615503n-124.02510672119926n^3\right)\mathrm{Sin}[n\pi]\right)}{c^2n^7}$$

Equation (8) for the sum of the first five terms is,

```
u20Sum[x,t]=Table[{γ,Sum[(ℓ^2/nπc)u20[t,n]⟦γ⟧Sin[(nπ/ℓ)x]/.values,{n,1,5}]},{γ,1,4}]//Simplify//Chop;
```

E.g., the third term of the sum associated with x^3. An element of the polynomial is given below; however, the output is suppressed.

```
u20Sum[x,t]⟦1,2⟧//Simplify//Chop;
```

Tabulated numeric coefficients of the terms of this sum are,

```
TableForm[{Table[{u1Sum[x,t]⟦1,2,n,1⟧},{n,1,5}],Table[{-u20Sum[x,t]⟦1,2,n,1⟧},{n,1,5}]},TableHeadings->{{“u1Sum~”x^m,”u20~”e^"-""0.""t"x^m},{"c1,m=1","c2,m=2","c3, m=3","c4,m=4","c5"}}]
```

Table 1. Accuracy of the two different codes.

	c1, m=1	c2, m=2	c3, m=3	c4, m=4	c5
u1Sum~x^m	0.020532	−0.00128325	0.000253481	−0.000080203	0.0000328511
u20~$e^{-\theta\cdot t}x^m$	0.020532	−0.00128325	0.000253481	−0.000080203	0.0000328511

Table 1 shows that the two rows are the same. This justifies the claim that, although two different codes are used, the second row for the specific case $\alpha = 0$ yields the result of case 1a.

Within the desired time span, the graph of each mode can be animated. For instance, the mode associated with the 1st element of the polynomial is,

```
Manipulate[Plot[Evaluate[u20Sum[x,t]⟦1,2,1⟧/.t->τ],{x,0,1},PlotRange->{-0.05,0.05}(*{-0.03,0.05}*),GridLines->Automatic,AxesLabel->{"x","u20(x,t)"}],{τ,0,2,0.1}];
```

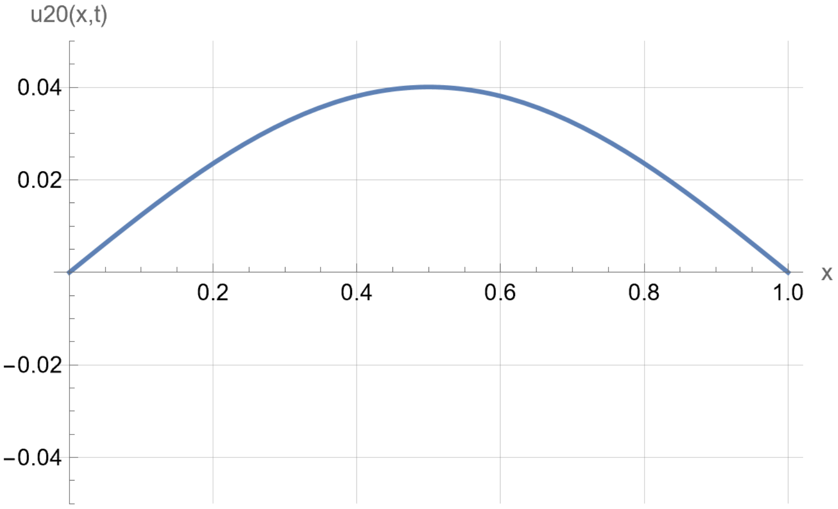


Figure 3. Display of the animation of the modes. Clockwise from the top left corner are $m = 1, 2, 3$, and 4. Amplitudes are always positive.

Next, for $\alpha = 0.5$, the code associated with (8) is modified,

u21[t_,n_]=Table[($\frac{\ell}{n\pi c}\int_0^t[\frac{n\pi c(t-\xi)}{\ell}]$(tab2m⟦m,3⟧/.{t->ξ,α->0.5})dξ),{m,1, 4}];

u21Sum[x,t]=Table[{γ,Sum[(ℓ^2/nπc)u21[t,n]⟦γ⟧Sin[(nπ/ℓ)x]/.values1,{n,1, 5}]},{γ,1,4}]//Simplify//Chop;

E.g., its fourth term is,

u21[t,n]⟦4⟧

$$\frac{\begin{gathered}2\ell^6\left(24-\left(24-12n^2\pi^2+n^4\pi^4\right)\mathrm{Cos}[n\pi]+4n\pi\left(-6+n^2\pi^2\right)\mathrm{Sin}[n\pi]\right)\\ \left(0.31830988618379064ce^{-0.5t}n-0.31830988618379064cn\,\mathrm{Cos}\left[\frac{(3.141592653589793+0\cdot i)cnt}{\ell}\right]\right.\\ \left.+0.050660591821168895l\,\mathrm{Sin}\left[\frac{(3.141592653589793+0\cdot i)cnt}{\ell}\right]\right)\end{gathered}}{cn^6\pi^6\left(1\cdot c^2n^2+0.025330295910584447\ell^2\right)}$$

u21Sum[x,t]⟦1,2⟧//Simplify//Chop;

TableForm[{Table[{u1Sum[x,t]⟦1,2,n,1⟧},{n,1,5}],Table[{u21Sum[x,t]⟦1,2, n,1⟧},{n,1,5}]},TableHeadings->{{"u1Sum~"x$^{\text{m}}$,"u21~"e$^{\text{"-""0.5""t"}}$x$^{\text{m}}$},{"c1,m=1", "c2,m =2","c3,m=3","c4,m=4","c5"}}]

Table 2. This table compares the coefficients of the terms in the Sum.

	c1, m=1	c2, m=2	c3, m=3	c4, m=4	c5
u1Sum~x^{m}	0.020532	−0.00128325	0.000253481	−0.000080203	0.0000328511
u21~e$^{-0.5t}$x^{m}	0.0629096	−0.00200304	0.0002647	−0.0000628917	0.0000206201

The first cell of the first column is the term of the polynomial of case 1a. The second cell of the first column is the time-dependent value of the previous cell for $\alpha = 0.5$. The common heading for both is the coefficients c associated with the order of the polynomial x^m for m = 1, 2, 3, 4.

As shown, both coefficients alternate in sign and progressively decrease in value.

Like Figure 3, each mode can be animated. The associated code, however, suppresses the individual output. The profile of all four cases at t=1.75 s is shown in Figure 4.

manip21=Manipulate[Table[Plot[Evaluate[u21Sum[x,t]⟦m,2⟧/.t->τ],{x,0,1}, PlotRange->{-0.02,0.05},PlotStyle->cc⟦m⟧(*,Filling→Axis,FillingStyle→Magenta*),GridLines->Automatic,AxesLabel->{"x","u21(x,t)"},GridLines->Automatic,PlotLabel->"α=0.5",PlotLegends->{f[x]==e$^{-0.5t}$x$^{\text{m}}$,"Expressions"}],{m, 1,4}],{τ,0.,2.}];

An atlas of the general animation is formed,

The impact of the time-dependent component of the applied force is evident. The animation shows that the amplitude becomes negative, and it crosses the horizontal axis.

And, finally, for $\alpha = 10$. The corresponding code to (8) is,

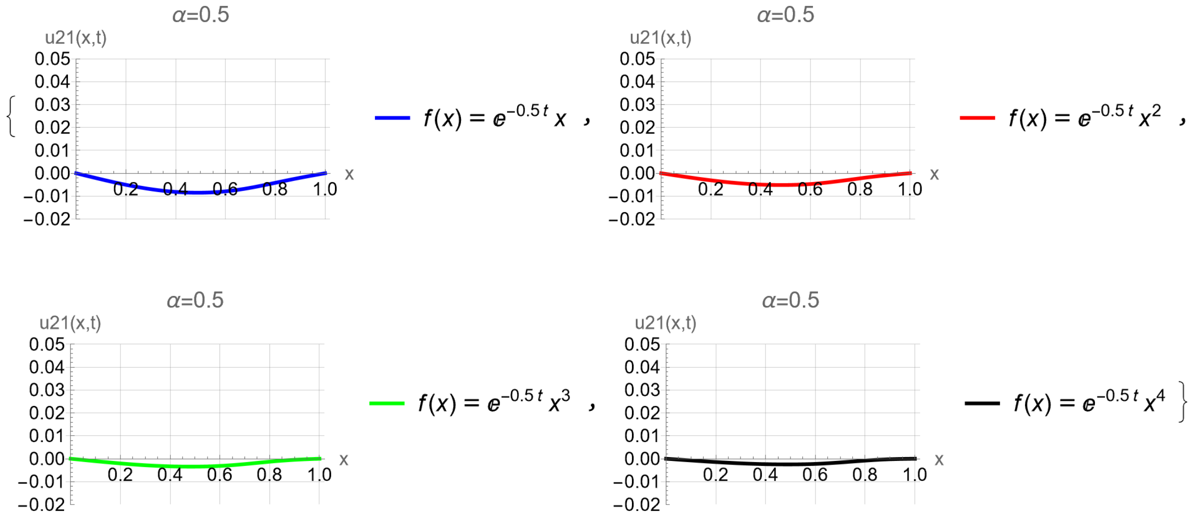


Figure 4. Similar description as **Figure 3**.

$$\mathbf{u22[t\_,n\_]=Table[(}\ \frac{\ell}{n\pi c}\int_0^t[\frac{n\pi c(t-\xi)}{\ell}]\ \mathbf{(tab2m[\![m,3]\!]/.\{t\text{->}\xi,\alpha\text{->}10.\})d\xi),\{m,1,4\}]}$$

The previous code associated with (8) is modified,

```
u22Sum[x,t]=Chop[Simplify[Table[{γ,Sum[(ℓ^2/(n*Pi*c))*u22[t,n][[γ]]*Sin
[((n*Pi)/ℓ)*x]/.values2,{n,1,5}]},{γ,1,4}]]];
```

Its 4th term is,

```
u22[t,n]⟦4⟧
```

$$\frac{\begin{array}{l}2\ell^6\left(24-\left(24-12n^2\pi^2+n^4\pi^4\right)\text{Cos}[n\pi]+4n\pi\left(-6+n^2\pi^2\right)\text{Sin}[n\pi]\right)\\ \left(0.3183098861837907ce^{-10t}n-0.3183098861837907cn\,\text{Cos}\left[\frac{(3.141592653589793+0\cdot i)cnt}{\ell}\right]\right.\\ \left.+1.013211836423378l\,\text{Sin}\left[\frac{(3.141592653589793+0\cdot i)cnt}{\ell}\right]\right)\end{array}}{cn^6\pi^6\left(1\cdot c^2n^2+10.13211836423378\ell^2\right)}$$

A sample of its indexed terms and the corresponding numeric coefficients is tabulated.

```
u22Sum[x,t]⟦1,2⟧//Simplify//Chop
```

$$\begin{array}{l}\left(0.0018443897053175005\text{e}^{-10t}-0.0018443897053175005\,\text{Cos}[3.141592653589793t]\right.\\ \left.+\,0.005870874771781689\,\text{Sin}[3.141592653589793t]\right)\text{Sin}[3.141592653589793x]\\ +\left(-0.0003632145581467093\text{e}^{-10t}+0.0003632145581467093\,\text{Cos}[6.283185307179586t]\right.\\ \left.-\,0.0005780739233198745\,\text{Sin}[6.283185307179586t]\right)\text{Sin}[6.283185307179586x]\\ +\,0.0001192408151830508\text{e}^{-10t}\,\text{Sin}[9.42477796076938x]\\ -\,0.0001192408151830508\,\text{Cos}[9.42477796076938t]\text{Sin}[9.42477796076938x]\\ +\,0.0001265184343645977\,5\text{Sin}[9.42477796076938t]\text{Sin}[9.42477796076938x]\\ -\,0.00004910615220509193\,5\text{e}^{-10t}\,\text{Sin}[12.566370614359172x]\\ +\,0.00004910615220509193\,5\text{Cos}[12.566370614359172t]\text{Sin}[12.566370614359172x]\end{array}$$

$$
\begin{aligned}
&-0.0000390774342983168\,\mathrm{Sin}[12.566370614359172t]\,\mathrm{Sin}[12.566370614359172x]\\
&+0.0000233768590854700 3e^{-10t}\,\mathrm{Sin}[15.707963267948966x]\\
&-0.00002337685908547003\,\mathrm{Cos}[15.707963267948966t]\,\mathrm{Sin}[15.707963267948966x]\\
&+0.0000148821707096609 6\,\mathrm{Sin}[15.707963267948966t]\,\mathrm{Sin}[15.707963267948966x]
\end{aligned}
$$

```
Table Form[{Table[u1Sum[x,t]⟦1,2,n,1⟧,{n,1,5}],Table[u21Sum[x,t]⟦1,2,n,1⟧,{n,1,5}],{u22Sum[x,t]⟦1,2,1,1,1,1⟧,u22Sum[x,t]⟦1,2,2,1,1,1⟧,u22Sum[x,t]⟦1,2,3,1⟧,u22Sum[x,t]⟦1,2,6,1⟧,u22Sum[x,t]⟦1,2,9,1⟧}},TableHeadings->{{"u1Sum~"x^m,"u21~"e^"-""0.5""t"x^m,"u22~"e^"""10""t"x^m},{"c1,m=1","c2,m=2","c3,m=3","c4,m=4","c5"}}]
```

Table 3. A comprehensive numeric table of the coefficients for the three studied cases.

	c1, m=1	c2, m=2	c3, m=3	c4, m=4	c5
u1Sum~x^m	0.020532	−0.00128325	0.000253481	−0.000080203	0.0000328511
u21~$e^{-0.5t}x^m$	0.0629096	−0.00200304	0.0002647	−0.0000628917	0.0000206201
u22~$e^{-10t}x^m$	0.00184439	−0.000363215	0.000119241	−0.0000491062	0.0000233769

See Table 2 for a detailed description of the numbers in Table 3. The third row is the numeric coefficients associated with the time-dependent e^{-10t}.

The general feature of Table 3 is self-explanatory. E.g., the coefficients, irrespective of the case of interest, have alternative signs. Their values are decreasing rapidly, indicating that the series is converging. The profile of all four cases, for instance, at t = 0.72 s is shown in Figure 5.

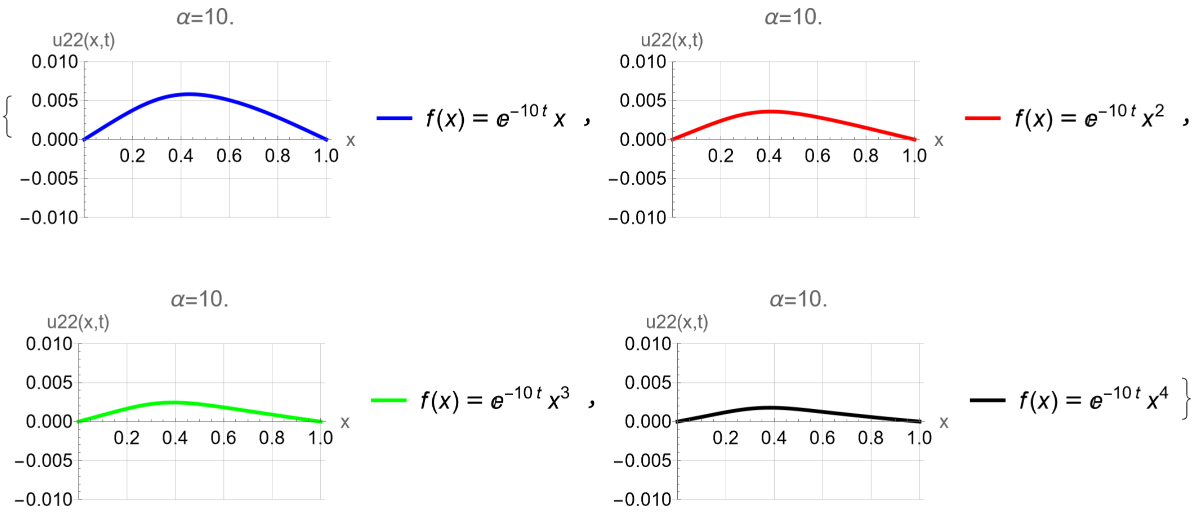
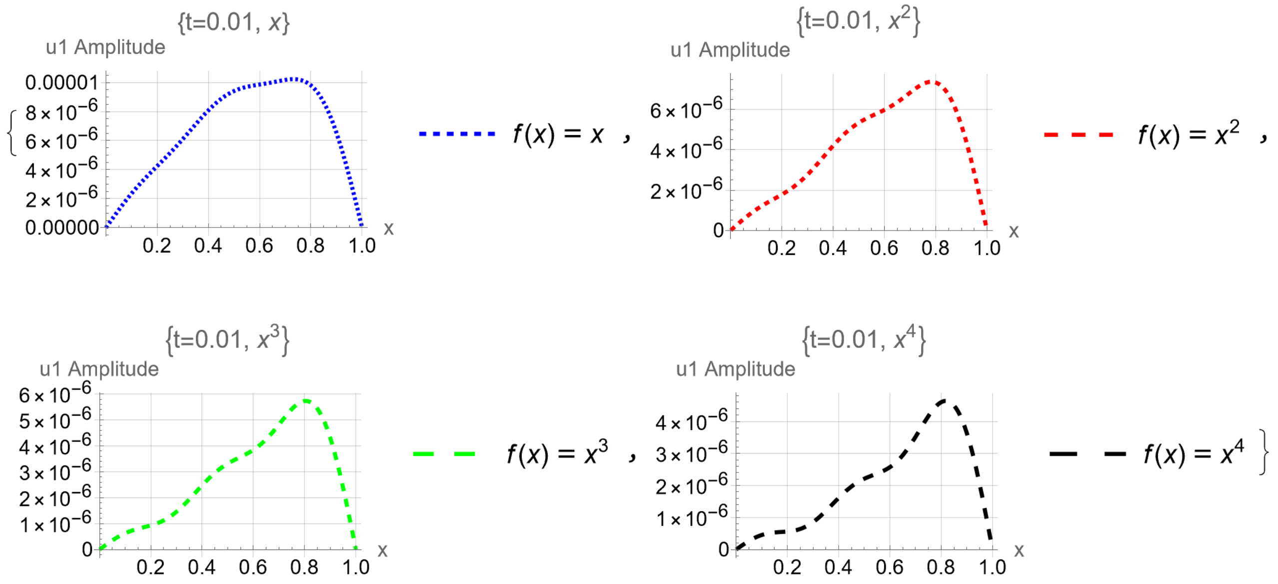


Figure 5. Description is like Figure 4.

We conclude this segment by displaying four samples of the impact of the time-independent polynomial forces, x^m. These graphs show that at the beginning of the vibrations, say, $t = 0.01$ s, the string undergoes turbulence; shortly afterward, it stabilizes, with differences in the impact of the applied forces evident.

3. Conclusions and Remarks

D'Alembert waves, solutions of the unrestricted partial differential equation with constant coefficients, have been modified by considering a finite-length string subject to external forces. Two different classes of forces, time-independent and time-dependent, are considered, subject to certain boundary restrictions. Most of the calculations are carried out utilizing a CAS, *Mathematica*. Codes are provided for reproduction. An atlas of the graphs for better understanding is provided. This report deviates from traditional reports by not only providing the formulation but also presenting numerical and graphical results coded in the report. The profiles of all four cases at t = 0.01 s are shown in **Figure 5**.



Noticing, the amplitudes are very small. Nonetheless, this shows different forces have distinct distinguishing effects.

Figure 6. Impact of the time-independent forces on the early stage of the vibrating string.

In the Introduction, it is suggested that the scope of the investigation can be extended, e.g., one may consider actual (2), *i.e.* $\frac{\partial}{\partial t}u(x,0)=g(x)$. The current report considered, $g(x)\equiv 0$.

Interested readers may find [6]-[8] resourceful for *Mathematica* coding.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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