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Experimental Quantization of Exact Wave Turbulence II: Temporal Quantization

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Abstract

The exact solutions for deterministic chaos, stochastic chaos, and wave turbulence have been developed in terms of exponential oscillons and pulsons, which are governed by the nonstationary three-dimensional Navier-Stokes equations. We have already treated theoretical quantization of the deterministic chaos in invariant structures and experimental quantization in spatial and temporal eigenfunctions using the inhomogeneous Fourier expansions. Theoretical quantization of the stochastic chaos and the wave turbulence has been considered together with experimental quantization of the stochastic chaos and the wave turbulence in spatial x-eigenfunctions. In the present paper, experimental quantization of the stochastic chaos and the wave turbulence in temporal eigenfunctions proceeds experimental quantization of the stochastic chaos and the wave turbulence in the spatial x-eigenfunctions. The method of inhomogeneous Fourier expansions in the spatial x-eigenfunctions has been extended to deterministic-random, random-deterministic, random, external, internal, and temporal eigenfunctions. Exact solutions for quantized oscillons and pulsons depend on 1-, 2-, 3-, 4-, 5-, 6-, 8-, 9-, 12-, 13-, 16-, and 32-tuples of the temporal eigenfunctions. Similar to spatial quantization, the vector, deterministic-random, external oscillons, the vector, random-deterministic, external oscillons, the vector, deterministic-random, internal oscillons, the vector, turbulent, external oscillons, the vector, turbulent, diagonal oscillons, the vector, turbulent, internal oscillons, and the vector, turbulent pulsons are computed with the help of the random model of oscillatory cn-noise. Computation is performed using experimental and theoretical programming in Maple. The obtained results show a strong dependence of the quantized oscillons and pulsons on the Reynolds number. Contrary to spatial quantization, where oscillons and pulsons are displayed as multi-mode waves, the quantized oscillons and pulsons in the case of temporal quantization are visualized as fringed waves, which qualitatively correlate with experimental data.

Keywords

Exact Solutions, Navier-Stokes Equations, Vector Deterministic-Random External Oscillon, Vector Random-Deterministic External Oscillon, Vector Deterministic-Random Internal Oscillon, Vector Turbulent External Oscillon, Vector Turbulent Diagonal Oscillon, Vector Turbulent Internal Oscillon, Vector Turbulent Pulson, 1-Tuple, 2-Tuple, 3-Tuple, 4-Tuple, 5-Tuple, 6-Tuple, 8-Tuple, 9-Tuple, 12-Tuple, 13-Tuple, 16-Tuple, 32-Tuple of Temporal Eigenfunctions

1. Introduction

The exact solution for deterministic chaos of exponential oscillons and pulsons governed by the nonstationary, three-dimensional (3-d) Navier-Stokes equations has been developed in [1] [2]. The exact solution for stochastic chaos of random exponential oscillons and pulsons controlled by the nonstationary, 3-d Navier-Stokes equations has been considered in [3]. The exact wave turbulence of exponential oscillons and pulsons has been treated in [4].

A comprehensive review of the previous work is provided in the first part of this paper [5]. The objective of the second part is to proceed theoretical quantization of the wave turbulence with experimental quantization in *t*-eigenfunctions of the deterministic-random, external, and internal interaction, the random-deterministic, external interaction, the random, external, and internal interaction, and the deterministic, external, and internal interaction and to study topology, periodicity, and visualizations of the quantized deterministic-random, random-deterministic, and turbulent oscillons and pulsons.

The contents of the current paper are following. In Section 2, the deterministic-random, random-deterministic, and random *t*-eigenfunctions are defined. Section 3 studies oscillons of the deterministic-random, external interaction, explicitly, deterministic-random, random-deterministic, external, elementary, wave, group, and kinetic-energy oscillons. Oscillons of the deterministic-random, internal interaction are treated in Section 4, namely, deterministic-random, internal, elementary, wave, group, and kinetic-energy oscillons. We study oscillons of the turbulent, external interaction in Section 5, specifically, turbulent, external, elementary, wave, group, and kinetic-energy oscillons. Oscillons of the turbulent, diagonal interaction and the turbulent, internal interaction are computed in Sections 6 and 7, explicitly, turbulent, diagonal, elementary, wave, group, and kinetic-energy oscillons and turbulent, internal, elementary, wave, group, and kinetic-energy oscillons. Turbulent and cumulative pulsons are considered in Section 8. Subsequently, turbulent, elementary, wave, group, and kinetic-energy pulsons and cumulative, deterministic, random, turbulent, kinetic-energy pulsons. A brief discussion of the main results in Section 9 concludes the paper.

2. Eigenfunctions of Inhomogeneous Fourier Expansions

2.1. Deterministic and Random Eigenfunctions of the Turbulent Velocity Potential

We define the deterministic, velocity-potential, elementary oscillons (the dpe oscillons for brevity, see (1)-(3) of [5]).

$$K_{o,d,a,m} = a_{d,m},\ K_{o,d,b,m} = b_{d,m},\ K_{o,d,c,m} = c_{d,m},\ K_{o,d,d,m} = d_{d,m} \tag{1}$$

using the experimental Deterministic Scalar Kinematic (eDSK) structures

$$\begin{aligned} a_{d,m} &= +Av_{d,m}\, sse_{d,m} + Bv_{d,m}\, cse_{d,m} + Cv_{d,m}\, sce_{d,m} + Dv_{d,m}\, cce_{d,m}, \\ b_{d,m} &= -Bv_{d,m}\, sse_{d,m} + Av_{d,m}\, cse_{d,m} - Dv_{d,m}\, sce_{d,m} + Cv_{d,m}\, cce_{d,m}, \\ c_{d,m} &= -Cv_{d,m}\, sse_{d,m} - Dv_{d,m}\, cse_{d,m} + Av_{d,m}\, sce_{d,m} + Bv_{d,m}\, cce_{d,m}, \\ d_{d,m} &= +Dv_{d,m}\, sse_{d,m} - Cv_{d,m}\, cse_{d,m} - Bv_{d,m}\, sce_{d,m} + Av_{d,m}\, cce_{d,m}, \end{aligned} \tag{2}$$

where $m = 1, 2, \cdots, M$ is an index of oscillons, $\left[Av_{d,m}, Bv_{d,m}, Cv_{d,m}, Dv_{d,m}\right]$ are deterministic functional amplitudes of a deterministic harmonic variable $v_d\left(x, y, z, t\right)$. The three-variables (3-v) eDSK functions $\left[sse_{d,m}, cse_{d,m}, sce_{d,m}, cce_{d,m}\right]\left(X_{d,m}, Y_{d,m}, z\right)$, namely,

$$\begin{aligned} sse_{d,m} &= sx_{d,m}\, sy_{d,m}\, ez_{d,m}, \quad cse_{d,m} = cx_{d,m}\, sy_{d,m}\, ez_{d,m}, \\ sce_{d,m} &= sx_{d,m}\, cy_{d,m}\, ez_{d,m}, \quad cce_{d,m} = cx_{d,m}\, cy_{d,m}\, ez_{d,m}, \end{aligned} \tag{3}$$

and the two-variables (2-v) eDSK functions $\left[ss_{d,m}, cs_{d,m}, sc_{d,m}, cc_{d,m}\right]\left(X_{d,m}, Y_{d,m}\right)$, specifically,

$$\begin{aligned} ss_{d,m} &= sx_{d,m}\, sy_{d,m}, \quad cs_{d,m} = cx_{d,m}\, sy_{d,m}, \\ sc_{d,m} &= sx_{d,m}\, cy_{d,m}, \quad cc_{d,m} = cx_{d,m}\, cy_{d,m}, \end{aligned} \tag{4}$$

are products of the one-variable (1-v) eDSK functions $\left[sx_{d,m}, cx_{d,m}\right]\left(X_{d,m}\right)$, $\left[sy_{d,m}, cy_{d,m}\right]\left(Y_{d,m}\right), ez_{d,m}\left(z\right)$, which are computed by

$$\begin{aligned} sx_{d,m} &= \sin\left(\kappa_{d,m} X_{d,m}\right), \qquad cx_{d,m} = \cos\left(\kappa_{d,m} X_{d,m}\right), \\ sy_{d,m} &= \sin\left(\lambda_{d,m} Y_{d,m}\right), \qquad cy_{d,m} = \cos\left(\lambda_{d,m} Y_{d,m}\right), \end{aligned} \tag{5}$$

$$ez_{d,m} = \exp\left(\left(-1\right)^{\eta} \mu_{d,m}\, z\right),\ \mu_{d,m} = \sqrt{\kappa_{d,m}^2 + \lambda_{d,m}^2}, \tag{6}$$

where $X_{d,m}\left(x, t\right), Y_{d,m}\left(y, t\right)$ are deterministic propagation variables, explicitly,

$$X_{d,m} = x - U_{d,m} t + X_{d,m,0},\ Y_{d,m} = y - V_{d,m} t + Y_{d,m,0}. \tag{7}$$

In Equations (1)-(7), $\left(x, y, z\right)$ is the Cartesian coordinate of a motionless frame of reference, t is time, $\left(X_{d,m}, Y_{d,m}, z\right)$ is the Cartesian coordinate of the *m*th frame of reference moving with the dpe oscillon, $\left(U_{d,m}, V_{d,m}, 0\right)$ is a celerity of the *m*th dpe oscillon, $\left(X_{d,m,0}, Y_{d,m,0}\right)$ is a reference value of $\left(X_{d,m}, Y_{d,m}\right)$ at $t = 0, x = 0, y = 0$, $\kappa_{d,m}, \lambda_{d,m}, \mu_{d,m}$ are deterministic wavenumbers of the *m*th dpe oscillon in the *x*-, *y*-, *z*-directions, and a sign parameter $\eta = 0$ for $z < 0$ and $\eta = 1$ for $z > 0$.

Substitution of the 3-v eDSK functions in terms of the 1-v eDSK functions and collection of $ez_{d,m}$ give

$$\begin{aligned} K_{o,d,a,m} &= \left(+Av_{d,m}\, ss_{d,m} + Bv_{d,m}\, cs_{d,m} + Cv_{d,m}\, sc_{d,m} + Dv_{d,m}\, cc_{d,m}\right) ez_{d,m}, \\ K_{o,d,b,m} &= \left(-Bv_{d,m}\, ss_{d,m} + Av_{d,m}\, cs_{d,m} - Dv_{d,m}\, sc_{d,m} + Cv_{d,m}\, cc_{d,m}\right) ez_{d,m}, \\ K_{o,d,c,m} &= \left(-Cv_{d,m}\, ss_{d,m} - Dv_{d,m}\, cs_{d,m} + Av_{d,m}\, sc_{d,m} + Bv_{d,m}\, cc_{d,m}\right) ez_{d,m}, \\ K_{o,d,d,m} &= \left(+Dv_{d,m}\, ss_{d,m} - Cv_{d,m}\, cs_{d,m} - Bv_{d,m}\, sc_{d,m} + Av_{d,m}\, cc_{d,m}\right) ez_{d,m}. \end{aligned} \tag{8}$$

We then define deterministic frequencies in the *x*- and *y*-directions

$$\omega_{d,x,m} = \kappa_{d,m} U_{d,m}, \quad \omega_{d,y,m} = \lambda_{d,m} V_{d,m}. \tag{9}$$

Calculating the Cartesian coordinates of the *m*th moving frame in time yields

$$\begin{aligned} \kappa_{d,m} X_{d,m} &= \kappa_{d,m}\left(x - U_{d,m} t + X_{d,m,0}\right) = -\omega_{d,x,m} t - s_{d,x,m}, \\ \lambda_{d,m} Y_{d,m} &= \lambda_{d,m}\left(y - V_{d,m} t + Y_{d,m,0}\right) = -\omega_{d,y,m} t - s_{d,y,m}, \end{aligned} \tag{10}$$

where

$$s_{d,x,m} = -\kappa_{d,m}\left(x + X_{d,m,0}\right), \quad s_{d,y,m} = -\lambda_{d,m}\left(y + Y_{d,m,0}\right) \tag{11}$$

are temporal shifts of arguments $\omega_{d,x,m} t$ and $\omega_{d,y,m} t$, respectively.
Substituting the Cartesian coordinates of the *m*th moving frame in the 1-v eDSK functions gives

$$\begin{aligned} sx_{d,m} &= -\sin\left(\omega_{d,x,m} t + s_{d,x,m}\right), \qquad cx_{d,m} = \cos\left(\omega_{d,x,m} t + s_{d,x,m}\right), \\ sy_{d,m} &= -\sin\left(\omega_{d,y,m} t + s_{d,y,m}\right), \qquad cy_{d,m} = \cos\left(\omega_{d,y,m} t + s_{d,y,m}\right). \end{aligned} \tag{12}$$

We then substitute the trigonometric form of the 1-v eDSK functions in the 2-v eDSK functions and expand products of trigonometric functions to obtain the following trigonometric form of the 2-v eDSK functions:

$$\begin{aligned} ss_{d,m} &= \frac{1}{2}\left[-\cos\left(\mathrm{A}_{d,1,m}\right) + \cos\left(\mathrm{A}_{d,2,m}\right)\right], \; cs_{d,m} = \frac{1}{2}\left[-\sin\left(\mathrm{A}_{d,1,m}\right) + \sin\left(\mathrm{A}_{d,2,m}\right)\right], \\ sc_{d,m} &= \frac{1}{2}\left[-\sin\left(\mathrm{A}_{d,1,m}\right) - \sin\left(\mathrm{A}_{d,2,m}\right)\right], \; cc_{d,m} = \frac{1}{2}\left[+\cos\left(\mathrm{A}_{d,1,m}\right) + \cos\left(\mathrm{A}_{d,2,m}\right)\right], \end{aligned} \tag{13}$$

where

$$\mathrm{A}_{d,1,m} = \omega_{d,t,1,m} t + s_{d,x,m} + s_{d,y,m}, \quad \mathrm{A}_{d,2,m} = \omega_{d,t,2,m} t + s_{d,x,m} - s_{d,y,m} \tag{14}$$

are arguments,

$$\omega_{d,t,1,m} = \omega_{d,x,m} + \omega_{d,y,m}, \quad \omega_{d,t,2,m} = \omega_{d,x,m} - \omega_{d,y,m} \tag{15}$$

are deterministic frequencies in time.

So, the dpe-oscillons may be reduced to the following trigonometric form:

$$\begin{aligned} K_{o,d,a,m} = -\frac{1}{2}\Big[&\left(Bv_{d,m} + Cv_{d,m}\right)\sin\left(\mathrm{A}_{d,1,m}\right) + \left(Av_{d,m} - Dv_{d,m}\right)\cos\left(\mathrm{A}_{d,1,m}\right) \\ &- \left(Bv_{d,m} - Cv_{d,m}\right)\sin\left(\mathrm{A}_{d,2,m}\right) - \left(Av_{d,m} + Dv_{d,m}\right)\cos\left(\mathrm{A}_{d,2,m}\right)\Big] ez_{d,m}, \\ K_{o,d,b,m} = -\frac{1}{2}\Big[&\left(Av_{d,m} - Dv_{d,m}\right)\sin\left(\mathrm{A}_{d,1,m}\right) - \left(Bv_{d,m} + Cv_{d,m}\right)\cos\left(\mathrm{A}_{d,1,m}\right) \\ &- \left(Av_{d,m} + Dv_{d,m}\right)\sin\left(\mathrm{A}_{d,2,m}\right) + \left(Bv_{d,m} - Cv_{d,m}\right)\cos\left(\mathrm{A}_{d,2,m}\right)\Big] ez_{d,m}, \end{aligned}$$

$$K_{o,d,c,m} = -\frac{1}{2}\Big[\big(Av_{d,m} - Dv_{d,m}\big)\sin\big(\mathrm{A}_{d,1,m}\big) - \big(Bv_{d,m} + Cv_{d,m}\big)\cos\big(\mathrm{A}_{d,1,m}\big)$$
$$+\big(Av_{d,m} + Dv_{d,m}\big)\sin\big(\mathrm{A}_{d,2,m}\big) - \big(Bv_{d,m} - Cv_{d,m}\big)\cos\big(\mathrm{A}_{d,2,m}\big)\Big]ez_{d,m},$$
$$K_{o,d,d,m} = +\frac{1}{2}\Big[\big(Bv_{d,m} + Cv_{d,m}\big)\sin\big(\mathrm{A}_{d,1,m}\big) + \big(Av_{d,m} - Dv_{d,m}\big)\cos\big(\mathrm{A}_{d,1,m}\big)$$
$$+\big(Bv_{d,m} - Cv_{d,m}\big)\sin\big(\mathrm{A}_{d,2,m}\big) + \big(Av_{d,m} + Dv_{d,m}\big)\cos\big(\mathrm{A}_{d,2,m}\big)\Big]ez_{d,m}. \quad (16)$$

To simplify further the dpe-oscillons, we define four deterministic trigonometric functions

$$\sin\big(\alpha_{d,1,m}\big) = \frac{Av_{d,m} - Dv_{d,m}}{Q_{d,m}}, \quad \cos\big(\alpha_{d,1,m}\big) = \frac{Bv_{d,m} + Cv_{d,m}}{Q_{d,m}},$$
$$\sin\big(\alpha_{d,2,m}\big) = \frac{Av_{d,m} + Dv_{d,m}}{R_{d,m}}, \quad \cos\big(\alpha_{d,2,m}\big) = \frac{Bv_{d,m} - Cv_{d,m}}{R_{d,m}} \quad (17)$$

of two deterministic arguments

$$\alpha_{d,1,m} = \arcsin\frac{Av_{d,m} - Dv_{d,m}}{Q_{d,m}}, \quad \alpha_{d,2,m} = \arcsin\frac{Av_{d,m} + Dv_{d,m}}{R_{d,m}}, \quad (18)$$

where deterministic amplitudes

$$Q_{d,m} = \sqrt{\big(Av_{d,m} - Dv_{d,m}\big)^2 + \big(Bv_{d,m} + Cv_{d,m}\big)^2},$$
$$R_{d,m} = \sqrt{\big(Av_{d,m} + Dv_{d,m}\big)^2 + \big(Bv_{d,m} - Cv_{d,m}\big)^2}. \quad (19)$$

Similar to the dpe-x oscillons, four dpe-t oscillons are grouped into two vector dpe-t oscillons

$$\boldsymbol{K}_{d,p,e,t,q} = \boldsymbol{K}_{d,p,e,t,q}\big(\boldsymbol{f}_{d,p,e,t,q}\big), \quad q = 1, 2, \quad (20)$$

which are formed by two 2-tuples of the dpe-t oscillons varying in time:

$$\boldsymbol{f}_{d,p,e,t,1} = \big\{f_{d,t,1,m}, f_{d,t,3,m}\big\} = \big\{f_{d,t,2k-1,m}\big\},$$
$$\boldsymbol{f}_{d,p,e,t,2} = \big\{f_{d,t,2,m}, f_{d,t,4,m}\big\} = \big\{f_{d,t,2k,m}\big\}. \quad (21)$$

Two-tuple $\boldsymbol{f}_{d,p,e,t,1}$ consists of two sine waves $f_{d,t,2k-1,m}$ with frequencies $\omega_{d,t,k,m}$ and 2-tuple $\boldsymbol{f}_{d,p,e,t,2}$ comprises two cosine waves $f_{d,t,2k,m}$ with frequencies $\omega_{d,t,k,m}$ for $k = 1, 2$, and each m. Here, deterministic eigenfunctions of the dpe-t oscillons in the motionless frame

$$f_{d,t,1,m} = \sin\big(\mathrm{A}_{d,t,1,m}\big), \quad f_{d,t,2,m} = \cos\big(\mathrm{A}_{d,t,1,m}\big),$$
$$f_{d,t,3,m} = \sin\big(\mathrm{A}_{d,t,2,m}\big), \quad f_{d,t,4,m} = \cos\big(\mathrm{A}_{d,t,2,m}\big) \quad (22)$$

depend on two arguments

$$\mathrm{A}_{d,t,1,m} = \mathrm{A}_{d,1,m} + \alpha_{d,1,m} = \omega_{d,t,1,m}t + \alpha_{d,1,m} + s_{d,x,m} + s_{d,y,m},$$
$$\mathrm{A}_{d,t,2,m} = \mathrm{A}_{d,2,m} + \alpha_{d,2,m} = \omega_{d,t,2,m}t + \alpha_{d,2,m} + s_{d,x,m} - s_{d,y,m}. \quad (23)$$

For any frozen $x = x_0, y = y_0, z = z_0$, the 1st vector dpe-$t$ oscillon

$$\boldsymbol{K}_{d,p,e,t,1} = \big[K_{o,d,a,m}, K_{o,d,d,m}\big]\big(\boldsymbol{f}_{d,p,e,t,1}\big) \quad (24)$$

is given by a list of two two-frequency (2-f), deterministic, neutral oscillons in t, which are produced by 2-tuple $\boldsymbol{f}_{d,p,e,t,1}$ as

$$K_{o,d,a,m} = -\frac{1}{2} ez_{d,m} \left(Q_{d,m} f_{d,t,1,m} - R_{d,m} f_{d,t,3,m} \right),$$
$$K_{o,d,d,m} = +\frac{1}{2} ez_{d,m} \left(Q_{d,m} f_{d,t,1,m} + R_{d,m} f_{d,t,3,m} \right). \quad (25)$$

The 2nd vector dpe-t oscillon

$$\boldsymbol{K}_{d,p,e,t,2} = \left[K_{o,d,b,m}, K_{o,d,c,m} \right] \left(\boldsymbol{f}_{d,p,e,t,2} \right) \quad (26)$$

is displayed by a list of two 2-f, deterministic, neutral oscillons in t, which are generated by 2-tuple $\boldsymbol{f}_{d,p,e,t,2}$ since

$$K_{o,d,b,m} = +\frac{1}{2} ez_{d,m} \left(Q_{d,m} f_{d,t,2,m} - R_{d,m} f_{d,t,4,m} \right),$$
$$K_{o,d,c,m} = +\frac{1}{2} ez_{d,m} \left(Q_{d,m} f_{d,t,2,m} + R_{d,m} f_{d,t,4,m} \right). \quad (27)$$

For all vector dpe-t oscillons, amplitudes of eigenfunctions depend on $Q_{d,m}, R_{d,m}, z_0$ via $ez_{d,m}(z_0)$ and temporal shifts of eigenfunctions are determined by $\omega_{d,t,1,m}, \omega_{d,t,2,m}, \alpha_{d,1,m}, \alpha_{d,2,m}, x_0, y_0$ through $s_{d,x,m}(x_0), s_{d,y,m}(y_0)$. Periods $T_{d,t,1,m}$ of eigenfunctions $f_{d,t,1,m}, f_{d,t,2,m}$ and $T_{d,t,2,m}$ of eigenfunctions $f_{d,t,3,m}, f_{d,t,4,m}$ are calculated by

$$T_{d,t,1,m} = \frac{2\pi}{\omega_{d,t,1,m}}, \; T_{d,t,2,m} = \frac{2\pi}{\omega_{d,t,2,m}}. \quad (28)$$

The period of the dpe-t oscillons

$$T_{o,d,t,m} = \mathrm{LCM}\left(T_{d,t,1,m}, T_{d,t,1,m} \right) = k_{d,t,1,m} T_{d,t,1,m} = k_{d,t,2,m} T_{d,t,2,m}, \quad (29)$$

where $\mathrm{LCM}(a,b)$ is a least common multiple of a and b, $k_{d,t,1,m}$ and $k_{d,t,2,m}$ are integers.

Because integrals of sine waves $f_{d,t,2k-1,m}$ and cosine waves $f_{d,t,2k,m}$ over the relevant periods vanish, the average of the dpe-t oscillons over $T_{o,d,t,m}$ vanishes, as well,

$$\frac{1}{T_{o,d,t,m}} \int_0^{T_{o,d,t,m}} K_{o,d,i,m} \mathrm{d}t = 0, \; i = [a,b,c,d], \quad (30)$$

i.e. the vector dpe-t oscillons are neutral.

The random, velocity-potential, elementary oscillons (the rpe oscillons for briefness, see (24)-(26) of [5]).

$$K_{o,r,a,m} = a_{r,m}, \; K_{o,r,b,m} = b_{r,m}, \; K_{o,r,c,m} = c_{r,m}, \; K_{o,r,d,m} = d_{r,m} \quad (31)$$

are expressed via the experimental Random Scalar Kinematic (eRSK) structures

$$a_{r,m} = +Av_{r,m}\, sse_{r,m} + Bv_{r,m}\, cse_{r,m} + Cv_{r,m}\, sce_{r,m} + Dv_{r,m}\, cce_{r,m},$$
$$b_{r,m} = -Bv_{r,m}\, sse_{r,m} + Av_{r,m}\, cse_{r,m} - Dv_{r,m}\, sce_{r,m} + Cv_{r,m}\, cce_{r,m},$$
$$c_{d,m} = -Cv_{r,m}\, sse_{r,m} - Dv_{r,m}\, cse_{r,m} + Av_{r,m}\, sce_{r,m} + Bv_{r,m}\, cce_{r,m},$$
$$d_{r,m} = +Dv_{r,m}\, sse_{r,m} - Cv_{r,m}\, cse_{r,m} - Bv_{r,m}\, sce_{r,m} + Av_{r,m}\, cce_{r,m}, \quad (32)$$

where $\left[Av_{r,m}, Bv_{r,m}, Cv_{r,m}, Dv_{r,m} \right](t)$ are random functional amplitudes of a random harmonic variable $v_r(x,y,z,t)$.

The 3-v eRSK functions $\left[sse_{r,m}, cse_{r,m}, sce_{r,m}, cce_{r,m}\right](X_{r,m}, Y_{r,m}, z)$, specifically,

$$sse_{r,m} = sx_{r,m}\, sy_{r,m}\, ez_{r,m}, \quad cse_{r,m} = cx_{r,m}\, sy_{r,m}\, ez_{r,m}, \\ sce_{r,m} = sx_{r,m}\, cy_{r,m}\, ez_{r,m}, \quad cce_{r,m} = cx_{r,m}\, cy_{r,m}\, ez_{r,m}, \tag{33}$$

and the 2-v eRSK functions $\left[ss_{r,m}, cs_{r,m}, sc_{r,m}, cc_{r,m}\right](X_{r,m}, Y_{r,m})$, explicitely,

$$ss_{r,m} = sx_{r,m}\, sy_{r,m}, \quad cs_{r,m} = cx_{r,m}\, sy_{r,m}, \\ sc_{r,m} = sx_{r,m}\, cy_{r,m}, \quad cc_{r,m} = cx_{r,m}\, cy_{r,m}, \tag{34}$$

are defined as the products of 1-v eRSK functions $\left[sx_{r,m}, cx_{r,m}\right](X_{r,m})$, $\left[sy_{r,m}, cy_{r,m}\right](Y_{r,m})$, $ez_{r,m}(z)$, which are computed by

$$sx_{r,m} = \sin(\kappa_{r,m} X_{r,m}), \qquad cx_{r,m} = \cos(\kappa_{r,m} X_{r,m}), \\ sy_{r,m} = \sin(\lambda_{r,m} Y_{r,m}), \qquad cy_{r,m} = \cos(\lambda_{r,m} Y_{r,m}), \tag{35}$$

$$ez_{r,m} = \exp\left((-1)^{\eta}\, \mu_{r,m}\, z\right), \ \mu_{r,m} = \sqrt{\kappa_{r,m}^2 + \lambda_{r,m}^2}, \tag{36}$$

where $X_{r,m}(x,t), Y_{r,m}(y,t)$ are random propagation variables, namely,

$$X_{r,m} = x - U_{r,m} t + X_{r,m,0}, \ Y_{r,m} = y - V_{r,m} t + Y_{r,m,0}. \tag{37}$$

In Equations (31)-(37), $(X_{r,m}, Y_{r,m}, z)$ is the Cartesian coordinate of the *m*th frame of reference moving with the rpe oscillon, $(U_{r,m}, V_{r,m}, 0)(t)$ is the celerity of the *m*th rpe oscillon, $(X_{r,m,0}, Y_{r,m,0})(t)$ is a reference value of $(X_{r,m}, Y_{r,m})$ at $t = 0, x = 0, y = 0$, and $\kappa_{r,m}, \lambda_{r,m}, \mu_{r,m}$ are the random wavenumbers of the *m*th rpe oscillon in the *x*-, *y*-, *z*-directions. Wave parameters

$$U_{r,m} = U_{r,m}(t), \ V_{r,m} = V_{r,m}(t), \ X_{r,m,0} = X_{r,m,0}(t), \ Y_{r,m,0} = Y_{r,m,0}(t) \tag{38}$$

together with functional amplitudes

$$Av_{r,m} = Av_{r,m}(t), \ Bv_{r,m} = Bv_{r,m}(t), \ Cv_{r,m} = Cv_{r,m}(t), \ Dv_{r,m} = Dv_{r,m}(t) \tag{39}$$

are smooth random functions of time from C^{∞}. Wavenumbers $\kappa_{r,m}, \lambda_{r,m}, \mu_{r,m}$ are random parameters.

Substituting the 3-v eRSK functions via the 1-v eRSK functions and collecting $ez_{r,m}$ yield

$$K_{o,r,a,m} = \left(+Av_{r,m}\, ss_{r,m} + Bv_{r,m}\, cs_{r,m} + Cv_{r,m}\, sc_{r,m} + Dv_{r,m}\, cc_{r,m}\right) ez_{r,m}, \\ K_{o,r,b,m} = \left(-Bv_{r,m}\, ss_{r,m} + Av_{r,m}\, cs_{r,m} - Dv_{r,m}\, sc_{r,m} + Cv_{r,m}\, cc_{r,m}\right) ez_{r,m}, \\ K_{o,r,c,m} = \left(-Cv_{r,m}\, ss_{r,m} - Dv_{r,m}\, cs_{r,m} + Av_{r,m}\, sc_{r,m} + Bv_{r,m}\, cc_{r,m}\right) ez_{r,m}, \\ K_{o,r,d,m} = \left(+Dv_{r,m}\, ss_{r,m} - Cv_{r,m}\, cs_{r,m} - Bv_{r,m}\, sc_{r,m} + Av_{r,m}\, cc_{r,m}\right) ez_{r,m}. \tag{40}$$

We then introduce random frequencies in the *x*- and *y*-directions

$$\omega_{r,x,m} = \kappa_{r,m} U_{r,m}, \ \omega_{r,y,m} = \lambda_{r,m} V_{r,m}. \tag{41}$$

Calculation of the Cartesian coordinates of the *m*th moving frame in time gives

$$\begin{aligned}\kappa_{r,m} X_{r,m} &= \kappa_{r,m}\left(x - U_{r,m} t + X_{r,m,0}\right) = -\omega_{r,x,m} t - s_{r,x,m},\\ \lambda_{r,m} Y_{r,m} &= \lambda_{r,m}\left(y - V_{r,m} t + Y_{r,m,0}\right) = -\omega_{r,y,m} t - s_{r,y,m},\end{aligned} \tag{42}$$

where

$$s_{r,x,m} = -\kappa_{r,m}\left(x + X_{r,m,0}\right),\quad s_{r,y,m} = -\lambda_{r,m}\left(y + Y_{r,m,0}\right) \tag{43}$$

are temporal shifts of arguments $\omega_{r,x,m} t$ and $\omega_{r,y,m} t$, correspondingly.

Substituting the Cartesian coordinates of the mth moving frame in the 1-v eRSK functions returns

$$\begin{aligned}sx_{r,m} &= -\sin\left(\omega_{r,x,m} t + s_{r,x,m}\right),\quad cx_{r,m} = \cos\left(\omega_{r,x,m} t + s_{r,x,m}\right),\\ sy_{r,m} &= -\sin\left(\omega_{r,y,m} t + s_{r,y,m}\right),\quad cy_{r,m} = \cos\left(\omega_{r,y,m} t + s_{r,y,m}\right).\end{aligned} \tag{44}$$

We then use the trigonometric form of the 1-v eRSK in the 2-v eRSK functions and expand products of trigonometric functions to compute the following trigonometric form of the 2-v eRSK functions:

$$\begin{aligned}ss_{r,m} &= \frac{1}{2}\left[-\cos\left(\mathrm{A}_{r,1,m}\right) + \cos\left(\mathrm{A}_{r,2,m}\right)\right],\ cs_{r,m} = \frac{1}{2}\left[-\sin\left(\mathrm{A}_{r,1,m}\right) + \sin\left(\mathrm{A}_{r,2,m}\right)\right],\\ sc_{r,m} &= \frac{1}{2}\left[-\sin\left(\mathrm{A}_{r,1,m}\right) - \sin\left(\mathrm{A}_{r,2,m}\right)\right],\ cc_{r,m} = \frac{1}{2}\left[+\cos\left(\mathrm{A}_{r,1,m}\right) + \cos\left(\mathrm{A}_{r,2,m}\right)\right],\end{aligned} \tag{45}$$

with arguments

$$\mathrm{A}_{r,1,m} = \omega_{r,t,1,m} t + s_{r,x,m} + s_{r,y,m},\ \mathrm{A}_{r,2,m} = \omega_{r,t,2,m} t + s_{r,x,m} - s_{r,y,m} \tag{46}$$

and random frequencies in time

$$\omega_{r,t,1,m} = \omega_{r,x,m} + \omega_{r,y,m},\ \omega_{r,t,2,m} = \omega_{r,x,m} - \omega_{r,y,m}. \tag{47}$$

Thus, therpe-oscillons may be written in the following trigonometric form:

$$\begin{aligned}K_{o,r,a,m} =& -\frac{1}{2}\Big[\left(Bv_{r,m} + Cv_{r,m}\right)\sin\left(\mathrm{A}_{r,1,m}\right) + \left(Av_{r,m} - Dv_{r,m}\right)\cos\left(\mathrm{A}_{r,1,m}\right)\\ & - \left(Bv_{r,m} - Cv_{r,m}\right)\sin\left(\mathrm{A}_{r,2,m}\right) - \left(Av_{r,m} + Dv_{r,m}\right)\cos\left(\mathrm{A}_{r,2,m}\right)\Big] ez_{r,m},\\ K_{o,r,b,m} =& -\frac{1}{2}\Big[\left(Av_{r,m} - Dv_{r,m}\right)\sin\left(\mathrm{A}_{r,1,m}\right) - \left(Bv_{r,m} + Cv_{r,m}\right)\cos\left(\mathrm{A}_{r,1,m}\right)\\ & - \left(Av_{r,m} + Dv_{r,m}\right)\sin\left(\mathrm{A}_{r,2,m}\right) + \left(Bv_{r,m} - Cv_{r,m}\right)\cos\left(\mathrm{A}_{r,2,m}\right)\Big] ez_{r,m},\\ K_{o,r,c,m} =& -\frac{1}{2}\Big[\left(Av_{r,m} - Dv_{r,m}\right)\sin\left(\mathrm{A}_{r,1,m}\right) - \left(Bv_{r,m} + Cv_{r,m}\right)\cos\left(\mathrm{A}_{r,1,m}\right)\\ & + \left(Av_{r,m} + Dv_{r,m}\right)\sin\left(\mathrm{A}_{r,2,m}\right) - \left(Bv_{r,m} - Cv_{r,m}\right)\cos\left(\mathrm{A}_{r,2,m}\right)\Big] ez_{r,m},\\ K_{o,r,d,m} =& +\frac{1}{2}\Big[\left(Bv_{r,m} + Cv_{r,m}\right)\sin\left(\mathrm{A}_{r,1,m}\right) + \left(Av_{r,m} - Dv_{r,m}\right)\cos\left(\mathrm{A}_{r,1,m}\right)\\ & + \left(Bv_{r,m} - Cv_{r,m}\right)\sin\left(\mathrm{A}_{r,2,m}\right) + \left(Av_{r,m} + Dv_{r,m}\right)\cos\left(\mathrm{A}_{r,2,m}\right)\Big] ez_{r,m}.\end{aligned} \tag{48}$$

To reduce the rpe-oscillons, we introduce four random trigonometric functions

$$\begin{aligned}\sin\left(\alpha_{r,1,m}\right) &= \frac{Av_{r,m} - Dv_{r,m}}{Q_{r,m}},\quad \cos\left(\alpha_{r,1,m}\right) = \frac{Bv_{r,m} + Cv_{r,m}}{Q_{r,m}},\\ \sin\left(\alpha_{r,2,m}\right) &= \frac{Av_{r,m} + Dv_{r,m}}{R_{r,m}},\quad \cos\left(\alpha_{r,2,m}\right) = \frac{Bv_{r,m} - Cv_{r,m}}{R_{r,m}}\end{aligned} \tag{49}$$

of two random arguments

$$\alpha_{r,1,m} = \arcsin \frac{Av_{r,m} - Dv_{r,m}}{Q_{r,m}}, \quad \alpha_{r,2,m} = \arcsin \frac{Av_{r,m} + Dv_{r,m}}{R_{r,m}}, \tag{50}$$

where random amplitudes

$$\begin{aligned} Q_{r,m} &= \sqrt{\left(Av_{r,m} - Dv_{r,m}\right)^2 + \left(Bv_{r,m} + Cv_{r,m}\right)^2}, \\ R_{r,m} &= \sqrt{\left(Av_{r,m} + Dv_{r,m}\right)^2 + \left(Bv_{r,m} - Cv_{r,m}\right)^2}. \end{aligned} \tag{51}$$

Similar to the dpe-t oscillons, four rpe-t oscillons are combined into two vector rpe-t oscillons

$$\boldsymbol{K}_{r,p,e,t,q} = \boldsymbol{K}_{r,p,e,t,q}\left(\boldsymbol{f}_{r,p,e,t,q}\right), \ q = 1,2, \tag{52}$$

which are established by two two-tuples of the rpe oscillons varying in time:

$$\begin{aligned} \boldsymbol{f}_{r,p,e,t,1} &= \left\{ f_{r,t,1,m}, f_{r,t,3,m} \right\} = \left\{ f_{r,t,2k-1,m} \right\}, \\ \boldsymbol{f}_{r,p,e,t,2} &= \left\{ f_{r,t,2,m}, f_{r,t,4,m} \right\} = \left\{ f_{r,t,2k,m} \right\}. \end{aligned} \tag{53}$$

Two-tuple $\boldsymbol{f}_{r,p,e,t,1}$ includes two sine waves $f_{r,t,2k-1,m}$ with frequency $\omega_{r,t,k,m}$ and 2-tuple $\boldsymbol{f}_{r,p,e,t,2}$ is composed of two cosine waves $f_{r,t,2k,m}$ with frequency $\omega_{r,t,k,m}$ for $k = 1,2$, and each m. Here, eigenfunctions of the rpe-t oscillons

$$\begin{aligned} f_{r,t,1,m} &= \sin\left(\mathrm{A}_{r,t,1,m}\right), \quad f_{r,t,2,m} = \cos\left(\mathrm{A}_{r,t,1,m}\right), \\ f_{r,t,3,m} &= \sin\left(\mathrm{A}_{r,t,2,m}\right), \quad f_{r,t,4,m} = \cos\left(\mathrm{A}_{r,t,2,m}\right) \end{aligned} \tag{54}$$

depend on two arguments

$$\begin{aligned} \mathrm{A}_{r,t,1,m} &= \mathrm{A}_{r,1,m} + \alpha_{r,1,m} = \omega_{r,t,1,m} t + \alpha_{r,1,m} + s_{r,x,m} + s_{r,y,m}, \\ \mathrm{A}_{r,t,2,m} &= \mathrm{A}_{r,2,m} + \alpha_{r,2,m} = \omega_{r,t,2,m} t + \alpha_{r,2,m} + s_{r,x,m} - s_{r,y,m}. \end{aligned} \tag{55}$$

For any frozen $x = x_0, y = y_0, z = z_0$, the 1st vector rpe-$t$ oscillon

$$\boldsymbol{K}_{r,p,e,t,1} = \left[K_{o,r,a,m}, K_{o,r,d,m}\right]\left(\boldsymbol{f}_{r,p,e,t,1}\right) \tag{56}$$

is presented by a list of two two-random frequency (2-rf), random oscillons in t, which are set by 2-tuple $\boldsymbol{f}_{r,p,e,t,1}$ because

$$\begin{aligned} K_{o,r,a,m} &= -\frac{1}{2} ez_{r,m}\left(Q_{r,m} f_{r,t,1,m} - R_{r,m} f_{r,t,3,m}\right), \\ K_{o,r,d,m} &= +\frac{1}{2} ez_{r,m}\left(Q_{r,m} f_{r,t,1,m} + R_{r,m} f_{r,t,3,m}\right). \end{aligned} \tag{57}$$

The 2nd vector rpe-t oscillon

$$\boldsymbol{K}_{r,p,e,t,2} = \left[K_{o,r,b,m}, K_{o,r,c,m}\right]\left(\boldsymbol{f}_{r,p,e,t,2}\right) \tag{58}$$

is visualized by a list of two 2-rf, random oscillons in t, which are formed by 2-tuple $\boldsymbol{f}_{r,p,e,t,2}$ as

$$\begin{aligned} K_{o,r,b,m} &= +\frac{1}{2} ez_{r,m}\left(Q_{r,m} f_{r,t,2,m} - R_{r,m} f_{r,t,4,m}\right), \\ K_{o,r,c,m} &= +\frac{1}{2} ez_{r,m}\left(Q_{r,m} f_{r,t,2,m} + R_{r,m} f_{r,t,4,m}\right). \end{aligned} \tag{59}$$

For all vector rpe-t oscillons, amplitudes of eigenfunctions are governed by z_0, t via $ez_{r,m}(z_0), Q_{r,m}(t), R_{r,m}(t)$ and temporal shifts of eigenfunctions are influenced by x_0, y_0, t through $s_{r,x,m}(x_0,t), s_{r,y,m}(y_0,t), \omega_{r,t,1,m}(t), \omega_{r,t,2,m}(t)$, $\alpha_{r,1,m}(t), \alpha_{r,2,m}(t)$.

Because of the time-dependent wave parameters and functional amplitudes,

$$\begin{aligned} &\omega_{r,x,m} = \omega_{r,x,m}(t), \ \omega_{r,y,m} = \omega_{r,y,m}(t), \ \omega_{r,t,l,m} = \omega_{r,t,l,m}(t), \\ &Q_{r,m} = Q_{r,m}(t), \ R_{r,m} = R_{r,m}(t), \ \alpha_{r,l,m} = \alpha_{r,l,m}(t), \ l = 1,2 \end{aligned} \tag{60}$$

are also smooth random functions of time from C^∞ and the following variables

$$\begin{aligned} &s_{r,x,m} = s_{r,x,m}(x,t), \qquad s_{r,y,m} = s_{r,y,m}(y,t), \\ &A_{r,l,m} = A_{r,l,m}(x,y,t), \ A_{r,t,l,m} = A_{r,t,l,m}(x,y,t), \ l = 1,2, \\ &f_{r,t,l,m} = f_{r,t,l,m}(x,y,t), \qquad l = 1,2,3,4 \end{aligned} \tag{61}$$

are smooth functions of space and smooth random functions of time from C^∞. For any frozen $x = x_0, y = y_0, z = z_0$, the rpe-$t$ oscillons are smooth random functions of time with an unbounded period.

2.2. Eigenfunctions of Deterministic-Random, External Interaction

Consider the mth deterministic eigenfunctions $f_{d,t,i,m}$ of the dpe-t oscillons and the nth random eigenfunctions $f_{r,t,j,n}$ of the rpe-t oscillons, where $i = 1,2,3,4$ and $j = 1,2,3,4$,

$$\begin{aligned} f_{d,t,1,m} &= \sin(A_{d,t,1,m}), \quad f_{d,t,2,m} = \cos(A_{d,t,1,m}), \\ f_{d,t,3,m} &= \sin(A_{d,t,2,m}), \quad f_{d,t,4,m} = \cos(A_{d,t,2,m}), \\ f_{r,t,1,n} &= \sin(A_{r,t,1,n}), \quad f_{r,t,2,n} = \cos(A_{r,t,1,n}), \\ f_{r,t,3,n} &= \sin(A_{r,t,2,n}), \quad f_{r,t,4,n} = \cos(A_{r,t,2,n}) \end{aligned} \tag{62}$$

of four arguments

$$\begin{aligned} A_{d,t,1,m} &= \omega_{d,t,1,m} t + \alpha_{d,1,m} + s_{d,x,m} + s_{d,y,m}, \\ A_{d,t,2,m} &= \omega_{d,t,2,m} t + \alpha_{d,2,m} + s_{d,x,m} - s_{d,y,m}, \\ A_{r,t,1,n} &= \omega_{r,t,1,n} t + \alpha_{r,1,n} + s_{r,x,n} + s_{r,y,n}, \\ A_{r,t,2,n} &= \omega_{r,t,2,n} t + \alpha_{r,2,n} + s_{r,x,n} - s_{r,y,n} \end{aligned} \tag{63}$$

with two deterministic frequencies $\omega_{d,t,1,m}, \omega_{d,t,2,m}$ and two random frequencies $\omega_{r,t,1,n}, \omega_{r,t,2,n}$ for $m = 1,2,\cdots,M$ and $n = 1,2,\cdots,M$.
Matrix $\boldsymbol{f}_{d,t,i,m,r,t,j,n}$ of the deterministic-random, external interaction $(n \neq m)$ between eigenfunctions $f_{d,t,i,m}$ and $f_{r,t,j,n}$ takes the following form:

$$\boldsymbol{f}_{d,t,i,m,r,t,j,n} = \begin{bmatrix} f_{d,t,1,m} f_{r,t,1,n} & f_{d,t,1,m} f_{r,t,2,n} & f_{d,t,1,m} f_{r,t,3,n} & f_{d,t,1,m} f_{r,t,4,n} \\ f_{d,t,2,m} f_{r,t,1,n} & f_{d,t,2,m} f_{r,t,2,n} & f_{d,t,2,m} f_{r,t,3,n} & f_{d,t,2,m} f_{r,t,4,n} \\ f_{d,t,3,m} f_{r,t,1,n} & f_{d,t,3,m} f_{r,t,2,n} & f_{d,t,3,m} f_{r,t,3,n} & f_{d,t,3,m} f_{r,t,4,n} \\ f_{d,t,4,m} f_{r,t,1,n} & f_{d,t,4,m} f_{r,t,2,n} & f_{d,t,4,m} f_{r,t,3,n} & f_{d,t,4,m} f_{r,t,4,n} \end{bmatrix}. \tag{64}$$

Because of the trigonometric identities for products of sine and cosine (the product identities), there are 16 deterministic-random, external eigenfunctions $f_{d,r,t,l,m,n}$, specifically,

$$
\begin{aligned}
&f_{d,r,t,1,m,n} = \sin\left(\mathrm{A}_{d,r,t,1,m,n}\right), \ f_{d,r,t,2,m,n} = \cos\left(\mathrm{A}_{d,r,t,1,m,n}\right),\\
&f_{d,r,t,3,m,n} = \sin\left(\mathrm{A}_{d,r,t,2,m,n}\right), \ f_{d,r,t,4,m,n} = \cos\left(\mathrm{A}_{d,r,t,2,m,n}\right),\\
&f_{d,r,t,5,m,n} = \sin\left(\mathrm{A}_{d,r,t,3,m,n}\right), \ f_{d,r,t,6,m,n} = \cos\left(\mathrm{A}_{d,r,t,3,m,n}\right),\\
&f_{d,r,t,7,m,n} = \sin\left(\mathrm{A}_{d,r,t,4,m,n}\right), \ f_{d,r,t,8,m,n} = \cos\left(\mathrm{A}_{d,r,t,4,m,n}\right),\\
&f_{d,r,t,9,m,n} = \sin\left(\mathrm{A}_{d,r,t,5,m,n}\right), \ f_{d,r,t,10,m,n} = \cos\left(\mathrm{A}_{d,r,t,5,m,n}\right),\\
&f_{d,r,t,11,m,n} = \sin\left(\mathrm{A}_{d,r,t,6,m,n}\right), \ f_{d,r,t,12,m,n} = \cos\left(\mathrm{A}_{d,r,t,6,m,n}\right),\\
&f_{d,r,t,13,m,n} = \sin\left(\mathrm{A}_{d,r,t,7,m,n}\right), \ f_{d,r,t,14,m,n} = \cos\left(\mathrm{A}_{d,r,t,7,m,n}\right),\\
&f_{d,r,t,15,m,n} = \sin\left(\mathrm{A}_{d,r,t,8,m,n}\right), \ f_{d,r,t,16,m,n} = \cos\left(\mathrm{A}_{d,r,t,8,m,n}\right),
\end{aligned}
\tag{65}
$$

of eight deterministic-random, external arguments $\mathrm{A}_{d,r,t,k,m,n}$, namely,

$$
\begin{aligned}
\mathrm{A}_{d,r,t,1,m,n} &= \omega_{d,r,t,1,m,n}t + \alpha_{d,1,m} + \alpha_{r,1,n} + s_{d,x,m} + s_{d,y,m} + s_{r,x,n} + s_{r,y,n},\\
\mathrm{A}_{d,r,t,2,m,n} &= \omega_{d,r,t,2,m,n}t + \alpha_{d,1,m} - \alpha_{r,1,n} + s_{d,x,m} + s_{d,y,m} - s_{r,x,n} - s_{r,y,n},\\
\mathrm{A}_{d,r,t,3,m,n} &= \omega_{d,r,t,3,m,n}t + \alpha_{d,1,m} + \alpha_{r,2,n} + s_{d,x,m} + s_{d,y,m} + s_{r,x,n} - s_{r,y,n},\\
\mathrm{A}_{d,r,t,4,m,n} &= \omega_{d,r,t,4,m,n}t + \alpha_{d,1,m} - \alpha_{r,2,n} + s_{d,x,m} + s_{d,y,m} - s_{r,x,n} + s_{r,y,n},\\
\mathrm{A}_{d,r,t,5,m,n} &= \omega_{d,r,t,5,m,n}t + \alpha_{d,2,m} + \alpha_{r,1,n} + s_{d,x,m} - s_{d,y,m} + s_{r,x,n} + s_{r,y,n},\\
\mathrm{A}_{d,r,t,6,m,n} &= \omega_{d,r,t,6,m,n}t + \alpha_{d,2,m} - \alpha_{r,1,n} + s_{d,x,m} - s_{d,y,m} - s_{r,x,n} - s_{r,y,n},\\
\mathrm{A}_{d,r,t,7,m,n} &= \omega_{d,r,t,7,m,n}t + \alpha_{d,2,m} + \alpha_{r,2,n} + s_{d,x,m} - s_{d,y,m} + s_{r,x,n} - s_{r,y,n},\\
\mathrm{A}_{d,r,t,8,m,n} &= \omega_{d,r,t,8,m,n}t + \alpha_{d,2,m} - \alpha_{r,2,n} + s_{d,x,m} - s_{d,y,m} - s_{r,x,n} + s_{r,y,n},
\end{aligned}
\tag{66}
$$

with eight deterministic-random, external frequencies

$$
\begin{aligned}
\omega_{d,r,t,1,m,n} &= \omega_{d,x,m} + \omega_{d,y,m} + \omega_{r,x,n} + \omega_{r,y,n},\\
\omega_{d,r,t,2,m,n} &= \omega_{d,x,m} + \omega_{d,y,m} - \omega_{r,x,n} - \omega_{r,y,n},\\
\omega_{d,r,t,3,m,n} &= \omega_{d,x,m} + \omega_{d,y,m} + \omega_{r,x,n} - \omega_{r,y,n},\\
\omega_{d,r,t,4,m,n} &= \omega_{d,x,m} + \omega_{d,y,m} - \omega_{r,x,n} + \omega_{r,y,n},\\
\omega_{d,r,t,5,m,n} &= \omega_{d,x,m} - \omega_{d,y,m} + \omega_{r,x,n} + \omega_{r,y,n},\\
\omega_{d,r,t,6,m,n} &= \omega_{d,x,m} - \omega_{d,y,m} - \omega_{r,x,n} - \omega_{r,y,n},\\
\omega_{d,r,t,7,m,n} &= \omega_{d,x,m} - \omega_{d,y,m} + \omega_{r,x,n} - \omega_{r,y,n},\\
\omega_{d,r,t,8,m,n} &= \omega_{d,x,m} - \omega_{d,y,m} - \omega_{r,x,n} + \omega_{r,y,n}.
\end{aligned}
\tag{67}
$$

Deterministic-random, external eigenfunctions $f_{d,r,t,l,m,n}$ include sine and cosine waves with all combinations of deterministic-random *x*-frequencies, *y*-frequencies, *x*-shifts, and *y*-shifts.

Computing matrix $\boldsymbol{f}_{d,t,i,m,r,t,j,n}$ yields the following Fourier expansions in eigenfunctions $f_{d,r,t,l,m,n}$:

$$
\begin{aligned}
2 f_{d,t,1,m} f_{r,t,1,n} &= - f_{d,r,t,2,m,n} + f_{d,r,t,4,m,n},\\
2 f_{d,t,1,m} f_{r,t,2,n} &= + f_{d,r,t,1,m,n} + f_{d,r,t,3,m,n},\\
2 f_{d,t,1,m} f_{r,t,3,n} &= - f_{d,r,t,6,m,n} + f_{d,r,t,8,m,n},\\
2 f_{d,t,1,m} f_{r,t,4,n} &= + f_{d,r,t,5,m,n} + f_{d,r,t,7,m,n},\\
2 f_{d,t,2,m} f_{r,t,1,n} &= + f_{d,r,t,1,m,n} - f_{d,r,t,3,m,n},\\
2 f_{d,t,2,m} f_{r,t,2,n} &= + f_{d,r,t,2,m,n} + f_{d,r,t,4,m,n},\\
2 f_{d,t,2,m} f_{r,t,3,n} &= + f_{d,r,t,5,m,n} - f_{d,r,t,7,m,n},\\
2 f_{d,t,2,m} f_{r,t,4,n} &= + f_{d,r,t,6,m,n} + f_{d,r,t,8,m,n},
\end{aligned}
$$

$$
\begin{aligned}
2 f_{d,t,3,m} f_{r,t,1,n} &= -f_{d,r,t,10,m,n} + f_{d,r,t,12,m,n},\\
2 f_{d,t,3,m} f_{r,t,2,n} &= +f_{d,r,t,9,m,n} + f_{d,r,t,11,m,n},\\
2 f_{d,t,3,m} f_{r,t,3,n} &= -f_{d,r,t,14,m,n} + f_{d,r,t,16,m,n},\\
2 f_{d,t,3,m} f_{r,t,4,n} &= +f_{d,r,t,13,m,n} + f_{d,r,t,15,m,n},\\
2 f_{d,t,4,m} f_{r,t,1,n} &= +f_{d,r,t,9,m,n} - f_{d,r,t,11,m,n},\\
2 f_{d,t,4,m} f_{r,t,2,n} &= +f_{d,r,t,10,m,n} + f_{d,r,t,12,m,n},\\
2 f_{d,t,4,m} f_{r,t,3,n} &= +f_{d,r,t,13,m,n} - f_{d,r,t,15,m,n},\\
2 f_{d,t,4,m} f_{r,t,4,n} &= +f_{d,r,t,14,m,n} + f_{d,r,t,16,m,n}.
\end{aligned}
\tag{68}
$$

Due to the time-dependent wave parameters and functional amplitudes,

$$\omega_{d,r,t,k,m,n} = \omega_{d,r,t,k,m,n}(t),\ k = 1,2,\cdots,8 \tag{69}$$

are also smooth random functions of time from C^{∞} and

$$
\begin{aligned}
A_{d,r,t,k,m,n} &= A_{d,r,t,k,m,n}(x,y,t),\ k = 1,2,\cdots,8,\\
f_{d,r,t,l,m,n} &= f_{d,r,t,l,m,n}(x,y,t),\quad l = 1,2,\cdots,16
\end{aligned}
\tag{70}
$$

are smooth functions of space and smooth random functions of time from C^{∞}.

2.3. Eigenfunctions of Random-Deterministic, External Interaction

Consider the mth random eigenfunctions $f_{r,t,i,m}$ of the rpe-t oscillons and the nth deterministic eigenfunctions $f_{d,t,j,n}$ of the dpe-t oscillons, where $i = 1,2,3,4$ and $j = 1,2,3,4$,

$$
\begin{aligned}
f_{r,t,1,m} &= \sin(A_{r,t,1,m}), & f_{r,t,2,m} &= \cos(A_{r,t,1,m}),\\
f_{r,t,3,m} &= \sin(A_{r,t,2,m}), & f_{r,t,4,m} &= \cos(A_{r,t,2,m}),\\
f_{d,t,1,n} &= \sin(A_{d,t,1,n}), & f_{d,t,2,n} &= \cos(A_{d,t,1,n}),\\
f_{d,t,3,n} &= \sin(A_{d,t,2,n}), & f_{d,t,4,n} &= \cos(A_{d,t,2,n})
\end{aligned}
\tag{71}
$$

of four arguments

$$
\begin{aligned}
A_{r,t,1,m} &= \omega_{r,t,1,m} t + \alpha_{r,1,m} + s_{r,x,m} + s_{r,y,m},\\
A_{r,t,2,m} &= \omega_{r,t,2,m} t + \alpha_{r,2,m} + s_{r,x,m} - s_{r,y,m},\\
A_{d,t,1,n} &= \omega_{d,t,1,n} t + \alpha_{d,1,n} + s_{d,x,n} + s_{d,y,n},\\
A_{d,t,2,n} &= \omega_{d,t,2,n} t + \alpha_{d,2,n} + s_{d,x,n} - s_{d,y,n}
\end{aligned}
\tag{72}
$$

with two random frequencies $\omega_{r,t,1,m}, \omega_{r,t,2,m}$ and two deterministic frequencies $\omega_{d,t,1,n}, \omega_{d,t,2,n}$ for $m = 1,2,\cdots,M$ and $n = 1,2,\cdots,M$.

Matrix $\boldsymbol{f}_{r,t,i,m,d,t,j,n}$ of the random-deterministic, external interaction $(n \neq m)$ between eigenfunctions $f_{r,t,i,m}$ and $f_{d,t,j,n}$ becomes

$$
\boldsymbol{f}_{r,t,i,m,d,t,j,n} = \begin{bmatrix}
f_{r,t,1,m} f_{d,t,1,n} & f_{r,t,1,m} f_{d,t,2,n} & f_{r,t,1,m} f_{d,t,3,n} & f_{r,t,1,m} f_{d,t,4,n}\\
f_{r,t,2,m} f_{d,t,1,n} & f_{r,t,2,m} f_{d,t,2,n} & f_{r,t,2,m} f_{d,t,3,n} & f_{r,t,2,m} f_{d,t,4,n}\\
f_{r,t,3,m} f_{d,t,1,n} & f_{r,t,3,m} f_{d,t,2,n} & f_{r,t,3,m} f_{d,t,3,n} & f_{r,t,3,m} f_{d,t,4,n}\\
f_{r,t,4,m} f_{d,t,1,n} & f_{r,t,4,m} f_{d,t,2,n} & f_{r,t,4,m} f_{d,t,3,n} & f_{r,t,4,m} f_{d,t,4,n}
\end{bmatrix}.
\tag{73}
$$

In the view of the product identities, there are 16 random-deterministic, external eigenfunctions $f_{r,d,t,l,m,n}$, namely,

$$\begin{aligned}
f_{r,d,t,1,m,n} &= \sin\left(\mathrm{A}_{r,d,t,1,m,n}\right), & f_{r,d,t,2,m,n} &= \cos\left(\mathrm{A}_{r,d,t,1,m,n}\right),\\
f_{r,d,t,3,m,n} &= \sin\left(\mathrm{A}_{r,d,t,2,m,n}\right), & f_{r,d,t,4,m,n} &= \cos\left(\mathrm{A}_{r,d,t,2,m,n}\right),\\
f_{r,d,t,5,m,n} &= \sin\left(\mathrm{A}_{r,d,t,3,m,n}\right), & f_{r,d,t,6,m,n} &= \cos\left(\mathrm{A}_{r,d,t,3,m,n}\right),\\
f_{r,d,t,7,m,n} &= \sin\left(\mathrm{A}_{r,d,t,4,m,n}\right), & f_{r,d,t,8,m,n} &= \cos\left(\mathrm{A}_{r,d,t,4,m,n}\right),\\
f_{r,d,t,9,m,n} &= \sin\left(\mathrm{A}_{r,d,t,5,m,n}\right), & f_{r,d,t,10,m,n} &= \cos\left(\mathrm{A}_{r,d,t,5,m,n}\right),\\
f_{r,d,t,11,m,n} &= \sin\left(\mathrm{A}_{r,d,t,6,m,n}\right), & f_{r,d,t,12,m,n} &= \cos\left(\mathrm{A}_{r,d,t,6,m,n}\right),\\
f_{r,d,t,13,m,n} &= \sin\left(\mathrm{A}_{r,d,t,7,m,n}\right), & f_{r,d,t,14,m,n} &= \cos\left(\mathrm{A}_{r,d,t,7,m,n}\right),\\
f_{r,d,t,15,m,n} &= \sin\left(\mathrm{A}_{r,d,t,8,m,n}\right), & f_{r,d,t,16,m,n} &= \cos\left(\mathrm{A}_{r,d,t,8,m,n}\right),
\end{aligned} \tag{74}$$

of eight random-deterministic, external arguments $\mathrm{A}_{r,d,t,k,m,n}$, explicitly,

$$\begin{aligned}
\mathrm{A}_{r,d,t,1,m,n} &= \omega_{r,d,t,1,m,n}t + \alpha_{r,1,m} + \alpha_{d,1,n} + s_{r,x,m} + s_{r,y,m} + s_{d,x,n} + s_{d,y,n},\\
\mathrm{A}_{r,d,t,2,m,n} &= \omega_{r,d,t,2,m,n}t + \alpha_{r,1,m} - \alpha_{d,1,n} + s_{r,x,m} + s_{r,y,m} - s_{d,x,n} - s_{d,y,n},\\
\mathrm{A}_{r,d,t,3,m,n} &= \omega_{r,d,t,3,m,n}t + \alpha_{r,1,m} + \alpha_{d,2,n} + s_{r,x,m} + s_{r,y,m} + s_{d,x,n} - s_{d,y,n},\\
\mathrm{A}_{r,d,t,4,m,n} &= \omega_{r,d,t,4,m,n}t + \alpha_{r,1,m} - \alpha_{d,2,n} + s_{r,x,m} + s_{r,y,m} - s_{d,x,n} + s_{d,y,n},\\
\mathrm{A}_{r,d,t,5,m,n} &= \omega_{r,d,t,5,m,n}t + \alpha_{r,2,m} + \alpha_{d,1,n} + s_{r,x,m} - s_{r,y,m} + s_{d,x,n} + s_{d,y,n},\\
\mathrm{A}_{r,d,t,6,m,n} &= \omega_{r,d,t,6,m,n}t + \alpha_{r,2,m} - \alpha_{d,1,n} + s_{r,x,m} - s_{r,y,m} - s_{d,x,n} - s_{d,y,n},\\
\mathrm{A}_{r,d,t,7,m,n} &= \omega_{r,d,t,7,m,n}t + \alpha_{r,2,m} + \alpha_{d,2,n} + s_{r,x,m} - s_{r,y,m} + s_{d,x,n} - s_{d,y,n},\\
\mathrm{A}_{r,d,t,8,m,n} &= \omega_{r,d,t,8,m,n}t + \alpha_{r,2,m} - \alpha_{d,2,n} + s_{r,x,m} - s_{r,y,m} - s_{d,x,n} + s_{d,y,n},
\end{aligned} \tag{75}$$

which are parametrized by eight random-deterministic, external frequencies

$$\begin{aligned}
\omega_{r,d,t,1,m,n} &= \omega_{r,x,m} + \omega_{r,y,m} + \omega_{d,x,n} + \omega_{d,y,n},\\
\omega_{r,d,t,2,m,n} &= \omega_{r,x,m} + \omega_{r,y,m} - \omega_{d,x,n} - \omega_{d,y,n},\\
\omega_{r,d,t,3,m,n} &= \omega_{r,x,m} + \omega_{r,y,m} + \omega_{d,x,n} - \omega_{d,y,n},\\
\omega_{r,d,t,4,m,n} &= \omega_{r,x,m} + \omega_{r,y,m} - \omega_{d,x,n} + \omega_{d,y,n},\\
\omega_{r,d,t,5,m,n} &= \omega_{r,x,m} - \omega_{r,y,m} + \omega_{d,x,n} + \omega_{d,y,n},\\
\omega_{r,d,t,6,m,n} &= \omega_{r,x,m} - \omega_{r,y,m} - \omega_{d,x,n} - \omega_{d,y,n},\\
\omega_{r,d,t,7,m,n} &= \omega_{r,x,m} - \omega_{r,y,m} + \omega_{d,x,n} - \omega_{d,y,n},\\
\omega_{r,d,t,8,m,n} &= \omega_{r,x,m} - \omega_{r,y,m} - \omega_{d,x,n} + \omega_{d,y,n}.
\end{aligned} \tag{76}$$

Random-deterministic, external eigenfunctions $f_{r,d,t,l,m,n}$ contain sine and cosine waves with all combinations of random-deterministic, external *x*-frequencies, *y*-frequencies, *x*-shifts, and *y*-shifts.

Computing elements of matrix $\boldsymbol{f}_{r,t,i,m,d,t,j,n}$ gives the following Fourier expansions in eigenfunctions $f_{r,d,t,l,m,n}$:

$$
\begin{aligned}
2 f_{r,t,1,m} f_{d,t,1,n} &= - f_{r,d,t,2,m,n} + f_{r,d,t,4,m,n}, \\
2 f_{r,t,1,m} f_{d,t,2,n} &= + f_{r,d,t,1,m,n} + f_{r,d,t,3,m,n}, \\
2 f_{r,t,1,m} f_{d,t,3,n} &= - f_{r,d,t,6,m,n} + f_{r,d,t,8,m,n}, \\
2 f_{r,t,1,m} f_{d,t,4,n} &= + f_{r,d,t,5,m,n} + f_{r,d,t,7,m,n}, \\
2 f_{r,t,2,m} f_{d,t,1,n} &= + f_{r,d,t,1,m,n} - f_{r,d,t,3,m,n}, \\
2 f_{r,t,2,m} f_{d,t,2,n} &= + f_{r,d,t,2,m,n} + f_{r,d,t,4,m,n}, \\
2 f_{r,t,2,m} f_{d,t,3,n} &= + f_{r,d,t,5,m,n} - f_{r,d,t,7,m,n}, \\
2 f_{r,t,2,m} f_{d,t,4,n} &= + f_{r,d,t,6,m,n} + f_{r,d,t,8,m,n}, \\
2 f_{r,t,3,m} f_{d,t,1,n} &= - f_{r,d,t,10,m,n} + f_{r,d,t,12,m,n}, \\
2 f_{r,t,3,m} f_{d,t,2,n} &= + f_{r,d,t,9,m,n} + f_{r,d,t,11,m,n}, \\
2 f_{r,t,3,m} f_{d,t,3,n} &= - f_{r,d,t,14,m,n} + f_{r,d,t,16,m,n}, \\
2 f_{r,t,3,m} f_{d,t,4,n} &= + f_{r,d,t,13,m,n} + f_{r,d,t,15,m,n}, \\
2 f_{r,t,4,m} f_{d,t,1,n} &= + f_{r,d,t,9,m,n} - f_{r,d,t,11,m,n}, \\
2 f_{r,t,4,m} f_{d,t,2,n} &= + f_{r,d,t,10,m,n} + f_{r,d,t,12,m,n}, \\
2 f_{r,t,4,m} f_{d,t,3,n} &= + f_{r,d,t,13,m,n} - f_{r,d,t,15,m,n}, \\
2 f_{r,t,4,m} f_{d,t,4,n} &= + f_{r,d,t,14,m,n} + f_{r,d,t,16,m,n}.
\end{aligned} \tag{77}
$$

In agreement with the time-dependent wave parameters and functional amplitudes,

$$\omega_{r,d,t,k,m,n} = \omega_{r,d,t,k,m,n}(t), \quad k = 1, 2, \cdots, 8 \tag{78}$$

are also smooth random functions of time from C^∞ and

$$
\begin{aligned}
A_{r,d,t,k,m,n} &= A_{r,d,t,k,m,n}(x, y, t), \quad k = 1, 2, \cdots, 8, \\
f_{r,d,t,l,m,n} &= f_{r,d,t,l,m,n}(x, y, t), \quad l = 1, 2, \cdots, 16
\end{aligned} \tag{79}
$$

are smooth functions of space and smooth random functions of time from C^∞.

It is a straightforward matter to show that eigenfunctions $f_{d,r,t,l,m,n}$ and $f_{r,d,t,l,m,n}$ coincide up to the changing of index l and sign of eigenfunctions. Consequently, inhomogeneous Fourier expansions in either $f_{d,r,t,l,m,n}$ or $f_{r,d,t,l,m,n}$ are structurally invariant since they will differ only in order and signs of functional amplitudes.

2.4. Eigenfunctions of Deterministic-Random, Internal Interaction

If $n = m$, then eight frequencies $\omega_{d,r,t,k,m,n}$ of $f_{d,r,t,l,m,n}$ are transformed into eight deterministic-random, internal frequencies

$$
\begin{aligned}
\omega_{d,r,t,1,m,m} &= \omega_{d,x,m} + \omega_{d,y,m} + \omega_{r,x,m} + \omega_{r,y,m}, \\
\omega_{d,r,t,2,m,m} &= \omega_{d,x,m} + \omega_{d,y,m} - \omega_{r,x,m} - \omega_{r,y,m}, \\
\omega_{d,r,t,3,m,m} &= \omega_{d,x,m} + \omega_{d,y,m} + \omega_{r,x,m} - \omega_{r,y,m}, \\
\omega_{d,r,t,4,m,m} &= \omega_{d,x,m} + \omega_{d,y,m} - \omega_{r,x,m} + \omega_{r,y,m}, \\
\omega_{d,r,t,5,m,m} &= \omega_{d,x,m} - \omega_{d,y,m} + \omega_{r,x,m} + \omega_{r,y,m}, \\
\omega_{d,r,t,6,m,m} &= \omega_{d,x,m} - \omega_{d,y,m} - \omega_{r,x,m} - \omega_{r,y,m}, \\
\omega_{d,r,t,7,m,m} &= \omega_{d,x,m} - \omega_{d,y,m} + \omega_{r,x,m} - \omega_{r,y,m}, \\
\omega_{d,r,t,8,m,m} &= \omega_{d,x,m} - \omega_{d,y,m} - \omega_{r,x,m} + \omega_{r,y,m}.
\end{aligned} \tag{80}
$$

Therefore, we introduce the following eight deterministic-random, internal arguments $\mathrm{B}_{d,r,t,k,m,m}$, namely,

$$\begin{aligned}
\mathrm{B}_{d,r,t,1,m,m} &= \omega_{d,r,t,1,m,m}t + \alpha_{d,1,m} + \alpha_{r,1,m} + s_{d,x,m} + s_{d,y,m} + s_{r,x,m} + s_{r,y,m},\\
\mathrm{B}_{d,r,t,2,m,m} &= \omega_{d,r,t,2,m,m}t + \alpha_{d,1,m} - \alpha_{r,1,m} + s_{d,x,m} + s_{d,y,m} - s_{r,x,m} - s_{r,y,m},\\
\mathrm{B}_{d,r,t,3,m,m} &= \omega_{d,r,t,3,m,m}t + \alpha_{d,1,m} + \alpha_{r,2,m} + s_{d,x,m} + s_{d,y,m} + s_{r,x,m} - s_{r,y,m},\\
\mathrm{B}_{d,r,t,4,m,m} &= \omega_{d,r,t,4,m,m}t + \alpha_{d,1,m} - \alpha_{r,2,m} + s_{d,x,m} + s_{d,y,m} - s_{r,x,m} + s_{r,y,m},\\
\mathrm{B}_{d,r,t,5,m,m} &= \omega_{d,r,t,5,m,m}t + \alpha_{d,2,m} + \alpha_{r,1,m} + s_{d,x,m} - s_{d,y,m} + s_{r,x,m} + s_{r,y,m},\\
\mathrm{B}_{d,r,t,6,m,m} &= \omega_{d,r,t,6,m,m}t + \alpha_{d,2,m} - \alpha_{r,1,m} + s_{d,x,m} - s_{d,y,m} - s_{r,x,m} - s_{r,y,m},\\
\mathrm{B}_{d,r,t,7,m,m} &= \omega_{d,r,t,7,m,m}t + \alpha_{d,2,m} + \alpha_{r,2,m} + s_{d,x,m} - s_{d,y,m} + s_{r,x,m} - s_{r,y,m},\\
\mathrm{B}_{d,r,t,8,m,m} &= \omega_{d,r,t,8,m,m}t + \alpha_{d,2,m} - \alpha_{r,2,m} + s_{d,x,m} - s_{d,y,m} - s_{r,x,m} + s_{r,y,m}.
\end{aligned} \tag{81}$$

Consequently, 16 deterministic-random, internal eigenfunctions $g_{d,r,t,l,m,m}$ are defined as follows:

$$\begin{aligned}
g_{d,r,t,1,m,m} &= \sin\left(\mathrm{B}_{d,r,t,1,m,m}\right), & g_{d,r,t,2,m,m} &= \cos\left(\mathrm{B}_{d,r,t,1,m,m}\right),\\
g_{d,r,t,3,m,m} &= \sin\left(\mathrm{B}_{d,r,t,2,m,m}\right), & g_{d,r,t,4,m,m} &= \cos\left(\mathrm{B}_{d,r,t,2,m,m}\right),\\
g_{d,r,t,5,m,m} &= \sin\left(\mathrm{B}_{d,r,t,3,m,m}\right), & g_{d,r,t,6,m,m} &= \cos\left(\mathrm{B}_{d,r,t,3,m,m}\right),\\
g_{d,r,t,7,m,m} &= \sin\left(\mathrm{B}_{d,r,t,4,m,m}\right), & g_{d,r,t,8,m,m} &= \cos\left(\mathrm{B}_{d,r,t,4,m,m}\right),\\
g_{d,r,t,9,m,m} &= \sin\left(\mathrm{B}_{d,r,t,5,m,m}\right), & g_{d,r,t,10,m,m} &= \cos\left(\mathrm{B}_{d,r,t,5,m,m}\right),\\
g_{d,r,t,11,m,m} &= \sin\left(\mathrm{B}_{d,r,t,6,m,m}\right), & g_{d,r,t,12,m,m} &= \cos\left(\mathrm{B}_{d,r,t,6,m,m}\right),\\
g_{d,r,t,13,m,m} &= \sin\left(\mathrm{B}_{d,r,t,7,m,m}\right), & g_{d,r,t,14,m,m} &= \cos\left(\mathrm{B}_{d,r,t,7,m,m}\right),\\
g_{d,r,t,15,m,m} &= \sin\left(\mathrm{B}_{d,r,t,8,m,m}\right), & g_{d,r,t,16,m,m} &= \cos\left(\mathrm{B}_{d,r,t,8,m,m}\right).
\end{aligned} \tag{82}$$

Matrix $\boldsymbol{f}_{d,t,i,m,r,t,j,m}$ of the deterministic-random, internal interaction $(n = m)$ between eigenfunctions $f_{d,t,i,m}$ and $f_{r,t,j,m}$ is not a symmetrical matrix, explicitly,

$$\boldsymbol{f}_{d,t,i,m,r,t,j,m} = \begin{bmatrix}
f_{d,t,1,m} f_{r,t,1,m} & f_{d,t,1,m} f_{r,t,2,m} & f_{d,t,1,m} f_{r,t,3,m} & f_{d,t,1,m} f_{r,t,4,m}\\
f_{d,t,2,m} f_{r,t,1,m} & f_{d,t,2,m} f_{r,t,2,m} & f_{d,t,2,m} f_{r,t,3,m} & f_{d,t,2,m} f_{r,t,4,m}\\
f_{d,t,3,m} f_{r,t,1,m} & f_{d,t,3,m} f_{r,t,2,m} & f_{d,t,3,m} f_{r,t,3,m} & f_{d,t,3,m} f_{r,t,4,m}\\
f_{d,t,4,m} f_{r,t,1,m} & f_{d,t,4,m} f_{r,t,2,m} & f_{d,t,4,m} f_{r,t,3,m} & f_{d,t,4,m} f_{r,t,4,m}
\end{bmatrix}. \tag{83}$$

Computation of matrix $\boldsymbol{f}_{d,t,i,m,r,t,j,m}$ via eigenfunctions $g_{d,r,t,l,m,m}$ returns

$$\begin{aligned}
2 f_{d,t,1,m} f_{r,t,1,m} &= -g_{d,r,t,2,m,m} + g_{d,r,t,4,m,m},\\
2 f_{d,t,1,m} f_{r,t,2,m} &= +g_{d,r,t,1,m,m} + g_{d,r,t,3,m,m},\\
2 f_{d,t,1,m} f_{r,t,3,m} &= -g_{d,r,t,6,m,m} + g_{d,r,t,8,m,m},\\
2 f_{d,t,1,m} f_{r,t,4,m} &= +g_{d,r,t,5,m,m} + g_{d,r,t,7,m,m},\\
2 f_{d,t,2,m} f_{r,t,1,m} &= +g_{d,r,t,1,m,m} - g_{d,r,t,3,m,m},\\
2 f_{d,t,2,m} f_{r,t,2,m} &= +g_{d,r,t,2,m,m} + g_{d,r,t,4,m,m},\\
2 f_{d,t,2,m} f_{r,t,3,m} &= +g_{d,r,t,5,m,m} - g_{d,r,t,7,m,m},\\
2 f_{d,t,2,m} f_{r,t,4,m} &= +g_{d,r,t,6,m,m} + g_{d,r,t,8,m,m},
\end{aligned}$$

$$\begin{aligned}
2 f_{d,t,3,m} f_{r,t,1,m} &= -g_{d,r,t,10,m,m} + g_{d,r,t,12,m,m}, \\
2 f_{d,t,3,m} f_{r,t,2,m} &= +g_{d,r,t,9,m,m} + g_{d,r,t,11,m,m}, \\
2 f_{d,t,3,m} f_{r,t,3,m} &= -g_{d,r,t,14,m,m} + g_{d,r,t,16,m,m}, \\
2 f_{d,t,3,m} f_{r,t,4,m} &= +g_{d,r,t,13,m,m} + g_{d,r,t,15,m,m}, \\
2 f_{d,t,4,m} f_{r,t,1,m} &= +g_{d,r,t,9,m,m} - g_{d,r,t,11,m,m}, \\
2 f_{d,t,4,m} f_{r,t,2,m} &= +g_{d,r,t,10,m,m} + g_{d,r,t,12,m,m}, \\
2 f_{d,t,4,m} f_{r,t,3,m} &= +g_{d,r,t,13,m,m} - g_{d,r,t,15,m,m}, \\
2 f_{d,t,4,m} f_{r,t,4,m} &= +g_{d,r,t,14,m,m} + g_{d,r,t,16,m,m}.
\end{aligned} \tag{84}$$

In accordance with the time-dependent wave parameters,

$$\omega_{d,r,t,k,m,m} = \omega_{d,r,t,k,m,m}(t), \ k = 1, 2, \cdots, 8 \tag{85}$$

are also smooth random functions of time from C^∞ and

$$\begin{aligned}
B_{d,r,t,k,m,m} &= B_{d,r,t,k,m,m}(x, y, t), \ k = 1, 2, \cdots, 8, \\
g_{d,r,t,l,m,m} &= g_{d,r,t,l,m,m}(x, y, t), \ l = 1, 2, \cdots, 16
\end{aligned} \tag{86}$$

are smooth functions of space and smooth random functions of time from C^∞.

It may be shown that eigenfunctions $g_{d,r,t,l,m,m}$ and $g_{r,d,t,l,m,m}$ also coincide up to the changing of index l and sign of eigenfunctions. Consequently, inhomogeneous Fourier decompositions in either $g_{d,r,t,l,m,m}$ or $g_{r,d,t,l,m,m}$ are structurally invariant since they will differ only in order and signs of functional amplitudes, as well.

2.5. Eigenfunctions of Random, External Interaction

Matrix $\boldsymbol{f}_{r,t,i,m,r,t,j,n}$ of the random, external interaction $(n \neq m)$ between eigenfunctions $f_{r,t,i,m}$ and $f_{r,t,j,n}$ may be written as follows:

$$\boldsymbol{f}_{r,t,i,m,r,t,j,n} = \begin{bmatrix}
f_{r,t,1,m} f_{r,t,1,n} & f_{r,t,1,m} f_{r,t,2,n} & f_{r,t,1,m} f_{r,t,3,n} & f_{r,t,1,m} f_{r,t,4,n} \\
f_{r,t,2,m} f_{r,t,1,n} & f_{r,t,2,m} f_{r,t,2,n} & f_{r,t,2,m} f_{r,t,3,n} & f_{r,t,2,m} f_{r,t,4,n} \\
f_{r,t,3,m} f_{r,t,1,n} & f_{r,t,3,m} f_{r,t,2,n} & f_{r,t,3,m} f_{r,t,3,n} & f_{r,t,3,m} f_{r,t,4,n} \\
f_{r,t,4,m} f_{r,t,1,n} & f_{r,t,4,m} f_{r,t,2,n} & f_{r,t,4,m} f_{r,t,3,n} & f_{r,t,4,m} f_{r,t,4,n}
\end{bmatrix}. \tag{87}$$

In agreement with the product identities, there are 16 random, external eigenfunctions $f_{r,t,l,m,n}$, specifically,

$$\begin{aligned}
f_{r,t,1,m,n} &= \sin(A_{r,t,1,m,n}), & f_{r,t,2,m,n} &= \cos(A_{r,t,1,m,n}), \\
f_{r,t,3,m,n} &= \sin(A_{r,t,2,m,n}), & f_{r,t,4,m,n} &= \cos(A_{r,t,2,m,n}), \\
f_{r,t,5,m,n} &= \sin(A_{r,t,3,m,n}), & f_{r,t,6,m,n} &= \cos(A_{r,t,3,m,n}), \\
f_{r,t,7,m,n} &= \sin(A_{r,t,4,m,n}), & f_{r,t,8,m,n} &= \cos(A_{r,t,4,m,n}), \\
f_{r,t,9,m,n} &= \sin(A_{r,t,5,m,n}), & f_{r,t,10,m,n} &= \cos(A_{r,t,5,m,n}), \\
f_{r,t,11,m,n} &= \sin(A_{r,t,6,m,n}), & f_{r,t,12,m,n} &= \cos(A_{r,t,6,m,n}), \\
f_{r,t,13,m,n} &= \sin(A_{r,t,7,m,n}), & f_{r,t,14,m,n} &= \cos(A_{r,t,7,m,n}), \\
f_{r,t,15,m,n} &= \sin(A_{r,t,8,m,n}), & f_{r,t,16,m,n} &= \cos(A_{r,t,8,m,n}),
\end{aligned} \tag{88}$$

of eight random, external arguments $\mathrm{A}_{r,t,k,m,n}$, explicitly,

$$\begin{aligned}
\mathrm{A}_{r,t,1,m,n} &= \omega_{r,t,1,m,n}t + \alpha_{r,1,m} + \alpha_{r,1,n} + s_{r,x,m} + s_{r,y,m} + s_{r,x,n} + s_{r,y,n}, \\
\mathrm{A}_{r,t,2,m,n} &= \omega_{r,t,2,m,n}t + \alpha_{r,1,m} - \alpha_{r,1,n} + s_{r,x,m} + s_{r,y,m} - s_{r,x,n} - s_{r,y,n}, \\
\mathrm{A}_{r,t,3,m,n} &= \omega_{r,t,3,m,n}t + \alpha_{r,1,m} + \alpha_{r,2,n} + s_{r,x,m} + s_{r,y,m} + s_{r,x,n} - s_{r,y,n}, \\
\mathrm{A}_{r,t,4,m,n} &= \omega_{r,t,4,m,n}t + \alpha_{r,1,m} - \alpha_{r,2,n} + s_{r,x,m} + s_{r,y,m} - s_{r,x,n} + s_{r,y,n}, \\
\mathrm{A}_{r,t,5,m,n} &= \omega_{r,t,5,m,n}t + \alpha_{r,2,m} + \alpha_{r,1,n} + s_{r,x,m} - s_{r,y,m} + s_{r,x,n} + s_{r,y,n}, \\
\mathrm{A}_{r,t,6,m,n} &= \omega_{r,t,6,m,n}t + \alpha_{r,2,m} - \alpha_{r,1,n} + s_{r,x,m} - s_{r,y,m} - s_{r,x,n} - s_{r,y,n}, \\
\mathrm{A}_{r,t,7,m,n} &= \omega_{r,t,7,m,n}t + \alpha_{r,2,m} + \alpha_{r,2,n} + s_{r,x,m} - s_{r,y,m} + s_{r,x,n} - s_{r,y,n}, \\
\mathrm{A}_{r,t,8,m,n} &= \omega_{r,t,8,m,n}t + \alpha_{r,2,m} - \alpha_{r,2,n} + s_{r,x,m} - s_{r,y,m} - s_{r,x,n} + s_{r,y,n},
\end{aligned} \tag{89}$$

which include eight random, external frequencies

$$\begin{aligned}
\omega_{r,t,1,m,n} &= \omega_{r,x,m} + \omega_{r,y,m} + \omega_{r,x,n} + \omega_{r,y,n}, \\
\omega_{r,t,2,m,n} &= \omega_{r,x,m} + \omega_{r,y,m} - \omega_{r,x,n} - \omega_{r,y,n}, \\
\omega_{r,t,3,m,n} &= \omega_{r,x,m} + \omega_{r,y,m} + \omega_{r,x,n} - \omega_{r,y,n}, \\
\omega_{r,t,4,m,n} &= \omega_{r,x,m} + \omega_{r,y,m} - \omega_{r,x,n} + \omega_{r,y,n}, \\
\omega_{r,t,5,m,n} &= \omega_{r,x,m} - \omega_{r,y,m} + \omega_{r,x,n} + \omega_{r,y,n}, \\
\omega_{r,t,6,m,n} &= \omega_{r,x,m} - \omega_{r,y,m} - \omega_{r,x,n} - \omega_{r,y,n}, \\
\omega_{r,t,7,m,n} &= \omega_{r,x,m} - \omega_{r,y,m} + \omega_{r,x,n} - \omega_{r,y,n}, \\
\omega_{r,t,8,m,n} &= \omega_{r,x,m} - \omega_{r,y,m} - \omega_{r,x,n} + \omega_{r,y,n}.
\end{aligned} \tag{90}$$

Random, external eigenfunctions $f_{r,t,l,m,n}$ include random sine and cosine waves of all combinations of random *x*-frequencies, *y*-frequencies, *x*-shifts, and *y*-shifts.

Computation of matrix $\boldsymbol{f}_{r,t,i,m,r,t,j,n}$ returns the following Fourier expansions in eigenfunctions $f_{r,t,l,m,n}$:

$$\begin{aligned}
&2f_{r,t,1,m}f_{r,t,1,n} = -f_{r,t,2,m,n} + f_{r,t,4,m,n}, \quad 2f_{r,t,1,m}f_{r,t,2,n} = +f_{r,t,1,m,n} + f_{r,t,3,m,n}, \\
&2f_{r,t,1,m}f_{r,t,3,n} = -f_{r,t,6,m,n} + f_{r,t,8,m,n}, \quad 2f_{r,t,1,m}f_{r,t,4,n} = +f_{r,t,5,m,n} + f_{r,t,7,m,n}, \\
&2f_{r,t,2,m}f_{r,t,1,n} = +f_{r,t,1,m,n} - f_{r,t,3,m,n}, \quad 2f_{r,t,2,m}f_{r,t,2,n} = +f_{r,t,2,m,n} + f_{r,t,4,m,n}, \\
&2f_{r,t,2,m}f_{r,t,3,n} = +f_{r,t,5,m,n} - f_{r,t,7,m,n}, \quad 2f_{r,t,2,m}f_{r,t,4,n} = +f_{r,t,6,m,n} + f_{r,t,8,m,n}, \\
&2f_{r,t,3,m}f_{r,t,1,n} = -f_{r,t,10,m,n} + f_{r,t,12,m,n}, \quad 2f_{r,t,3,m}f_{r,t,2,n} = +f_{r,t,9,m,n} + f_{r,t,11,m,n}, \\
&2f_{r,t,3,m}f_{r,t,3,n} = -f_{r,t,14,m,n} + f_{r,t,16,m,n}, \quad 2f_{r,t,3,m}f_{r,t,4,n} = +f_{r,t,13,m,n} + f_{r,t,15,m,n}, \\
&2f_{r,t,4,m}f_{r,t,1,n} = +f_{r,t,9,m,n} - f_{r,t,11,m,n}, \quad 2f_{r,t,4,m}f_{r,t,2,n} = +f_{r,t,10,m,n} + f_{r,t,12,m,n}, \\
&2f_{r,t,4,m}f_{r,t,3,n} = +f_{r,t,13,m,n} - f_{r,t,15,m,n}, \quad 2f_{r,t,4,m}f_{r,t,4,n} = +f_{r,t,14,m,n} + f_{r,t,16,m,n}.
\end{aligned} \tag{91}$$

Replacing index r with d results in the deterministic, external eigenfunctions $f_{d,t,l,m,n}$, see (113)-(120) of [6].

Because of the time-dependent wave parameters and functional amplitudes,

$$\omega_{r,t,k,m,n} = \omega_{r,t,k,m,n}(t), \;\; k = 1,2,\cdots,8 \tag{92}$$

are smooth random functions of time from C^{∞} and

$$\begin{aligned}
\mathrm{A}_{r,t,k,m,n} &= \mathrm{A}_{r,t,k,m,n}(x,y,t), \;\; k = 1,2,\cdots,8, \\
f_{r,t,l,m,n} &= f_{r,t,l,m,n}(x,y,t), \quad l = 1,2,\cdots,16
\end{aligned} \tag{93}$$

are smooth functions of space and smooth random functions of time from C^∞.

2.6. Eigenfunctions of Random, Internal Interaction

If $n = m$, we then define four random, internal frequencies $\omega_{r,t,k,m,m}$, namely,

$$\begin{aligned} \omega_{r,t,1,m,m} &= 2\omega_{r,x,m} + 2\omega_{r,y,m}, \\ \omega_{r,t,2,m,m} &= 2\omega_{r,x,m}, \\ \omega_{r,t,3,m,m} &= 2\omega_{r,y,m}, \\ \omega_{r,t,4,m,m} &= 2\omega_{r,x,m} - 2\omega_{r,y,m}. \end{aligned} \tag{94}$$

Consequently, we introduce four random, internal arguments $\mathrm{B}_{r,t,k,m,m}$ as follows:

$$\begin{aligned} \mathrm{B}_{r,t,1,m,m} &= \omega_{r,t,1,m,m} t + 2\alpha_{r,1,m} + 2s_{r,x,m} + 2s_{r,y,m}, \\ \mathrm{B}_{r,t,2,m,m} &= \omega_{r,t,2,m,m} t + \alpha_{r,1,m} + \alpha_{r,2,m} + 2s_{r,x,m}, \\ \mathrm{B}_{r,t,3,m,m} &= \omega_{r,t,3,m,m} t + \alpha_{r,1,m} - \alpha_{r,2,m} + 2s_{r,y,m}, \\ \mathrm{B}_{r,t,4,m,m} &= \omega_{r,t,4,m,m} t + 2\alpha_{r,2,m} + 2s_{r,x,m} - 2s_{r,y,m}. \end{aligned} \tag{95}$$

Sixteen random, external eigenfunctions $f_{r,t,l,m,n}$ are converted into eight random, internal eigenfunctions $g_{r,t,l,m,m}$, which are set in the following form:

$$\begin{aligned} g_{r,t,1,m,m} &= \sin\left(\mathrm{B}_{r,t,1,m,m}\right), \; g_{r,t,2,m,m} = \cos\left(\mathrm{B}_{r,t,1,m,m}\right), \\ g_{r,t,3,m,m} &= \sin\left(\mathrm{B}_{r,t,2,m,m}\right), \; g_{r,t,4,m,m} = \cos\left(\mathrm{B}_{r,t,2,m,m}\right), \\ g_{r,t,5,m,m} &= \sin\left(\mathrm{B}_{r,t,3,m,m}\right), \; g_{r,t,6,m,m} = \cos\left(\mathrm{B}_{r,t,3,m,m}\right), \\ g_{r,t,7,m,m} &= \sin\left(\mathrm{B}_{r,t,4,m,m}\right), \; g_{r,t,8,m,m} = \cos\left(\mathrm{B}_{r,t,4,m,m}\right). \end{aligned} \tag{96}$$

Matrix $\boldsymbol{f}_{r,t,i,m,r,t,j,m}$ of the random, internal interaction $(n = m)$ between eigenfunctions $f_{r,t,i,m}$ and $f_{r,t,j,m}$ becomes a symmetrical one, explicitly,

$$\boldsymbol{f}_{r,t,i,m,r,t,j,m} = \begin{bmatrix} f_{r,t,1,m}^2 & f_{r,t,1,m} f_{r,t,2,m} & f_{r,t,1,m} f_{r,t,3,m} & f_{r,t,1,m} f_{r,t,4,m} \\ f_{r,t,2,m} f_{r,t,1,m} & f_{r,t,2,m}^2 & f_{r,t,2,m} f_{r,t,3,m} & f_{r,t,2,m} f_{r,t,4,m} \\ f_{r,t,3,m} f_{r,t,1,m} & f_{r,t,3,m} f_{r,t,2,m} & f_{r,t,3,m}^2 & f_{r,t,3,m} f_{r,t,4,m} \\ f_{r,t,4,m} f_{r,t,1,m} & f_{r,t,4,m} f_{r,t,2,m} & f_{r,t,4,m} f_{r,t,3,m} & f_{r,t,4,m}^2 \end{bmatrix}. \tag{97}$$

We then compute the following Fourier expansions of matrix $\boldsymbol{f}_{r,t,i,m,r,t,j,m}$ in $g_{r,t,l,m,m}$:

$$\begin{aligned} &2 f_{r,t,1,m}^2 = -g_{r,t,2,m,m} + 1, \qquad 2 f_{r,t,1,m} f_{r,t,2,m} = g_{r,t,1,m,m}, \\ &2 f_{r,t,1,m} f_{r,t,3,m} = -g_{r,t,4,m,m} + g_{r,t,6,m,m}, \; 2 f_{r,t,1,m} f_{r,t,4,m} = g_{r,t,3,m,m} + g_{r,t,5,m,m}, \\ &2 f_{r,t,2,m}^2 = +g_{r,t,2,m,m} + 1, \\ &2 f_{r,t,2,m} f_{r,t,3,m} = +g_{r,t,3,m,m} - g_{r,t,5,m,m}, \; 2 f_{r,t,2,m} f_{r,t,4,m} = g_{r,t,4,m,m} + g_{r,t,6,m,m}, \\ &2 f_{r,t,3,m}^2 = -g_{r,t,8,m,m} + 1, \qquad 2 f_{r,t,3,m} f_{r,t,4,m} = g_{r,t,7,m,m}, \\ &2 f_{r,t,4,m}^2 = +g_{r,t,8,m,m} + 1. \end{aligned} \tag{98}$$

Replacement of index r with d produces the deterministic, internal eigenfunctions $g_{d,t,l,m,m}$, see (121)-(129) of [6].

Due to to the time-dependent wave parameters,

$$\omega_{r,t,k,m,m} = \omega_{r,t,k,m,m}(t), \; k = 1,2,3,4 \tag{99}$$

are also smooth random functions of time from C^{∞} and

$$\begin{aligned} \mathrm{B}_{r,t,k,m,m} &= \mathrm{B}_{r,t,k,m,m}(x,y,t), \; k = 1,2,3,4, \\ g_{r,t,l,m,m} &= g_{r,t,l,m,m}(x,y,t), \; l = 1,2,\cdots,8 \end{aligned} \tag{100}$$

are smooth functions of space and smooth random functions of time from C^{∞}.

3. Oscillons of Deterministic-Random, External Interaction

3.1. The DREE Oscillons

Because of the identity resonance, 16 deterministic-random, external, elementary oscillons (dree oscillons for shortness, see (194) of [7]) are grouped into two vector dree-t oscillons

$$\boldsymbol{K}_{d,r,e,e,t,q} = \boldsymbol{K}_{d,r,e,e,t,q}\left(\boldsymbol{f}_{d,r,e,e,t,q}\right), \; q = 1,2, \tag{101}$$

which are formed by the following two 8-tuples of the deterministic-random, external, elementary interaction in t:

$$\boldsymbol{f}_{d,r,e,e,t,1} = \left\{ f_{d,r,t,2k-1,m,n} \right\}, \quad \boldsymbol{f}_{d,r,e,e,t,2} = \left\{ f_{d,r,t,2k,m,n} \right\}. \tag{102}$$

Eight-tuple $\boldsymbol{f}_{d,r,e,e,t,1}$ is composed of eight sine waves $f_{d,r,t,2k-1,m,n}$ with frequencies $\omega_{d,r,t,k,m,n}(t)$ and 8-tuple $\boldsymbol{f}_{d,r,e,e,t,2}$ consists of eight cosine waves $f_{d,r,t,2k,m,n}$ with frequencies $\omega_{d,r,t,k,m,n}(t)$ for $k = 1,2,\cdots,8$ and each m, n.

For any frozen $x = x_0, y = y_0, z = z_0$, usage of matrix $\boldsymbol{f}_{d,t,i,m,r,t,j,n}$ gives that the 1st vector dree-t oscillon

$$\begin{aligned} \boldsymbol{K}_{d,r,e,e,t,1} = \big[& K_{o,d,b,m,r,a,n}, K_{o,d,a,m,r,b,n}, K_{o,d,b,m,r,d,n}, K_{o,d,a,m,r,c,n}, \\ & K_{o,d,c,m,r,a,n}, K_{o,d,d,m,r,b,n}, K_{o,d,c,m,r,d,n}, K_{o,d,d,m,r,c,n} \big] \left(\boldsymbol{f}_{d,r,e,e,t,1} \right) \end{aligned} \tag{103}$$

is displayed as a list of eight 8-rf, deterministic-random oscillons in t, which are produced by 8-tuple $\boldsymbol{f}_{d,r,e,e,t,1}$ as

$$\begin{aligned} K_{o,d,b,m,r,a,n} = & -\frac{\rho_c}{8} ez_{d,m} ez_{r,n} \\ & \times \Big[Q_{d,m} Q_{r,n} \left(f_{d,r,t,1,m,n} - f_{d,r,t,3,m,n} \right) - Q_{d,m} R_{r,n} \left(f_{d,r,t,5,m,n} - f_{d,r,t,7,m,n} \right) \\ & - R_{d,m} Q_{r,n} \left(f_{d,r,t,9,m,n} - f_{d,r,t,11,m,n} \right) + R_{d,m} R_{r,n} \left(f_{d,r,t,13,m,n} - f_{d,r,t,15,m,n} \right) \Big], \end{aligned}$$

$$\begin{aligned} K_{o,d,a,m,r,b,n} = & -\frac{\rho_c}{8} ez_{d,m} ez_{r,n} \\ & \times \Big[Q_{d,m} Q_{r,n} \left(f_{d,r,t,1,m,n} + f_{d,r,t,3,m,n} \right) - Q_{d,m} R_{r,n} \left(f_{d,r,t,5,m,n} + f_{d,r,t,7,m,n} \right) \\ & - R_{d,m} Q_{r,n} \left(f_{d,r,t,9,m,n} + f_{d,r,t,11,m,n} \right) + R_{d,m} R_{r,n} \left(f_{d,r,t,13,m,n} + f_{d,r,t,15,m,n} \right) \Big], \end{aligned}$$

$$\begin{aligned} K_{o,d,b,m,r,d,n} = & +\frac{\rho_c}{8} ez_{d,m} ez_{r,n} \\ & \times \Big[Q_{d,m} Q_{r,n} \left(f_{d,r,t,1,m,n} - f_{d,r,t,3,m,n} \right) + Q_{d,m} R_{r,n} \left(f_{d,r,t,5,m,n} - f_{d,r,t,7,m,n} \right) \\ & - R_{d,m} Q_{r,n} \left(f_{d,r,t,9,m,n} - f_{d,r,t,11,m,n} \right) - R_{d,m} R_{r,n} \left(f_{d,r,t,13,m,n} - f_{d,r,t,15,m,n} \right) \Big], \end{aligned}$$

$$K_{o,d,a,m,r,c,n}=-\frac{\rho_c}{8}ez_{d,m}ez_{r,n}$$
$$\times\Big[Q_{d,m}Q_{r,n}\left(f_{d,r,t,1,m,n}+f_{d,r,t,3,m,n}\right)+Q_{d,m}R_{r,n}\left(f_{d,r,t,5,m,n}+f_{d,r,t,7,m,n}\right)$$
$$-R_{d,m}Q_{r,n}\left(f_{d,r,t,9,m,n}+f_{d,r,t,11,m,n}\right)-R_{d,m}R_{r,n}\left(f_{d,r,t,13,m,n}+f_{d,r,t,15,m,n}\right)\Big],$$

$$K_{o,d,c,m,r,a,n}=-\frac{\rho_c}{8}ez_{d,m}ez_{r,n}$$
$$\times\Big[Q_{d,m}Q_{r,n}\left(f_{d,r,t,1,m,n}-f_{d,r,t,3,m,n}\right)-Q_{d,m}R_{r,n}\left(f_{d,r,t,5,m,n}-f_{d,r,t,7,m,n}\right)$$
$$+R_{d,m}Q_{r,n}\left(f_{d,r,t,9,m,n}-f_{d,r,t,11,m,n}\right)-R_{d,m}R_{r,n}\left(f_{d,r,t,13,m,n}-f_{d,r,t,15,m,n}\right)\Big],$$

$$K_{o,d,d,m,r,b,n}=+\frac{\rho_c}{8}ez_{d,m}ez_{r,n}$$
$$\times\Big[Q_{d,m}Q_{r,n}\left(f_{d,r,t,1,m,n}+f_{d,r,t,3,m,n}\right)-Q_{d,m}R_{r,n}\left(f_{d,r,t,5,m,n}+f_{d,r,t,7,m,n}\right)$$
$$+R_{d,m}Q_{r,n}\left(f_{d,r,t,9,m,n}+f_{d,r,t,11,m,n}\right)-R_{d,m}R_{r,n}\left(f_{d,r,t,13,m,n}+f_{d,r,t,15,m,n}\right)\Big],$$

$$K_{o,d,c,m,r,d,n}=+\frac{\rho_c}{8}ez_{d,m}ez_{r,n}$$
$$\times\Big[Q_{d,m}Q_{r,n}\left(f_{d,r,t,1,m,n}-f_{d,r,t,3,m,n}\right)+Q_{d,m}R_{r,n}\left(f_{d,r,t,5,m,n}-f_{d,r,t,7,m,n}\right)$$
$$+R_{d,m}Q_{r,n}\left(f_{d,r,t,9,m,n}-f_{d,r,t,11,m,n}\right)+R_{d,m}R_{r,n}\left(f_{d,r,t,13,m,n}-f_{d,r,t,15,m,n}\right)\Big],$$

$$K_{o,d,d,m,r,c,n}=+\frac{\rho_c}{8}ez_{d,m}ez_{r,n}$$
$$\times\Big[Q_{d,m}Q_{r,n}\left(f_{d,r,t,1,m,n}+f_{d,r,t,3,m,n}\right)+Q_{d,m}R_{r,n}\left(f_{d,r,t,5,m,n}+f_{d,r,t,7,m,n}\right) \quad (104)$$
$$+R_{d,m}Q_{r,n}\left(f_{d,r,t,9,m,n}+f_{d,r,t,11,m,n}\right)+R_{d,m}R_{r,n}\left(f_{d,r,t,13,m,n}+f_{d,r,t,15,m,n}\right)\Big].$$

The 2nd vector dree-t oscillon

$$\boldsymbol{K}_{d,r,e,e,t,2}=\big[K_{o,d,a,m,r,a,n},K_{o,d,b,m,r,b,n},K_{o,d,a,m,r,d,n},K_{o,d,b,m,r,c,n},$$
$$K_{o,d,d,m,r,a,n},K_{o,d,c,m,r,b,n},K_{o,d,d,m,r,d,n},K_{o,d,c,m,r,c,n}\big]\left(\boldsymbol{f}_{d,r,e,e,t,2}\right) \quad (105)$$

is represented by a list of eight 8-rf deterministic-random oscillons in t, which are generated by 8-tuple $\boldsymbol{f}_{d,r,e,e,t,2}$ since

$$K_{o,d,a,m,r,a,n}=-\frac{\rho_c}{8}ez_{d,m}ez_{r,n}$$
$$\times\Big[Q_{d,m}Q_{r,n}\left(f_{d,r,t,2,m,n}-f_{d,r,t,4,m,n}\right)-Q_{d,m}R_{r,n}\left(f_{d,r,t,6,m,n}-f_{d,r,t,8,m,n}\right)$$
$$-R_{d,m}Q_{r,n}\left(f_{d,r,t,10,m,n}-f_{d,r,t,12,m,n}\right)+R_{d,m}R_{r,n}\left(f_{d,r,t,14,m,n}-f_{d,r,t,16,m,n}\right)\Big],$$

$$K_{o,d,b,m,r,b,n}=+\frac{\rho_c}{8}ez_{d,m}ez_{r,n}$$
$$\times\Big[Q_{d,m}Q_{r,n}\left(f_{d,r,t,2,m,n}+f_{d,r,t,4,m,n}\right)-Q_{d,m}R_{r,n}\left(f_{d,r,t,6,m,n}+f_{d,r,t,8,m,n}\right)$$
$$-R_{d,m}Q_{r,n}\left(f_{d,r,t,10,m,n}+f_{d,r,t,12,m,n}\right)+R_{d,m}R_{r,n}\left(f_{d,r,t,14,m,n}+f_{d,r,t,16,m,n}\right)\Big],$$

$$K_{o,d,a,m,r,d,n}=+\frac{\rho_c}{8}ez_{d,m}ez_{r,n}$$
$$\times\Big[Q_{d,m}Q_{r,n}\left(f_{d,r,t,2,m,n}-f_{d,r,t,4,m,n}\right)+Q_{d,m}R_{r,n}\left(f_{d,r,t,6,m,n}-f_{d,r,t,8,m,n}\right)$$
$$-R_{d,m}Q_{r,n}\left(f_{d,r,t,10,m,n}-f_{d,r,t,12,m,n}\right)-R_{d,m}R_{r,n}\left(f_{d,r,t,14,m,n}-f_{d,r,t,16,m,n}\right)\Big],$$

$$K_{o,d,b,m,r,c,n} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,n}$$
$$\times\Big[Q_{d,m} Q_{r,n}\left(f_{d,r,t,2,m,n} + f_{d,r,t,4,m,n}\right) + Q_{d,m} R_{r,n}\left(f_{d,r,t,6,m,n} + f_{d,r,t,8,m,n}\right)$$
$$- R_{d,m} Q_{r,n}\left(f_{d,r,t,10,m,n} + f_{d,r,t,12,m,n}\right) - R_{d,m} R_{r,n}\left(f_{d,r,t,14,m,n} + f_{d,r,t,16,m,n}\right)\Big],$$

$$K_{o,d,d,m,r,a,n} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,n}$$
$$\times\Big[Q_{d,m} Q_{r,n}\left(f_{d,r,t,2,m,n} - f_{d,r,t,4,m,n}\right) - Q_{d,m} R_{r,n}\left(f_{d,r,t,6,m,n} - f_{d,r,t,8,m,n}\right)$$
$$+ R_{d,m} Q_{r,n}\left(f_{d,r,t,10,m,n} - f_{d,r,t,12,m,n}\right) - R_{d,m} R_{r,n}\left(f_{d,r,t,14,m,n} - f_{d,r,t,16,m,n}\right)\Big],$$

$$K_{o,d,c,m,r,b,n} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,n}$$
$$\times\Big[Q_{d,m} Q_{r,n}\left(f_{d,r,t,2,m,n} + f_{d,r,t,4,m,n}\right) - Q_{d,m} R_{r,n}\left(f_{d,r,t,6,m,n} + f_{d,r,t,8,m,n}\right)$$
$$+ R_{d,m} Q_{r,n}\left(f_{d,r,t,10,m,n} + f_{d,r,t,12,m,n}\right) - R_{d,m} R_{r,n}\left(f_{d,r,t,14,m,n} + f_{d,r,t,16,m,n}\right)\Big],$$

$$K_{o,d,d,m,r,d,n} = -\frac{\rho_c}{8} ez_{d,m} ez_{r,n}$$
$$\times\Big[Q_{d,m} Q_{r,n}\left(f_{d,r,t,2,m,n} - f_{d,r,t,4,m,n}\right) + Q_{d,m} R_{r,n}\left(f_{d,r,t,6,m,n} - f_{d,r,t,8,m,n}\right)$$
$$+ R_{d,m} Q_{r,n}\left(f_{d,r,t,10,m,n} - f_{d,r,t,12,m,n}\right) + R_{d,m} R_{r,n}\left(f_{d,r,t,14,m,n} - f_{d,r,t,16,m,n}\right)\Big],$$

$$K_{o,d,c,m,r,c,n} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,n}$$
$$\times\Big[Q_{d,m} Q_{r,n}\left(f_{d,r,t,2,m,n} + f_{d,r,t,4,m,n}\right) + Q_{d,m} R_{r,n}\left(f_{d,r,t,6,m,n} + f_{d,r,t,8,m,n}\right) \quad (106)$$
$$+ R_{d,m} Q_{r,n}\left(f_{d,r,t,10,m,n} + f_{d,r,t,12,m,n}\right) + R_{d,m} R_{r,n}\left(f_{d,r,t,14,m,n} + f_{d,r,t,16,m,n}\right)\Big].$$

For all vector dree-t oscillons, amplitudes of eigenfunctions depend on $\rho_c, Q_{d,m}, R_{d,m}, z_0, t$ via $ez_{d,m}(z_0), ez_{r,n}(z_0), Q_{r,n}(t), R_{r,n}(t)$ and temporal shifts of eigenfunctions are determined by $\alpha_{d,1,m}, \alpha_{d,2,m}, x_0, y_0, t$ through $s_{d,x,m}(x_0), s_{d,y,m}(y_0), s_{r,x,n}(x_0,t), s_{r,y,n}(y_0,t), \omega_{d,r,t,k,m,n}(t), \alpha_{r,1,n}(t), \alpha_{r,2,n}(t)$. The dree-$t$ oscillons are smooth random functions of time with an unbounded period.

3.2. The RDEE Oscillons

Due to the identity resonance, 16 random-deterministic, external, elementary oscillons (rdee-oscillons for concision, see (195) of [7]) are combined into two vector rdee-t oscillons

$$\boldsymbol{K}_{r,d,e,e,t,q} = \boldsymbol{K}_{r,d,e,e,t,q}\left(\boldsymbol{f}_{r,d,e,e,t,q}\right), \; q = 1,2, \quad (107)$$

which are produced by two 8-tuples of the random-deterministic, external, elementary interaction in t:

$$\boldsymbol{f}_{r,d,e,e,t,1} = \left\{f_{r,d,t,2k-1,m,n}\right\}, \quad \boldsymbol{f}_{r,d,e,e,t,2} = \left\{f_{r,d,t,2k,m,n}\right\}. \quad (108)$$

Eight-tuple $\boldsymbol{f}_{r,d,e,e,t,1}$ is constructed of eight sine waves $f_{r,d,t,2k-1,m,n}$ with frequencies $\omega_{r,d,t,k,m,n}(t)$ and 8-tuple $\boldsymbol{f}_{r,d,e,e,t,2}$ comprises eight cosine waves $f_{r,d,t,2k,m,n}$ with frequencies $\omega_{r,d,t,k,m,n}(t)$ for $k = 1,2,\cdots,8$ and each m, n.

For any frozen $x = x_0, y = y_0, z = z_0$, we use matrix $\boldsymbol{f}_{r,t,i,m,d,t,j,n}$ to compute that the 1st vector rdee-t oscillon

$$\boldsymbol{K}_{r,d,e,e,t,1} = \left[K_{o,r,b,m,d,a,n}, K_{o,r,a,m,d,b,n}, K_{o,r,b,m,d,d,n}, K_{o,r,a,m,d,c,n}, K_{o,r,c,m,d,a,n}, K_{o,r,d,m,d,b,n}, K_{o,r,c,m,d,d,n}, K_{o,r,d,m,d,c,n} \right] \left(\boldsymbol{f}_{r,d,e,e,t,1} \right) \tag{109}$$

is exposed as a list of eight 8-rf, random-deterministic oscillons in t, which depend on by 8-tuple $\boldsymbol{f}_{r,d,e,e,t,1}$ in the view of

$$\begin{aligned}
K_{o,r,b,m,d,a,n} &= -\frac{\rho_c}{8} ez_{r,m} ez_{d,n} \\
&\times \Big[Q_{r,m} Q_{d,n} \left(f_{r,d,t,1,m,n} - f_{r,d,t,3,m,n} \right) - Q_{r,m} R_{d,n} \left(f_{r,d,t,5,m,n} - f_{r,d,t,7,m,n} \right) \\
&- R_{r,m} Q_{d,n} \left(f_{r,d,t,9,m,n} - f_{r,d,t,11,m,n} \right) + R_{r,m} R_{d,n} \left(f_{r,d,t,13,m,n} - f_{r,d,t,15,m,n} \right) \Big], \\
K_{o,r,a,m,d,b,n} &= -\frac{\rho_c}{8} ez_{r,m} ez_{d,n} \\
&\times \Big[Q_{r,m} Q_{d,n} \left(f_{r,d,t,1,m,n} + f_{r,d,t,3,m,n} \right) - Q_{r,m} R_{d,n} \left(f_{r,d,t,5,m,n} + f_{r,d,t,7,m,n} \right) \\
&- R_{r,m} Q_{d,n} \left(f_{r,d,t,9,m,n} + f_{r,d,t,11,m,n} \right) + R_{r,m} R_{d,n} \left(f_{r,d,t,13,m,n} + f_{r,d,t,15,m,n} \right) \Big], \\
K_{o,r,b,m,d,d,n} &= +\frac{\rho_c}{8} ez_{r,m} ez_{d,n} \\
&\times \Big[Q_{r,m} Q_{d,n} \left(f_{r,d,t,1,m,n} - f_{r,d,t,3,m,n} \right) + Q_{r,m} R_{d,n} \left(f_{r,d,t,5,m,n} - f_{r,d,t,7,m,n} \right) \\
&- R_{r,m} Q_{d,n} \left(f_{r,d,t,9,m,n} - f_{r,d,t,11,m,n} \right) - R_{r,m} R_{r,n} \left(f_{r,d,t,13,m,n} - f_{r,d,t,15,m,n} \right) \Big], \\
K_{o,r,a,m,d,c,n} &= -\frac{\rho_c}{8} ez_{r,m} ez_{d,n} \\
&\times \Big[Q_{r,m} Q_{d,n} \left(f_{r,d,t,1,m,n} + f_{r,d,t,3,m,n} \right) + Q_{r,m} R_{d,n} \left(f_{r,d,t,5,m,n} + f_{r,d,t,7,m,n} \right) \\
&- R_{r,m} Q_{d,n} \left(f_{r,d,t,9,m,n} + f_{r,d,t,11,m,n} \right) - R_{r,m} R_{d,n} \left(f_{r,d,t,13,m,n} + f_{r,d,t,15,m,n} \right) \Big], \\
K_{o,r,c,m,d,a,n} &= -\frac{\rho_c}{8} ez_{r,m} ez_{d,n} \\
&\times \Big[Q_{r,m} Q_{d,n} \left(f_{r,d,t,1,m,n} - f_{r,d,t,3,m,n} \right) - Q_{r,m} R_{d,n} \left(f_{r,d,t,5,m,n} - f_{r,d,t,7,m,n} \right) \\
&+ R_{r,m} Q_{d,n} \left(f_{r,d,t,9,m,n} - f_{r,d,t,11,m,n} \right) - R_{r,m} R_{d,n} \left(f_{r,d,t,13,m,n} - f_{r,d,t,15,m,n} \right) \Big], \\
K_{o,r,d,m,d,b,n} &= +\frac{\rho_c}{8} ez_{r,m} ez_{d,n} \\
&\times \Big[Q_{r,m} Q_{d,n} \left(f_{r,d,t,1,m,n} + f_{r,d,t,3,m,n} \right) - Q_{r,m} R_{d,n} \left(f_{r,d,t,5,m,n} + f_{r,d,t,7,m,n} \right) \\
&+ R_{r,m} Q_{d,n} \left(f_{r,d,t,9,m,n} + f_{r,d,t,11,m,n} \right) - R_{r,m} R_{d,n} \left(f_{r,d,t,13,m,n} + f_{r,d,t,15,m,n} \right) \Big], \\
K_{o,r,c,m,d,d,n} &= +\frac{\rho_c}{8} ez_{r,m} ez_{d,n} \\
&\times \Big[Q_{r,m} Q_{d,n} \left(f_{r,d,t,1,m,n} - f_{r,d,t,3,m,n} \right) + Q_{r,m} R_{d,n} \left(f_{r,d,t,5,m,n} - f_{r,d,t,7,m,n} \right) \\
&+ R_{r,m} Q_{d,n} \left(f_{r,d,t,9,m,n} - f_{r,d,t,11,m,n} \right) + R_{r,m} R_{d,n} \left(f_{r,d,t,13,m,n} - f_{r,d,t,15,m,n} \right) \Big], \\
K_{o,r,d,m,d,c,n} &= +\frac{\rho_c}{8} ez_{r,m} ez_{d,n} \\
&\times \Big[Q_{r,m} Q_{d,n} \left(f_{r,d,t,1,m,n} + f_{r,d,t,3,m,n} \right) + Q_{r,m} R_{d,n} \left(f_{r,d,t,5,m,n} + f_{r,d,t,7,m,n} \right) \\
&+ R_{r,m} Q_{d,n} \left(f_{r,d,t,9,m,n} + f_{r,d,t,11,m,n} \right) + R_{r,m} R_{d,n} \left(f_{r,d,t,13,m,n} + f_{r,d,t,15,m,n} \right) \Big].
\end{aligned} \tag{110}$$

The 2nd vector rdee-t oscillon

$$\boldsymbol{K}_{r,d,e,e,t,2}=\left[K_{o,r,a,m,d,a,n},K_{o,r,b,m,d,b,n},K_{o,r,a,m,d,d,n},K_{o,r,b,m,d,c,n},\right.$$
$$\left.K_{o,r,d,m,d,a,n},K_{o,r,c,m,d,b,n},K_{o,r,d,m,d,d,n},K_{o,r,c,m,d,c,n}\right]\left(\boldsymbol{f}_{r,d,e,e,t,2}\right) \quad (111)$$

is exhibited by a list of eight 8-rf, random-deterministic oscillons in t, which are created by 8-tuple $\boldsymbol{f}_{r,d,e,e,t,2}$ because

$$K_{o,r,a,m,d,a,n}=-\frac{\rho_c}{8}ez_{r,m}ez_{d,n}$$
$$\times\left[Q_{r,m}Q_{d,n}\left(f_{r,d,t,2,m,n}-f_{r,d,t,4,m,n}\right)-Q_{r,m}R_{d,n}\left(f_{r,d,t,6,m,n}-f_{r,d,t,8,m,n}\right)\right.$$
$$\left.-R_{r,m}Q_{d,n}\left(f_{r,d,t,10,m,n}-f_{r,d,t,12,m,n}\right)+R_{r,m}R_{d,n}\left(f_{r,d,t,14,m,n}-f_{r,d,t,16,m,n}\right)\right],$$

$$K_{o,r,b,m,d,b,n}=+\frac{\rho_c}{8}ez_{r,m}ez_{d,n}$$
$$\times\left[Q_{r,m}Q_{d,n}\left(f_{r,d,t,2,m,n}+f_{r,d,t,4,m,n}\right)-Q_{r,m}R_{d,n}\left(f_{r,d,t,6,m,n}+f_{r,d,t,8,m,n}\right)\right.$$
$$\left.-R_{r,m}Q_{d,n}\left(f_{r,d,t,10,m,n}+f_{r,d,t,12,m,n}\right)+R_{r,m}R_{r,n}\left(f_{r,d,t,14,m,n}+f_{r,d,t,16,m,n}\right)\right],$$

$$K_{o,r,a,m,d,d,n}=+\frac{\rho_c}{8}ez_{r,m}ez_{d,n}$$
$$\times\left[Q_{r,m}Q_{d,n}\left(f_{r,d,t,2,m,n}-f_{r,d,t,4,m,n}\right)+Q_{r,m}R_{d,n}\left(f_{r,d,t,6,m,n}-f_{r,d,t,8,m,n}\right)\right.$$
$$\left.-R_{r,m}Q_{d,n}\left(f_{r,d,t,10,m,n}-f_{r,d,t,12,m,n}\right)-R_{r,m}R_{d,n}\left(f_{r,d,t,14,m,n}-f_{r,d,t,16,m,n}\right)\right],$$

$$K_{o,r,b,m,d,c,n}=+\frac{\rho_c}{8}ez_{r,m}ez_{d,n}$$
$$\times\left[Q_{r,m}Q_{d,n}\left(f_{r,d,t,2,m,n}+f_{r,d,t,4,m,n}\right)+Q_{r,m}R_{d,n}\left(f_{r,d,t,6,m,n}+f_{r,d,t,8,m,n}\right)\right.$$
$$\left.-R_{r,m}Q_{d,n}\left(f_{r,d,t,10,m,n}+f_{r,d,t,12,m,n}\right)-R_{r,m}R_{r,n}\left(f_{r,d,t,14,m,n}+f_{r,d,t,16,m,n}\right)\right],$$

$$K_{o,r,d,m,d,a,n}=+\frac{\rho_c}{8}ez_{r,m}ez_{d,n}$$
$$\times\left[Q_{r,m}Q_{d,n}\left(f_{r,d,t,2,m,n}-f_{r,d,t,4,m,n}\right)-Q_{r,m}R_{d,n}\left(f_{r,d,t,6,m,n}-f_{r,d,t,8,m,n}\right)\right.$$
$$\left.+R_{r,m}Q_{d,n}\left(f_{r,d,t,10,m,n}-f_{r,d,t,12,m,n}\right)-R_{r,m}R_{d,n}\left(f_{r,d,t,14,m,n}-f_{r,d,t,16,m,n}\right)\right],$$

$$K_{o,r,c,m,d,b,n}=+\frac{\rho_c}{8}ez_{r,m}ez_{d,n}$$
$$\times\left[Q_{r,m}Q_{d,n}\left(f_{r,d,t,2,m,n}+f_{r,d,t,4,m,n}\right)-Q_{r,m}R_{d,n}\left(f_{r,d,t,6,m,n}+f_{r,d,t,8,m,n}\right)\right.$$
$$\left.+R_{r,m}Q_{d,n}\left(f_{r,d,t,10,m,n}+f_{r,d,t,12,m,n}\right)-R_{r,m}R_{d,n}\left(f_{r,d,t,14,m,n}+f_{r,d,t,16,m,n}\right)\right],$$

$$K_{o,r,d,m,d,d,n}=-\frac{\rho_c}{8}ez_{r,m}ez_{d,n}$$
$$\times\left[Q_{r,m}Q_{d,n}\left(f_{r,d,t,2,m,n}-f_{r,d,t,4,m,n}\right)+Q_{r,m}R_{d,n}\left(f_{r,d,t,6,m,n}-f_{r,d,t,8,m,n}\right)\right.$$
$$\left.+R_{r,m}Q_{d,n}\left(f_{r,d,t,10,m,n}-f_{r,d,t,12,m,n}\right)+R_{r,m}R_{d,n}\left(f_{r,d,t,14,m,n}-f_{r,d,t,16,m,n}\right)\right],$$

$$K_{o,r,c,m,d,c,n}=+\frac{\rho_c}{8}ez_{r,m}ez_{d,n}$$
$$\times\left[Q_{r,m}Q_{d,n}\left(f_{r,d,t,2,m,n}+f_{r,d,t,4,m,n}\right)+Q_{r,m}R_{d,n}\left(f_{r,d,t,6,m,n}+f_{r,d,t,8,m,n}\right)\right. \quad (112)$$
$$\left.+R_{r,m}Q_{d,n}\left(f_{r,d,t,10,m,n}+f_{r,d,t,12,m,n}\right)+R_{r,m}R_{d,n}\left(f_{r,d,t,14,m,n}+f_{r,d,t,16,m,n}\right)\right].$$

For all vector rdee-t oscillons, amplitudes of eigenfunctions are determined by $\rho_c, Q_{d,n}, R_{d,n}, z_0, t$ via $ez_{d,n}(z_0), ez_{r,m}(z_0), Q_{r,m}(t), R_{r,m}(t)$ temporal shifts of eigenfunctions are governed by $\alpha_{d,1,n}, \alpha_{d,2,n}, x_0, y_0, t$ through $s_{d,x,n}(x_0)$, $s_{d,y,n}(y_0), s_{r,x,m}(x_0,t), s_{r,y,m}(y_0,t), \omega_{r,d,t,k,m,n}(t), \alpha_{r,1,m}(t), \alpha_{r,2,m}(t)$. Similar to the dree-$t$ oscillons, the rdee-t oscillons are smooth random functions of time with an unbounded period.

3.3. The DREW Oscillons

We use the decomposition of deterministic-random, external, wave oscillons (drew oscillons for conciseness, see (197 of [7]) via the dree and rdee oscillons and substitute the inhomogeneous Fourier expansion of the vector dree-t and rdee-t oscillons to find that 16 drew oscillons are grouped into two vector drew-t oscillons

$$\boldsymbol{K}_{d,r,e,w,t,q} = \boldsymbol{K}_{d,r,e,w,t,q}\left(\boldsymbol{f}_{d,r,e,w,t,q}\right),\ q = 1,2, \tag{113}$$

which are formed by two 16-tuples of the deterministic-random, external wave interaction in t:

$$\boldsymbol{f}_{d,r,e,w,t,1} = \left\{ f_{d,r,t,2k-1,m,n}, f_{r,d,t,2k-1,m,n} \right\},\ \boldsymbol{f}_{d,r,e,w,t,2} = \left\{ f_{d,r,t,2k,m,n}, f_{r,d,t,2k,m,n} \right\}. \tag{114}$$

Sixteen-tuple $\boldsymbol{f}_{d,r,e,w,t,1}$ is constructed of eight sine waves $f_{d,r,t,2k-1,m,n}$ with frequencies $\omega_{d,r,t,k,m,n}(t)$ and eight sine waves $f_{r,d,t,2k-1,m,n}$ with frequencies $\omega_{r,d,t,k,m,n}(t)$ for $k = 1,2,\cdots,8$ and each m, n. Sixteen-tuple $\boldsymbol{f}_{d,r,e,w,t,2}$ consists of eight cosine waves $f_{d,r,t,2k,m,n}$ with frequencies $\omega_{d,r,t,k,m,n}(t)$ and eight cosine waves $f_{r,d,t,2k,m,n}$ with frequencies $\omega_{r,d,t,k,m,n}(t)$ for $k = 1,2,\cdots,8$ and each m, n.

Due to the identity resonance of the vector dree-t and rdee-t oscillons for any frozen $x = x_0, y = y_0, z = z_0$, the 1st vector drew-$t$ oscillon

$$\begin{aligned} \boldsymbol{K}_{d,r,e,w,t,1} = \big[& K_{w,d,b,m,r,a,n}, K_{w,d,a,m,r,b,n}, K_{w,d,b,m,r,d,n}, K_{w,d,a,m,r,c,n}, \\ & K_{w,d,c,m,r,a,n}, K_{w,d,d,m,r,b,n}, K_{w,d,c,m,r,d,n}, K_{w,d,d,m,r,c,n} \big]\left(\boldsymbol{f}_{d,r,e,w,t,1}\right) \end{aligned} \tag{115}$$

is visualized by a list of eight 16-rf, deterministic-random, random-deterministic oscillons in t, which are produced by 16-tuple $\boldsymbol{f}_{d,r,e,w,t,1}$ as

$$\begin{aligned} & K_{w,d,b,m,r,a,n} = \\ & +\frac{\rho_c}{8}\Big\{ ez_{d,m} ez_{r,n} \Big[Q_{d,m} Q_{r,n} \left(\mathrm{N}_{d,m,r,n} f_{d,r,t,1,m,n} + \mathrm{M}_{d,m,r,n} f_{d,r,t,3,m,n} \right) \\ & \qquad + Q_{d,m} R_{r,n} \left(\Lambda_{d,m,r,n} f_{d,r,t,5,m,n} - \mathrm{K}_{d,m,r,n} f_{d,r,t,7,m,n} \right) \\ & \qquad + R_{d,m} Q_{r,n} \left(\Lambda_{d,m,r,n} f_{d,r,t,9,m,n} - \mathrm{K}_{d,m,r,n} f_{d,r,t,11,m,n} \right) \\ & \qquad + R_{d,m} R_{r,n} \left(\mathrm{N}_{d,m,r,n} f_{d,r,t,13,m,n} + \mathrm{M}_{d,m,r,n} f_{d,r,t,15,m,n} \right) \Big] \\ & \quad + ez_{r,m} ez_{d,n} \Big[Q_{r,m} Q_{d,n} \left(\mathrm{N}_{r,m,d,n} f_{r,d,t,1,m,n} - \mathrm{M}_{r,m,d,n} f_{r,d,t,3,m,n} \right) \\ & \qquad + Q_{r,m} R_{d,n} \left(\Lambda_{r,m,d,n} f_{r,d,t,5,m,n} + \mathrm{K}_{r,m,d,n} f_{r,d,t,7,m,n} \right) \\ & \qquad + R_{r,m} Q_{d,n} \left(\Lambda_{r,m,d,n} f_{r,d,t,9,m,n} + \mathrm{K}_{r,m,d,n} f_{r,d,t,11,m,n} \right) \\ & \qquad + R_{r,m} R_{d,n} \left(\mathrm{N}_{r,m,d,n} f_{r,d,t,13,m,n} - \mathrm{M}_{r,m,d,n} f_{r,d,t,15,m,n} \right) \Big] \Big\}, \end{aligned}$$

$$
\begin{aligned}
&K_{w,d,a,m,r,b,n} = \\
&+\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,1,m,n}-\mathrm{M}_{d,m,r,n}\,f_{d,r,t,3,m,n}\right)\\
&\qquad+Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,5,m,n}+\mathrm{K}_{d,m,r,n}\,f_{d,r,t,7,m,n}\right)\\
&\qquad+R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,9,m,n}+\mathrm{K}_{d,m,r,n}\,f_{d,r,t,11,m,n}\right)\\
&\qquad+R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,13,m,n}-\mathrm{M}_{d,m,r,n}\,f_{d,r,t,15,m,n}\right)\Big]\\
&\quad+ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,1,m,n}+\mathrm{M}_{r,m,d,n}\,f_{r,d,t,3,m,n}\right)\\
&\qquad+Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,5,m,n}-\mathrm{K}_{r,m,d,n}\,f_{r,d,t,7,m,n}\right)\\
&\qquad+R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,9,m,n}-\mathrm{K}_{r,m,d,n}\,f_{r,d,t,11,m,n}\right)\\
&\qquad+R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,13,m,n}+\mathrm{M}_{r,m,d,n}\,f_{r,d,t,15,m,n}\right)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&K_{w,d,b,m,r,d,n} = \\
&-\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,1,m,n}+\mathrm{M}_{d,m,r,n}\,f_{d,r,t,3,m,n}\right)\\
&\qquad-Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,5,m,n}-\mathrm{K}_{d,m,r,n}\,f_{d,r,t,7,m,n}\right)\\
&\qquad+R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,9,m,n}-\mathrm{K}_{d,m,r,n}\,f_{d,r,t,11,m,n}\right)\\
&\qquad-R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,13,m,n}+\mathrm{M}_{d,m,r,n}\,f_{d,r,t,15,m,n}\right)\Big]\\
&\quad+ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,1,m,n}-\mathrm{M}_{r,m,d,n}\,f_{r,d,t,3,m,n}\right)\\
&\qquad+Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,5,m,n}+\mathrm{K}_{r,m,d,n}\,f_{r,d,t,7,m,n}\right)\\
&\qquad-R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,9,m,n}+\mathrm{K}_{r,m,d,n}\,f_{r,d,t,11,m,n}\right)\\
&\qquad-R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,13,m,n}-\mathrm{M}_{r,m,d,n}\,f_{r,d,t,15,m,n}\right)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&K_{w,d,a,m,r,c,n} = \\
&+\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,1,m,n}-\mathrm{M}_{d,m,r,n}\,f_{d,r,t,3,m,n}\right)\\
&\qquad-Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,5,m,n}+\mathrm{K}_{d,m,r,n}\,f_{d,r,t,7,m,n}\right)\\
&\qquad+R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,9,m,n}+\mathrm{K}_{d,m,r,n}\,f_{d,r,t,11,m,n}\right)\\
&\qquad-R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,13,m,n}-\mathrm{M}_{d,m,r,n}\,f_{d,r,t,15,m,n}\right)\Big]\\
&\quad+ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,1,m,n}+\mathrm{M}_{r,m,d,n}\,f_{r,d,t,3,m,n}\right)\\
&\qquad+Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,5,m,n}-\mathrm{K}_{r,m,d,n}\,f_{r,d,t,7,m,n}\right)\\
&\qquad-R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,9,m,n}-\mathrm{K}_{r,m,d,n}\,f_{r,d,t,11,m,n}\right)\\
&\qquad-R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,13,m,n}+\mathrm{M}_{r,m,d,n}\,f_{r,d,t,15,m,n}\right)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&K_{w,d,c,m,r,a,n} = \\
&+\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,1,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,3,m,n}\right) \\
&\qquad + Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,5,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,7,m,n}\right) \\
&\qquad - R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,9,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,11,m,n}\right) \\
&\qquad - R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,13,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,15,m,n}\right)\Big] \\
&\quad + ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,1,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,3,m,n}\right) \\
&\qquad - Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,5,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,7,m,n}\right) \\
&\qquad + R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,9,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,11,m,n}\right) \\
&\qquad - R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,13,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,15,m,n}\right)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&K_{w,d,d,m,r,b,n} = \\
&-\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,1,m,n} - \mathrm{M}_{d,m,r,n}\,f_{d,r,t,3,m,n}\right) \\
&\qquad + Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,5,m,n} + \mathrm{K}_{d,m,r,n}\,f_{d,r,t,7,m,n}\right) \\
&\qquad - R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,9,m,n} + \mathrm{K}_{d,m,r,n}\,f_{d,r,t,11,m,n}\right) \\
&\qquad - R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,13,m,n} - \mathrm{M}_{d,m,r,n}\,f_{d,r,t,15,m,n}\right)\Big] \\
&\quad + ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,1,m,n} + \mathrm{M}_{r,m,d,n}\,f_{r,d,t,3,m,n}\right) \\
&\qquad - Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,5,m,n} - \mathrm{K}_{r,m,d,n}\,f_{r,d,t,7,m,n}\right) \\
&\qquad + R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,9,m,n} - \mathrm{K}_{r,m,d,n}\,f_{r,d,t,11,m,n}\right) \\
&\qquad - R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,13,m,n} + \mathrm{M}_{r,m,d,n}\,f_{r,d,t,15,m,n}\right)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&K_{w,d,c,m,r,d,n} = \\
&-\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,1,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,3,m,n}\right) \\
&\qquad - Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,5,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,7,m,n}\right) \\
&\qquad - R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,9,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,11,m,n}\right) \\
&\qquad + R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,13,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,15,m,n}\right)\Big] \\
&\quad + ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,1,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,3,m,n}\right) \\
&\qquad - Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,5,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,7,m,n}\right) \\
&\qquad - R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,9,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,11,m,n}\right) \\
&\qquad + R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,13,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,15,m,n}\right)\Big]\Big\},
\end{aligned}
$$

$$K_{w,d,d,m,r,c,n} =$$
$$-\frac{\rho_c}{8}\{ez_{d,m}ez_{r,n}\left[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,1,m,n}\;-\mathrm{M}_{d,m,r,n}\,f_{d,r,t,3,m,n}\right)\right.$$
$$-\,Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,5,m,n}\;+\mathrm{K}_{d,m,r,n}\,f_{d,r,t,7,m,n}\right)$$
$$-\,R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,9,m,n}\;+\mathrm{K}_{d,m,r,n}\,f_{d,r,t,11,m,n}\right)$$
$$\left.+\,R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,13,m,n}-\mathrm{M}_{d,m,r,n}\,f_{d,r,t,15,m,n}\right)\right] \tag{116}$$
$$+\,ez_{r,m}ez_{d,n}\left[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,1,m,n}\;+\mathrm{M}_{r,m,d,n}\,f_{r,d,t,3,m,n}\right)\right.$$
$$-\,Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,5,m,n}\;-\mathrm{K}_{r,m,d,n}\,f_{r,d,t,7,m,n}\right)$$
$$-\,R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,9,m,n}\;-\mathrm{K}_{r,m,d,n}\,f_{r,d,t,11,m,n}\right)$$
$$\left.+\,R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,13,m,n}+\mathrm{M}_{r,m,d,n}\,f_{r,d,t,15,m,n}\right)\right]\}.$$

The 2nd vector drew-t oscillon

$$\boldsymbol{K}_{d,r,e,w,t,2} = \left[K_{w,d,a,m,r,a,n}, K_{w,d,b,m,r,b,n}, K_{w,d,a,m,r,d,n}, K_{w,d,b,m,r,c,n},\right.$$
$$\left.K_{w,d,d,m,r,a,n}, K_{w,d,c,m,r,b,n}, K_{w,d,d,m,r,d,n}, K_{w,d,c,m,r,c,n}\right]\left(\boldsymbol{f}_{d,r,e,w,t,2}\right) \tag{117}$$

is presented by a list of eight 16-rf, deterministic-random, random-deterministic oscillons in t, which are generated by 16-tuple $\boldsymbol{f}_{d,r,e,w,t,2}$ because

$$K_{w,d,a,m,r,a,n} =$$
$$+\frac{\rho_c}{8}\{ez_{d,m}ez_{r,n}\left[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,2,m,n}\;+\mathrm{M}_{d,m,r,n}\,f_{d,r,t,4,m,n}\right)\right.$$
$$+\,Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,6,m,n}\;-\mathrm{K}_{d,m,r,n}\,f_{d,r,t,8,m,n}\right)$$
$$+\,R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,10,m,n}-\mathrm{K}_{d,m,r,n}\,f_{d,r,t,12,m,n}\right)$$
$$\left.+\,R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,14,m,n}+\mathrm{M}_{d,m,r,n}\,f_{d,r,t,16,m,n}\right)\right]$$
$$+\,ez_{r,m}ez_{d,n}\left[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,2,m,n}\;+\mathrm{M}_{r,m,d,n}\,f_{r,d,t,4,m,n}\right)\right.$$
$$+\,Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,6,m,n}\;-\mathrm{K}_{r,m,d,n}\,f_{r,d,t,8,m,n}\right)$$
$$+\,R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,10,m,n}-\mathrm{K}_{r,m,d,n}\,f_{r,d,t,12,m,n}\right)$$
$$\left.+\,R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,14,m,n}+\mathrm{M}_{r,m,d,n}\,f_{r,d,t,16,m,n}\right)\right]\},$$

$$K_{w,d,b,m,r,b,n} =$$
$$-\frac{\rho_c}{8}\{ez_{d,m}ez_{r,n}\left[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,2,m,n}\;-\mathrm{M}_{d,m,r,n}\,f_{d,r,t,4,m,n}\right)\right.$$
$$+\,Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,6,m,n}\;+\mathrm{K}_{d,m,r,n}\,f_{d,r,t,8,m,n}\right)$$
$$+\,R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,10,m,n}+\mathrm{K}_{d,m,r,n}\,f_{d,r,t,12,m,n}\right)$$
$$\left.+\,R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,14,m,n}-\mathrm{M}_{d,m,r,n}\,f_{d,r,t,16,m,n}\right)\right]$$
$$+\,ez_{r,m}ez_{d,n}\left[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,2,m,n}\;-\mathrm{M}_{r,m,d,n}\,f_{r,d,t,4,m,n}\right)\right.$$
$$+\,Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,6,m,n}\;+\mathrm{K}_{r,m,d,n}\,f_{r,d,t,8,m,n}\right)$$
$$+\,R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,10,m,n}+\mathrm{K}_{r,m,d,n}\,f_{r,d,t,12,m,n}\right)$$
$$\left.+\,R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,14,m,n}-\mathrm{M}_{r,m,d,n}\,f_{r,d,t,16,m,n}\right)\right]\},$$

$$
\begin{aligned}
&K_{w,d,a,m,r,d,n} = \\
&-\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,2,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,4,m,n}\right) \\
&\qquad - Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,6,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,8,m,n}\right) \\
&\qquad + R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,10,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,12,m,n}\right) \\
&\qquad - R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,14,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,16,m,n}\right)\Big] \\
&\quad + ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,2,m,n} + \mathrm{M}_{r,m,d,n}\,f_{r,d,t,4,m,n}\right) \\
&\qquad + Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,6,m,n} - \mathrm{K}_{r,m,d,n}\,f_{r,d,t,8,m,n}\right) \\
&\qquad - R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,10,m,n} - \mathrm{K}_{r,m,d,n}\,f_{r,d,t,12,m,n}\right) \\
&\qquad - R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,14,m,n} + \mathrm{M}_{r,m,d,n}\,f_{r,d,t,16,m,n}\right)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&K_{w,d,b,m,r,c,n} = \\
&-\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,2,m,n} - \mathrm{M}_{d,m,r,n}\,f_{d,r,t,4,m,n}\right) \\
&\qquad - Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,6,m,n} + \mathrm{K}_{d,m,r,n}\,f_{d,r,t,8,m,n}\right) \\
&\qquad + R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,10,m,n} + \mathrm{K}_{d,m,r,n}\,f_{d,r,t,12,m,n}\right) \\
&\qquad - R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,14,m,n} - \mathrm{M}_{d,m,r,n}\,f_{d,r,t,16,m,n}\right)\Big] \\
&\quad + ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,2,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,4,m,n}\right) \\
&\qquad + Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,6,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,8,m,n}\right) \\
&\qquad - R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,10,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,12,m,n}\right) \\
&\qquad - R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,14,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,16,m,n}\right)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&K_{w,d,d,m,r,a,n} = \\
&-\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,2,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,4,m,n}\right) \\
&\qquad + Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,6,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,8,m,n}\right) \\
&\qquad - R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,10,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,12,m,n}\right) \\
&\qquad - R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,14,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,16,m,n}\right)\Big] \\
&\quad + ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,2,m,n} + \mathrm{M}_{r,m,d,n}\,f_{r,d,t,4,m,n}\right) \\
&\qquad - Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,6,m,n} - \mathrm{K}_{r,m,d,n}\,f_{r,d,t,8,m,n}\right) \\
&\qquad + R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,10,m,n} - \mathrm{K}_{r,m,d,n}\,f_{r,d,t,12,m,n}\right) \\
&\qquad - R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,14,m,n} + \mathrm{M}_{r,m,d,n}\,f_{r,d,t,16,m,n}\right)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&K_{w,d,c,m,r,b,n} = \\
&-\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,2,m,n} - \mathrm{M}_{d,m,r,n}\,f_{d,r,t,4,m,n}\right) \\
&\qquad + Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,6,m,n} + \mathrm{K}_{d,m,r,n}\,f_{d,r,t,8,m,n}\right) \\
&\qquad - R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,10,m,n} + \mathrm{K}_{d,m,r,n}\,f_{d,r,t,12,m,n}\right) \\
&\qquad - R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,14,m,n} - \mathrm{M}_{d,m,r,n}\,f_{d,r,t,16,m,n}\right)\Big] \\
&\quad + ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,2,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,4,m,n}\right) \\
&\qquad - Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,6,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,8,m,n}\right) \\
&\qquad + R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,10,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,12,m,n}\right) \\
&\qquad - R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,14,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,16,m,n}\right)\Big]\Big\}, \\
&K_{w,d,d,m,r,d,n} = \\
&+\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,2,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,4,m,n}\right) \\
&\qquad - Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,6,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,8,m,n}\right) \\
&\qquad - R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,10,m,n} - \mathrm{K}_{d,m,r,n}\,f_{d,r,t,12,m,n}\right) \\
&\qquad + R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,14,m,n} + \mathrm{M}_{d,m,r,n}\,f_{d,r,t,16,m,n}\right)\Big] \\
&\quad + ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,2,m,n} + \mathrm{M}_{r,m,d,n}\,f_{r,d,t,4,m,n}\right) \\
&\qquad - Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,6,m,n} - \mathrm{K}_{r,m,d,n}\,f_{r,d,t,8,m,n}\right) \\
&\qquad - R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,10,m,n} - \mathrm{K}_{r,m,d,n}\,f_{r,d,t,12,m,n}\right) \\
&\qquad + R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,14,m,n} + \mathrm{M}_{r,m,d,n}\,f_{r,d,t,16,m,n}\right)\Big]\Big\}, \\
&K_{w,d,c,m,r,c,n} = \\
&-\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}\,Q_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,2,m,n} - \mathrm{M}_{d,m,r,n}\,f_{d,r,t,4,m,n}\right) \\
&\qquad - Q_{d,m}\,R_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,6,m,n} + \mathrm{K}_{d,m,r,n}\,f_{d,r,t,8,m,n}\right) \\
&\qquad - R_{d,m}\,Q_{r,n}\left(\Lambda_{d,m,r,n}\,f_{d,r,t,10,m,n} + \mathrm{K}_{d,m,r,n}\,f_{d,r,t,12,m,n}\right) \\
&\qquad + R_{d,m}\,R_{r,n}\left(\mathrm{N}_{d,m,r,n}\,f_{d,r,t,14,m,n} - \mathrm{M}_{d,m,r,n}\,f_{d,r,t,16,m,n}\right)\Big] \\
&\quad + ez_{r,m}ez_{d,n}\Big[Q_{r,m}\,Q_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,2,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,4,m,n}\right) \\
&\qquad - Q_{r,m}\,R_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,6,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,8,m,n}\right) \\
&\qquad - R_{r,m}\,Q_{d,n}\left(\Lambda_{r,m,d,n}\,f_{r,d,t,10,m,n} + \mathrm{K}_{r,m,d,n}\,f_{r,d,t,12,m,n}\right) \\
&\qquad + R_{r,m}\,R_{d,n}\left(\mathrm{N}_{r,m,d,n}\,f_{r,d,t,14,m,n} - \mathrm{M}_{r,m,d,n}\,f_{r,d,t,16,m,n}\right)\Big]\Big\}.
\end{aligned}
\tag{118}
$$

In Equations (116) and (118), nonlinear amplitudes

$$
\begin{aligned}
\mathrm{K}_{d,m,r,n} &= +\kappa_{d,m}\kappa_{r,n} - \lambda_{d,m}\lambda_{r,n} + \mu_{d,m}\mu_{r,n}, \\
\Lambda_{d,m,r,n} &= -\kappa_{d,m}\kappa_{r,n} + \lambda_{d,m}\lambda_{r,n} + \mu_{d,m}\mu_{r,n}, \\
\mathrm{M}_{d,m,r,n} &= +\kappa_{d,m}\kappa_{r,n} + \lambda_{d,m}\lambda_{r,n} + \mu_{d,m}\mu_{r,n}, \\
\mathrm{N}_{d,m,r,n} &= +\kappa_{d,m}\kappa_{r,n} + \lambda_{d,m}\lambda_{r,n} - \mu_{d,m}\mu_{r,n}
\end{aligned}
\tag{119}
$$

and

$$\begin{aligned}
\mathrm{K}_{r,m,d,n} &= +\kappa_{r,m}\kappa_{d,n} - \lambda_{r,m}\lambda_{d,n} + \mu_{r,m}\mu_{d,n},\\
\Lambda_{r,m,d,n} &= -\kappa_{r,m}\kappa_{d,n} + \lambda_{r,m}\lambda_{d,n} + \mu_{r,m}\mu_{d,n},\\
\mathrm{M}_{r,m,d,n} &= +\kappa_{r,m}\kappa_{d,n} + \lambda_{r,m}\lambda_{d,n} + \mu_{r,m}\mu_{d,n},\\
\mathrm{N}_{r,m,d,n} &= +\kappa_{r,m}\kappa_{d,n} + \lambda_{r,m}\lambda_{d,n} - \mu_{r,m}\mu_{d,n}
\end{aligned} \tag{120}$$

are produced by the deterministic and random wavenumbers.

For all vector drew-t oscillons, amplitudes of eigenfunctions are influenced by

$$\begin{gathered}
\rho_c, \mathrm{K}_{d,m,r,n}, \Lambda_{d,m,r,n}, \mathrm{M}_{d,m,r,n}, \mathrm{N}_{d,m,r,n},\\
\mathrm{K}_{r,m,d,n}, \Lambda_{r,m,d,n}, \mathrm{M}_{r,m,d,n}, \mathrm{N}_{r,m,d,n}, Q_{d,m}, Q_{d,n}, R_{d,m}, R_{d,n}, z_0, t
\end{gathered} \tag{121}$$

via

$$ez_{d,m}(z_0), ez_{d,n}(z_0), ez_{r,m}(z_0), ez_{r,n}(z_0), Q_{r,m}(t), Q_{r,n}(t), R_{r,m}(t), R_{r,n}(t) \tag{122}$$

and temporal shifts of eigenfunctions are controlled by

$$\alpha_{d,1,m}, \alpha_{d,1,n}, \alpha_{d,2,m}, \alpha_{d,2,n}, x_0, y_0, t \tag{123}$$

through

$$\begin{gathered}
s_{d,x,m}(x_0), s_{d,x,n}(x_0), s_{d,y,m}(y_0), s_{d,y,n}(y_0), s_{r,x,m}(x_0,t), s_{r,x,n}(x_0,t),\\
s_{r,y,m}(y_0,t), s_{r,y,n}(y_0,t), \omega_{d,r,t,k,m,n}(t), \omega_{r,d,t,k,m,n}(t),\\
\alpha_{r,1,m}(t), \alpha_{r,1,n}(t), \alpha_{r,2,m}(t), \alpha_{r,2,n}(t).
\end{gathered} \tag{124}$$

Similar to the dree-t and rdee-t oscillons, the drew-t oscillons are smooth random functions of time with an unbounded period.

3.4. The DREG Oscillon

The symmetry resonance of the dree-t and rdee-t oscillons results in reduction of the deterministic-random, external, group oscillon (the dreg oscillon for briefness, see (198) of [7]) to a 16-rf oscillon, which is produced by 16-tuple of the deterministic-random, external, group interaction in t

$$\boldsymbol{f}_{d,r,e,g,t} = \left\{ f_{d,r,t,2k-1,m,n}, f_{d,r,t,2l,m,n}, f_{r,d,t,2k-1,m,n} f_{r,d,t,2l,m,n} \right\}. \tag{125}$$

Sixteen-tuple $\boldsymbol{f}_{d,r,e,g,t}$ includes four sine waves $f_{d,r,t,2k-1,m,n}$ with frequencies $\omega_{d,r,t,k,m,n}(t)$, four cosine waves $f_{d,r,t,2l,m,n}$ with frequencies $\omega_{d,r,t,l,m,n}(t)$, four sine waves $f_{r,d,t,2k-1,m,n}$ with frequencies $\omega_{r,d,t,k,m,n}(t)$, and four cosine waves $f_{r,d,t,2l,m,n}$ with frequencies $\omega_{r,d,t,l,m,n}(t)$ for $k = 3,4,5,6$, $l = 1,2,7,8$, and each m, n.

Expressing the dreg oscillon via the dree and rdee oscillons and substituting the inhomogeneous Fourier expansion of the vector dree-t and rdee-t oscillons yields that the dreg-t oscillon for any frozen $x = x_0, y = y_0, z = z_0$, is displayed as the 16-rf, deterministic-random, random-deterministic oscillon in t, which is formed by 16-tuple $\boldsymbol{f}_{d,r,e,g,t}$,

$$K_{g,d,i,m,r,j,n} = K_{g,d,i,m,r,j,n}\left(\boldsymbol{f}_{d,r,e,g,t}\right) \tag{126}$$

because

$$
\begin{aligned}
&K_{g,d,i,m,r,j,n} = \\
&+\frac{\rho_c}{2}\Big\{ez_{d,m}ez_{r,n}\Big[Q_{d,m}R_{r,n}\left(\Lambda_{d,m,r,n}f_{d,r,t,5,m,n}-\mathrm{K}_{d,m,r,n}f_{d,r,t,7,m,n}\right)\\
&\qquad +R_{d,m}Q_{r,n}\left(\Lambda_{d,m,r,n}f_{d,r,t,9,m,n}+\mathrm{K}_{d,m,r,n}f_{d,r,t,11,m,n}\right)\\
&\qquad -Q_{d,m}Q_{r,n}\left(\mathrm{N}_{d,m,r,n}f_{d,r,t,2,m,n}-\mathrm{M}_{d,m,r,n}f_{d,r,t,4,m,n}\right)\\
&\qquad +R_{d,m}R_{r,n}\left(\mathrm{N}_{d,m,r,n}f_{d,r,t,14,m,n}+\mathrm{M}_{d,m,r,n}f_{d,r,t,16,m,n}\right)\Big]\\
&+ez_{r,m}ez_{d,n}\Big[Q_{r,m}R_{d,n}\left(\Lambda_{r,m,d,n}f_{r,d,t,5,m,n}-\mathrm{K}_{r,m,d,n}f_{r,d,t,7,m,n}\right)\\
&\qquad +R_{r,m}Q_{d,n}\left(\Lambda_{r,m,d,n}f_{r,d,t,9,m,n}+\mathrm{K}_{r,m,d,n}f_{r,d,t,11,m,n}\right)\\
&\qquad -Q_{r,m}Q_{d,n}\left(\mathrm{N}_{r,m,d,n}f_{r,d,t,2,m,n}-\mathrm{M}_{r,m,d,n}f_{r,d,t,4,m,n}\right)\\
&\qquad +R_{r,m}R_{d,n}\left(\mathrm{N}_{r,m,d,n}f_{r,d,t,14,m,n}+\mathrm{M}_{r,m,d,n}f_{r,d,t,16,m,n}\right)\Big]\Big\}.
\end{aligned}
\tag{127}
$$

Amplitudes of eigenfunctions depend on (121) via (122) and temporal shifts of eigenfunctions are determined by (123) through (124). Similar to the drew-t oscillons, the dreg-t oscillon is a smooth random function of time with an unbounded period.

3.5. The DREK Oscillon

The deterministic-random, external, kinetic-energy oscillon (the drek oscillon for pithiness, see (92) of [7]) takes the following form:

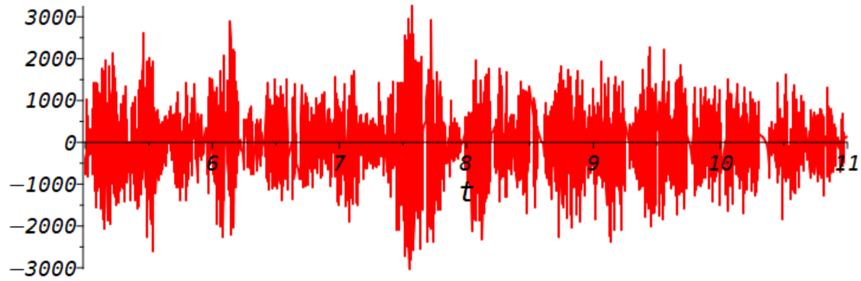
$$K_{e,d,i,m,r,j,n} = \sum_{m=1}^{M-1}\sum_{n=m+1}^{M} K_{g,d,i,m,r,j,n}. \tag{128}$$

If all frequencies of the drek-t oscillon are distinct smooth random functions of time, then the drek-t oscillon is displayed as an $8M(M-1)$-rf oscillon. For any frozen $x = x_0, y = y_0, z = z_0$, the drek-$t$ oscillon is represented by the $8M(M-1)$-rf, deterministic-random, random-deterministic oscillon in t, which is formed by $M(M-1)/2$ 16-tuples $\boldsymbol{f}_{d,r,e,g,t}$ with frequencies $\omega_{d,r,t,k,m,n}(t), \omega_{r,d,t,k,m,n}(t)$ for $k = 1,2,\cdots,8$, all m, n, Re, and wave parameters of the dreg-t oscillons. The drek-t oscillon is a smooth random function of time with an unbounded period, as well.

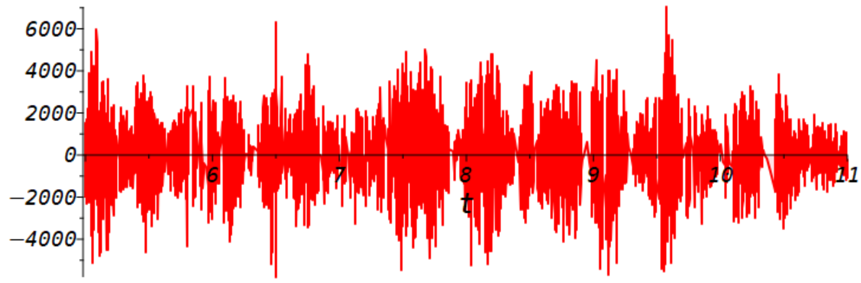
The drek-t oscillons for $x = x_0, y = y_0, z = z_0$, the Reynolds number $Re = 10^3$, and $Re = 10^5$ are shown in **Figure 1** on $[t_0, t_0 + T_{e,d,e}] = [5,11]$, where $t_0 = 5$ is the initial time, $T_{e,d,e} = 6$ is the period of the dek-t oscillon. We observe a substantial change in range and shape of the 48-rf, deterministic-random, random-deterministic drek-t oscillon with the Reynolds number.

In **Figure 1** and sequential figures, the independent deterministic parameters are the same as in [5], specifically,

$$
\begin{aligned}
&\rho_c = 1, \quad M = 3, \quad \eta = 0, \quad x_0 = 0, \quad y_0 = 0, \quad z_0 = 0, \quad t_0 = 5,\\
&\kappa_{d,1} = \frac{\pi}{4}, \quad \kappa_{d,2} = \frac{2\pi}{3}, \kappa_{d,3} = \frac{3\pi}{2}, \lambda_{d,1} = \frac{2\pi}{9}, \lambda_{d,2} = \frac{2\pi}{3}, \lambda_{d,3} = 2\pi,\\
&U_{d,1} = 4, \quad U_{d,2} = 3, \quad U_{d,3} = 2, \quad V_{d,1} = 3, \quad V_{d,2} = 2, \quad V_{d,3} = 1,\\
&X_{d,1,0} = 7, X_{d,2,0} = 5, \quad X_{d,3,0} = 3, \quad Y_{d,1,0} = 5, \quad Y_{d,2,0} = 3, \quad Y_{d,3,0} = 1,\\
&Av_{d,1} = 11, Av_{d,2} = 7, \quad Av_{d,3} = 3, \quad Bv_{d,1} = 6, \quad Bv_{d,2} = 4, \quad Bv_{d,3} = 2,\\
&Cv_{d,1} = 8, Cv_{d,2} = 6, \quad Cv_{d,3} = 4, \quad Dv_{d,1} = 7, \quad Dv_{d,2} = 4, \quad Dv_{d,3} = 1.
\end{aligned}
\tag{129}
$$



(a)



(b)

Figure 1. The drek-t oscillons: (a) $-K_{e,d,i,m,r,j,n}(t)$ for $Re = 10^3$, (b) $-K_{e,d,i,m,r,j,n}(t)$ for $Re = 10^5$.

With the help of the random oscillatory cn-noise [8], independent smooth random functions of time (38)-(39) are computed as follows:

$$\begin{aligned}
U_{r,m} &= S_U(Re)\sum_{l=1}^{5} A_{U,m,l}\,\mathrm{cn}\left(S_f(Re)\upsilon_{U,m,l}\,t - \mathrm{K}(\varepsilon),\varepsilon\right),\\
V_{r,m} &= S_V(Re)\sum_{l=1}^{5} A_{V,m,l}\,\mathrm{cn}\left(S_f(Re)\upsilon_{V,m,l}\,t - \mathrm{K}(\varepsilon),\varepsilon\right),\\
X_{r,m,0} &= S_X(Re)\sum_{l=1}^{5} A_{X,m,l}\,\mathrm{cn}\left(S_f(Re)\upsilon_{X,m,l}\,t - \mathrm{K}(\varepsilon),\varepsilon\right),\\
Y_{r,m,0} &= S_Y(Re)\sum_{l=1}^{5} A_{Y,m,l}\,\mathrm{cn}\left(S_f(Re)\upsilon_{Y,m,l}\,t - \mathrm{K}(\varepsilon),\varepsilon\right),\\
Av_{r,m} &= S_A(Re)\sum_{l=1}^{5} A_{A,m,l}\,\mathrm{cn}\left(S_f(Re)\upsilon_{A,m,l}\,t - \mathrm{K}(\varepsilon),\varepsilon\right),\\
Bv_{r,m} &= S_A(Re)\sum_{l=1}^{5} A_{B,m,l}\,\mathrm{cn}\left(S_f(Re)\upsilon_{B,m,l}\,t - \mathrm{K}(\varepsilon),\varepsilon\right),\\
Cv_{r,m} &= S_A(Re)\sum_{l=1}^{5} A_{C,m,l}\,\mathrm{cn}\left(S_f(Re)\upsilon_{C,m,l}\,t - \mathrm{K}(\varepsilon),\varepsilon\right),\\
Dv_{r,m} &= S_A(Re)\sum_{l=1}^{5} A_{D,m,l}\,\mathrm{cn}\left(S_f(Re)\upsilon_{D,m,l}\,t_0 - \mathrm{K}(\varepsilon),\varepsilon\right).
\end{aligned} \tag{130}$$

where $\mathrm{cn}(t,\varepsilon)$ is the elliptic cosine, $\varepsilon = 0.9999$ is the elliptic modulus, $\mathrm{K}(\varepsilon) = 5.64514827$ is the complete elliptic integral of the first kind,

$$A_{U,m,l}, A_{V,m,l}, A_{X,m,l}, A_{Y,m,l}, A_{A,m,l}, A_{B,m,l}, A_{C,m,l}, A_{D,m,l} \tag{131}$$

are random amplitudes selected from a list of 120 random numbers on [0,1],

$$\upsilon_{U,m,l}, \upsilon_{V,m,l}, \upsilon_{X,m,l}, \upsilon_{Y,m,l}, \upsilon_{A,m,l}, \upsilon_{B,m,l}, \upsilon_{C,m,l}, \upsilon_{D,m,l}, \upsilon_{\kappa,m}, \upsilon_{\lambda,m} \tag{132}$$

are random frequencies chosen from a list of reciprocals of first 120 prime numbers mixed up randomly,

$$\begin{aligned} &S_x(Re) = 696.7909371\,Re^{0.01952026874}, \quad S_y(Re) = 958.7435364\,Re^{0.04994726271}, \\ &S_f(Re) = 5529.843495\,Re^{0.007252001036}, \; S_A(Re) = 0.6787933393\,Re^{0.1215190243}, \\ &S_U(Re) = 0.5303300858\,Re^{0.1505149978}, \; S_V(Re) = 0.8838834764\,Re^{0.1505149978}, \\ &S_X(Re) = 1.237436866\,Re^{0.1505149978}, \quad S_Y(Re) = 1.590990258\,Re^{0.1505149978} \end{aligned} \tag{133}$$

are scales of the random parameters.

For $Re = 10^3$, the random wavenumbers have the same values as in [5], namely,

$$\begin{aligned} &\kappa_{r,1} = \frac{797}{383}, \quad \kappa_{r,2} = \frac{797}{599}, \quad \kappa_{r,3} = \frac{797}{163}, \\ &\lambda_{r,1} = \frac{1353}{467}, \; \lambda_{r,2} = \frac{1353}{349}, \; \lambda_{r,3} = \frac{1353}{317}. \end{aligned} \tag{134}$$

For $Re = 10^5$, the random wavenumbers become, similarly,

$$\begin{aligned} &\kappa_{r,1} = \frac{872}{383}, \quad \kappa_{r,2} = \frac{872}{599}, \quad \kappa_{r,3} = \frac{872}{163}, \\ &\lambda_{r,1} = \frac{1703}{467}, \; \lambda_{r,2} = \frac{1703}{349}, \; \lambda_{r,3} = \frac{1703}{317}. \end{aligned} \tag{135}$$

The shape of the drek-t oscillons in **Figure 1** qualitatively differs from the shape of the drek-x oscillons in **Figure 1** of [5]. The drek-x oscillons in **Figure 1** of [5] are multi-wavenumber oscillons because deterministic-random and random-deterministic, external wavenumbers are random constants at $t = t_0$. The drek-t oscillons in **Figure 1** are oscillons with deterministic-random and random-deterministic, external frequences, which are smooth random functions of time together with other wave parameters. So, we observe in **Figure 1** and sequential figures that random functions of time (130) produce random, fringe-like shapes of deterministic-random, random-deterministic, random, turbulent oscillons and pulsons. Maple programs for computing temporal quantization in the t-eigenfunctions will be published elsewhere.

4. Oscillons of Deterministic-Random, Internal Interaction

4.1. The DRIE Oscillons

In the view of the identity resonance, 16 deterministic-random, internal, elementary oscillons (the drie oscillons for briefness, see (188) of [7]) are arranged into two vector drie-t oscillons

$$\boldsymbol{K}_{d,r,i,e,t,q} = \boldsymbol{K}_{d,r,i,e,t,q}\left(\boldsymbol{f}_{d,r,i,e,t,q}\right), \; q = 1,2, \tag{136}$$

which are formed by two 8-tuples of the deterministic-random, internal, elementary interaction in t:

$$\boldsymbol{f}_{d,r,i,e,t,1} = \left\{g_{d,r,t,2k-1,m,m}\right\}, \; \boldsymbol{f}_{d,r,i,e,t,2} = \left\{g_{d,r,t,2k,m,m}\right\}. \tag{137}$$

Eight-tuple $\boldsymbol{f}_{d,r,i,e,t,1}$ comprises eight sine waves $g_{d,r,t,2k-1,m,m}$ with frequencies $\omega_{d,r,t,k,m,m}(t)$ and 8-tuple $\boldsymbol{f}_{d,r,i,e,t,2}$ consists of eight cosine waves $g_{d,r,t,2k,m,m}$ with frequencies $\omega_{d,r,t,k,m,m}(t)$ for $k=1,2,\cdots,8$ and each m.

For any frozen $x=x_0, y=y_0, z=z_0$, application of matrix $\boldsymbol{f}_{d,t,i,m,r,t,j,m}$ yields that the 1st vector drie-t oscillon

$$\boldsymbol{K}_{d,r,i,e,t,1}=\left[K_{o,d,b,m,r,a,m}, K_{o,d,a,m,r,b,m}, K_{o,d,b,m,r,d,m}, K_{o,d,a,m,r,c,m}, \right. \\ \left. K_{o,d,c,m,r,a,m}, K_{o,d,d,m,r,b,m}, K_{o,d,c,m,r,d,m}, K_{o,d,d,m,r,c,m}\right]\left(\boldsymbol{f}_{d,r,i,e,t,1}\right) \tag{138}$$

is exposed by a list of eight 8-rf, deterministic-random oscillons in t, which depend on 8-tuple $\boldsymbol{f}_{d,r,i,e,t,1}$ since

$$K_{o,d,b,m,r,a,m}=-\frac{\rho_c}{8}ez_{d,m}ez_{r,m} \\ \times\left[Q_{d,m}Q_{r,m}\left(g_{d,r,t,1,m,m}-g_{d,r,t,3,m,m}\right)-Q_{d,m}R_{r,m}\left(g_{d,r,t,5,m,m}-g_{d,r,t,7,m,m}\right)\right. \\ \left.-R_{d,m}Q_{r,m}\left(g_{d,r,t,9,m,m}-g_{d,r,t,11,m,m}\right)+R_{d,m}R_{r,m}\left(g_{d,r,t,13,m,m}-g_{d,r,t,15,m,m}\right)\right],$$

$$K_{o,d,a,m,r,b,m}=-\frac{\rho_c}{8}ez_{d,m}ez_{r,m} \\ \times\left[Q_{d,m}Q_{r,m}\left(g_{d,r,t,1,m,m}+g_{d,r,t,3,m,m}\right)-Q_{d,m}R_{r,m}\left(g_{d,r,t,5,m,m}+g_{d,r,t,7,m,m}\right)\right. \\ \left.-R_{d,m}Q_{r,m}\left(g_{d,r,t,9,m,m}+g_{d,r,t,11,m,m}\right)+R_{d,m}R_{r,m}\left(g_{d,r,t,13,m,m}+g_{d,r,t,15,m,m}\right)\right],$$

$$K_{o,d,b,m,r,d,m}=+\frac{\rho_c}{8}ez_{d,m}ez_{r,m} \\ \times\left[Q_{d,m}Q_{r,m}\left(g_{d,r,t,1,m,m}-g_{d,r,t,3,m,m}\right)+Q_{d,m}R_{r,m}\left(g_{d,r,t,5,m,m}-g_{d,r,t,7,m,m}\right)\right. \\ \left.-R_{d,m}Q_{r,m}\left(g_{d,r,t,9,m,m}-g_{d,r,t,11,m,m}\right)-R_{d,m}R_{r,m}\left(g_{d,r,t,13,m,m}-g_{d,r,t,15,m,m}\right)\right],$$

$$K_{o,d,a,m,r,c,m}=-\frac{\rho_c}{8}ez_{d,m}ez_{r,m} \\ \times\left[Q_{d,m}Q_{r,m}\left(g_{d,r,t,1,m,m}+g_{d,r,t,3,m,m}\right)+Q_{d,m}R_{r,m}\left(g_{d,r,t,5,m,m}+g_{d,r,t,7,m,m}\right)\right. \\ \left.-R_{d,m}Q_{r,m}\left(g_{d,r,t,9,m,m}+g_{d,r,t,11,m,m}\right)-R_{d,m}R_{r,m}\left(g_{d,r,t,13,m,m}+g_{d,r,t,15,m,m}\right)\right],$$

$$K_{o,d,c,m,r,a,m}=-\frac{\rho_c}{8}ez_{d,m}ez_{r,m} \\ \times\left[Q_{d,m}Q_{r,m}\left(g_{d,r,t,1,m,m}-g_{d,r,t,3,m,m}\right)-Q_{d,m}R_{r,m}\left(g_{d,r,t,5,m,m}-g_{d,r,t,7,m,m}\right)\right. \\ \left.+R_{d,m}Q_{r,m}\left(g_{d,r,t,9,m,m}-g_{d,r,t,11,m,m}\right)-R_{d,m}R_{r,m}\left(g_{d,r,t,13,m,m}-g_{d,r,t,15,m,m}\right)\right],$$

$$K_{o,d,d,m,r,b,m}=+\frac{\rho_c}{8}ez_{d,m}ez_{r,m} \\ \times\left[Q_{d,m}Q_{r,m}\left(g_{d,r,t,1,m,m}+g_{d,r,t,3,m,m}\right)-Q_{d,m}R_{r,m}\left(g_{d,r,t,5,m,m}+g_{d,r,t,7,m,m}\right)\right. \\ \left.+R_{d,m}Q_{r,m}\left(g_{d,r,t,9,m,m}+g_{d,r,t,11,m,m}\right)-R_{d,m}R_{r,m}\left(g_{d,r,t,13,m,m}+g_{d,r,t,15,m,m}\right)\right],$$

$$K_{o,d,c,m,r,d,m}=+\frac{\rho_c}{8}ez_{d,m}ez_{r,m} \\ \times\left[Q_{d,m}Q_{r,m}\left(g_{d,r,t,1,m,m}-g_{d,r,t,3,m,m}\right)+Q_{d,m}R_{r,m}\left(g_{d,r,t,5,m,m}-g_{d,r,t,7,m,m}\right)\right. \\ \left.+R_{d,m}Q_{r,m}\left(g_{d,r,t,9,m,m}-g_{d,r,t,11,m,m}\right)+R_{d,m}R_{r,m}\left(g_{d,r,t,13,m,m}-g_{d,r,t,15,m,m}\right)\right],$$

$$K_{o,d,d,m,r,c,m} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \times\left[Q_{d,m} Q_{r,m}\left(g_{d,r,t,1,m,m}+g_{d,r,t,3,m,m}\right) + Q_{d,m} R_{r,m}\left(g_{d,r,t,5,m,m}+g_{d,r,t,7,m,m}\right) + R_{d,m} Q_{r,m}\left(g_{d,r,t,9,m,m}+g_{d,r,t,11,m,m}\right) + R_{d,m} R_{r,m}\left(g_{d,r,t,13,m,m}+g_{d,r,t,15,m,m}\right)\right]. \quad (139)$$

The 2nd vector drie-t oscillon

$$\boldsymbol{K}_{d,r,i,e,t,2} = \left[K_{o,d,a,m,r,a,m}, K_{o,d,b,m,r,b,m}, K_{o,d,a,m,r,d,m}, K_{o,d,b,m,r,c,m}, K_{o,d,d,m,r,a,m}, K_{o,d,c,m,r,b,m}, K_{o,d,d,m,r,d,m}, K_{o,d,c,m,r,c,m}\right]\left(\boldsymbol{f}_{d,r,i,e,t,2}\right) \quad (140)$$

is exhibited by a list of eight 8-rf, deterministic-random oscillons in t, which are created by 8-tuple $\boldsymbol{f}_{d,r,i,e,t,2}$ as

$$K_{o,d,a,m,r,a,m} = -\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \times\left[Q_{d,m} Q_{r,m}\left(g_{d,r,t,2,m,m}-g_{d,r,t,4,m,m}\right) - Q_{d,m} R_{r,m}\left(g_{d,r,t,6,m,m}-g_{d,r,t,8,m,m}\right) - R_{d,m} Q_{r,m}\left(g_{d,r,t,10,m,m}-g_{d,r,t,12,m,m}\right) + R_{d,m} R_{r,m}\left(g_{d,r,t,14,m,m}-g_{d,r,t,16,m,m}\right)\right],$$

$$K_{o,d,b,m,r,b,m} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \times\left[Q_{d,m} Q_{r,m}\left(g_{d,r,t,2,m,m}+g_{d,r,t,4,m,m}\right) - Q_{d,m} R_{r,m}\left(g_{d,r,t,6,m,m}+g_{d,r,t,8,m,m}\right) - R_{d,m} Q_{r,m}\left(g_{d,r,t,10,m,m}+g_{d,r,t,12,m,m}\right) + R_{d,m} R_{r,m}\left(g_{d,r,t,14,m,m}+g_{d,r,t,16,m,m}\right)\right],$$

$$K_{o,d,a,m,r,d,m} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \times\left[Q_{d,m} Q_{r,m}\left(g_{d,r,t,2,m,m}-g_{d,r,t,4,m,m}\right) + Q_{d,m} R_{r,m}\left(g_{d,r,t,6,m,m}-g_{d,r,t,8,m,m}\right) - R_{d,m} Q_{r,m}\left(g_{d,r,t,10,m,m}-g_{d,r,t,12,m,m}\right) - R_{d,m} R_{r,m}\left(g_{d,r,t,14,m,m}-g_{d,r,t,16,m,m}\right)\right],$$

$$K_{o,d,b,m,r,c,m} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \times\left[Q_{d,m} Q_{r,m}\left(g_{d,r,t,2,m,m}+g_{d,r,t,4,m,m}\right) + Q_{d,m} R_{r,m}\left(g_{d,r,t,6,m,m}+g_{d,r,t,8,m,m}\right) - R_{d,m} Q_{r,m}\left(g_{d,r,t,10,m,m}+g_{d,r,t,12,m,m}\right) - R_{d,m} R_{r,m}\left(g_{d,r,t,14,m,m}+g_{d,r,t,16,m,m}\right)\right],$$

$$K_{o,d,d,m,r,a,m} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \times\left[Q_{d,m} Q_{r,m}\left(g_{d,r,t,2,m,m}-g_{d,r,t,4,m,m}\right) - Q_{d,m} R_{r,m}\left(g_{d,r,t,6,m,m}-g_{d,r,t,8,m,m}\right) + R_{d,m} Q_{r,m}\left(g_{d,r,t,10,m,m}-g_{d,r,t,12,m,m}\right) - R_{d,m} R_{r,m}\left(g_{d,r,t,14,m,m}-g_{d,r,t,16,m,m}\right)\right],$$

$$K_{o,d,c,m,r,b,m} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \times\left[Q_{d,m} Q_{r,m}\left(g_{d,r,t,2,m,m}+g_{d,r,t,4,m,m}\right) - Q_{d,m} R_{r,m}\left(g_{d,r,t,6,m,m}+g_{d,r,t,8,m,m}\right) + R_{d,m} Q_{r,m}\left(g_{d,r,t,10,m,m}+g_{d,r,t,12,m,m}\right) - R_{d,m} R_{r,m}\left(g_{d,r,t,14,m,m}+g_{d,r,t,16,m,m}\right)\right],$$

$$K_{o,d,d,m,r,d,m} = -\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \times\left[Q_{d,m} Q_{r,m}\left(g_{d,r,t,2,m,m}-g_{d,r,t,4,m,m}\right) + Q_{d,m} R_{r,m}\left(g_{d,r,t,6,m,m}-g_{d,r,t,8,m,m}\right) + R_{d,m} Q_{r,m}\left(g_{d,r,t,10,m,m}-g_{d,r,t,12,m,m}\right) + R_{d,m} R_{r,m}\left(g_{d,r,t,14,m,m}-g_{d,r,t,16,m,m}\right)\right],$$

$$K_{o,d,c,m,r,c,m} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,m}$$
$$\times\Big[Q_{d,m} Q_{r,m}\left(g_{d,r,t,2,m,m} + g_{d,r,t,4,m,m}\right) + Q_{d,m} R_{r,m}\left(g_{d,r,t,6,m,m} + g_{d,r,t,8,m,m}\right)$$
$$+R_{d,m} Q_{r,m}\left(g_{d,r,t,10,m,m} + g_{d,r,t,12,m,m}\right) + R_{d,m} R_{r,m}\left(g_{d,r,t,14,m,m} + g_{d,r,t,16,m,m}\right)\Big]. \quad (141)$$

For all vector drie-t oscillons, amplitudes of eigenfunctions are governed by $\rho_c, Q_{d,m}, R_{d,m}, z_0, t$ via $ez_{d,m}(z_0), ez_{r,m}(z_0), Q_{r,m}(t), R_{r,m}(t)$ and temporal shifts of eigenfunctions are influenced by $\alpha_{d,1,m}, \alpha_{d,2,m}, x_0, y_0, t$ through $s_{d,x,m}(x_0), s_{d,y,m}(y_0), s_{r,x,m}(x_0,t), s_{r,y,m}(y_0,t), \omega_{d,r,t,k,m,m}(t), \alpha_{r,1,m}(t), \alpha_{r,2,m}(t)$. Similar to the dree-$t$ oscillons, drie-t oscillons are smooth random functions of t with an unbounded period.

4.2. The DRIW Oscillons

We use the decompositions of the deterministic-random, internal, wave oscillons (the driw oscillons for straightforwardness, see (190) of [7]) via the drie oscillons and substitute the inhomogeneous Fourier expansion of the vector drie-t oscillons to find that 16 driw-t oscillons are assembled into two vector driw-t oscillons

$$\boldsymbol{K}_{d,r,i,w,t,q} = \boldsymbol{K}_{d,r,i,w,t,q}\left(\boldsymbol{f}_{d,r,i,e,t,q}\right), \ q = 1,2, \quad (142)$$

which are produced by 8-tuples $\boldsymbol{f}_{d,r,i,e,t,1}$ and $\boldsymbol{f}_{d,r,i,e,t,2}$.

For any frozen $x = x_0, y = y_0, z = z_0$, the 1st vector driw-$t$ oscillon

$$\boldsymbol{K}_{d,r,i,w,t,1} = \big[K_{w,d,b,m,r,a,m}, K_{w,d,a,m,r,b,m}, K_{w,d,b,m,r,d,m}, K_{w,d,a,m,r,c,m},$$
$$K_{w,d,c,m,r,a,m}, K_{w,d,d,m,r,b,m}, K_{w,d,c,m,r,d,m}, K_{w,d,d,m,r,c,m}\big]\left(\boldsymbol{f}_{d,r,i,e,t,1}\right) \quad (143)$$

is displayed by a list of eight 8-rf, deterministic-random oscillons in t, which depend on 8-tuple $\boldsymbol{f}_{d,r,i,e,t,1}$ in agreement with

$$K_{w,d,b,m,r,a,m} =$$
$$+\frac{\rho_c}{8} ez_{d,m} ez_{r,m}\Big[Q_{d,m} Q_{r,m}\left(\mathrm{N}_{d,m,r,m} g_{d,r,t,1,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,3,m,m}\right)$$
$$+ Q_{d,m} R_{r,m}\left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m}\right)$$
$$+ R_{d,m} Q_{r,m}\left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m}\right)$$
$$+ R_{d,m} R_{r,m}\left(\mathrm{N}_{d,m,r,m} g_{d,r,t,13,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,15,m,m}\right)\Big],$$

$$K_{w,d,a,m,r,b,m} =$$
$$+\frac{\rho_c}{8} ez_{d,m} ez_{r,m}\Big[Q_{d,m} Q_{r,m}\left(\mathrm{N}_{d,m,r,m} g_{d,r,t,1,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,3,m,m}\right)$$
$$+ Q_{d,m} R_{r,m}\left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m}\right)$$
$$+ R_{d,m} Q_{r,m}\left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m}\right)$$
$$+ R_{d,m} R_{r,m}\left(\mathrm{N}_{d,m,r,m} g_{d,r,t,13,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,15,m,m}\right)\Big],$$

$$
\begin{aligned}
&K_{w,d,b,m,r,d,m} = \\
&-\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,1,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,3,m,m} \right) \\
&\qquad - Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m} \right) \\
&\qquad + R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m} \right) \\
&\qquad - R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,13,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,15,m,m} \right) \Big], \\
&K_{w,d,a,m,r,c,m} = \\
&+\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,1,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,3,m,m} \right) \\
&\qquad - Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m} \right) \\
&\qquad + R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m} \right) \\
&\qquad - R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,13,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,15,m,m} \right) \Big], \\
&K_{w,d,c,m,r,a,m} = \\
&+\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,1,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,3,m,m} \right) \\
&\qquad + Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m} \right) \\
&\qquad - R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m} \right) \\
&\qquad - R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,13,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,15,m,m} \right) \Big], \\
&K_{w,d,d,m,r,b,m} = \\
&-\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,1,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,3,m,m} \right) \\
&\qquad + Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m} \right) \\
&\qquad - R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m} \right) \\
&\qquad - R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,13,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,15,m,m} \right) \Big], \\
&K_{w,d,c,m,r,d,m} = \\
&-\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,1,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,3,m,m} \right) \\
&\qquad - Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m} \right) \\
&\qquad - R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m} \right) \\
&\qquad + R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,13,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,15,m,m} \right) \Big], \\
&K_{w,d,d,m,r,c,m} = \\
&-\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,1,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,3,m,m} \right) \\
&\qquad - Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m} \right) \\
&\qquad - R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m} \right) \\
&\qquad + R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,13,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,15,m,m} \right) \Big].
\end{aligned} \tag{144}
$$

The 2nd vector driw-t oscillon

$$\boldsymbol{K}_{d,r,i,w,t,2} = \left[K_{w,d,a,m,r,a,n}, K_{w,d,b,m,r,b,n}, K_{w,d,a,m,r,d,n}, K_{w,d,b,m,r,c,n}, K_{w,d,d,m,r,a,n}, K_{w,d,c,m,r,b,n}, K_{w,d,d,m,r,d,n}, K_{w,d,c,m,r,c,n} \right] \left(\boldsymbol{f}_{d,r,i,e,t,2} \right) \quad (145)$$

is represented by a list of eight 8-rf, deterministic-random oscillons in t, which are formed by 8-tuple $\boldsymbol{f}_{d,r,i,e,t,2}$ in accordance with

$$K_{w,d,a,m,r,a,m} = +\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) + Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,6,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,8,m,m} \right) + R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,10,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,12,m,m} \right) + R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big],$$

$$K_{w,d,b,m,r,b,m} = -\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) + Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,6,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,8,m,m} \right) + R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,10,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,12,m,m} \right) + R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big],$$

$$K_{w,d,a,m,r,d,m} = -\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) - Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,6,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,8,m,m} \right) + R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,10,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,12,m,m} \right) - R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big],$$

$$K_{w,d,b,m,r,c,m} = -\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) - Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,6,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,8,m,m} \right) + R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,10,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,12,m,m} \right) - R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big],$$

$$K_{w,d,d,m,r,a,m} = -\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) + Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,6,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,8,m,m} \right) - R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,10,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,12,m,m} \right) - R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big],$$

$$
\begin{aligned}
&K_{w,d,c,m,r,b,m} = \\
&-\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) \\
&\qquad + Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,6,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,8,m,m} \right) \\
&\qquad - R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,10,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,12,m,m} \right) \\
&\qquad - R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big], \\
&K_{w,d,d,m,r,d,m} = \\
&+\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) \\
&\qquad - Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,6,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,8,m,m} \right) \\
&\qquad - R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,10,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,12,m,m} \right) \\
&\qquad + R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big], \\
&K_{w,d,c,m,r,b,m} = \\
&-\frac{\rho_c}{8} ez_{d,m} ez_{r,m} \Big[Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) \\
&\qquad - Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,6,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,8,m,m} \right) \\
&\qquad - R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,10,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,12,m,m} \right) \\
&\qquad + R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big].
\end{aligned} \tag{146}
$$

For all vector driw-t oscillons, amplitudes of eigenfunctions are controlled by

$$\rho_c, \mathrm{K}_{d,m,r,m}, \Lambda_{d,m,r,m}, \mathrm{M}_{d,m,r,m}, \mathrm{N}_{d,m,r,m}, Q_{d,m}, R_{d,m}, z_0, t \tag{147}$$

via $ez_{d,m}(z_0), ez_{r,m}(z_0), Q_{r,m}(t), R_{r,m}(t)$ and temporal shifts of eigenfunctions depend on $\alpha_{d,1,m}, \alpha_{d,2,m}, x_0, y_0, t$ through

$$
\begin{aligned}
&s_{d,x,m}(x_0), s_{d,y,m}(y_0), s_{r,x,m}(x_0,t), s_{r,y,m}(y_0,t), \\
&\omega_{d,r,t,k,m,m}(t), \alpha_{r,1,m}(t), \alpha_{r,1,m}(t), \alpha_{r,2,m}(t).
\end{aligned} \tag{148}
$$

Similar to the drie-t oscillons, driw-t oscillons are smooth random functions of time with an unbounded period.

4.3. The DRIG Oscillon

The symmetry resonance of the drie-t oscillons results in reduction of the deterministic-random, internal, group oscillon (the drig oscillon for easiness, see (191) of [7]) to an 8-rf oscillon, which is produced by 8-tuple of the deterministic-random, internal, group interaction in t

$$\boldsymbol{f}_{d,r,i,g,t} = \left\{ g_{d,r,t,2k-1,m,m}, g_{d,r,t,2l,m,m} \right\}. \tag{149}$$

Eight-tuple $\boldsymbol{f}_{d,r,i,g,t}$ consists of four sine waves $g_{d,r,t,2k-1,m,m}$ with frequencies $\omega_{d,r,t,k,m,m}(t)$ and four cosine waves $g_{d,r,t,2l,m,m}$ with frequencies $\omega_{d,r,t,l,m,m}(t)$ for $k = 3,4,5,6$, $l = 1,2,7,8$, and each m.

For any frozen $x = x_0, y = y_0, z = z_0$, expressing the drig oscillon via the drie oscillons and substituting the inhomogeneous Fourier expansion of the vector drie-t oscillons gives that the drig-t oscillon is transformed into the 8-rf deterministic-random oscillon in t, which is created by 8-tuple $\boldsymbol{f}_{d,r,i,g,t}$

$$K_{g,d,i,m,r,j,m} = K_{g,d,i,m,r,j,m}\left(\boldsymbol{f}_{d,r,i,g,t}\right) \quad (150)$$

since

$$\begin{aligned} &K_{g,d,i,m,r,j,m} = \\ &+\frac{\rho_c}{2} ez_{d,m} ez_{r,m} \Big[Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m} \right) \\ &\qquad + R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m} \right) \\ &\qquad - Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) \\ &\qquad + R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big]. \end{aligned} \quad (151)$$

Amplitudes of eigenfunctions are controlled by (147) via $ez_{d,m}(z_0), ez_{r,m}(z_0)$, $Q_{r,m}(t), R_{r,m}(t)$ and temporal shifts of eigenfunctions depend on $\alpha_{d,1,m}, \alpha_{d,2,m}, x_0, y_0, t$ through (148). Likewise the driw-t oscillons, the drig-t oscillon is a smooth random function of time with an unbounded period.

4.4. The DRIK Oscillon

The deterministic-random, internal, kinetic-energy oscillon (the drik oscillon for brevity, see (86) of [7]) may be written as follows:

$$K_{e,d,i,m,r,j,m} = \sum_{m=1}^{M} K_{g,d,i,m,r,j,m}. \quad (152)$$

Since all frequencies of the drik-t oscillon are distinct smooth random functions of time, the drik-t oscillon is visualized as an $8M$-rf oscillon. For any frozen $x = x_0, y = y_0, z = z_0$, the drik-$t$ oscillon is presented by the $8M$-rf, deterministic-random oscillon in t, which is formed by M 8-tuples $\boldsymbol{f}_{d,r,i,g,t}$ with frequencies $\omega_{d,r,t,k,m,m}(t)$ for $k = 1, 2, \cdots, 8$, all m, Re, and wave parameters of the drig-t oscillons. The drik-t oscillon is a smooth random function of time with an unbounded period, as well.

The drik-t oscillons for $x = x_0, y = y_0, z = z_0, Re = 10^3, Re = 10^5$, and wave parameters (129)-(135) are displayed on $\left[t_0, t_0 + T_{e,d,e}\right]$ in **Figure 2**. The Reynolds number strongly affects both the range and the shape of the 24-rf, deterministic-random drik-t oscillon.

5. Oscillons of Turbulent, External Interaction

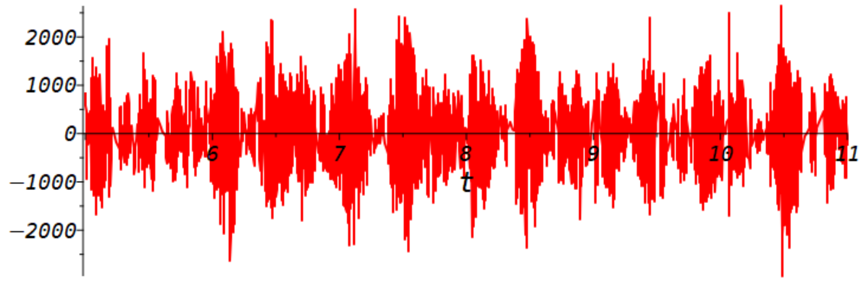
5.1. The TEE Oscillons

The turbulent, external, elementary oscillons (the tee oscillons for shortness) are computed as the superposition of the dee oscillons (see (149) of [7]) and ree oscillons (see (181) of [7]) as follows:

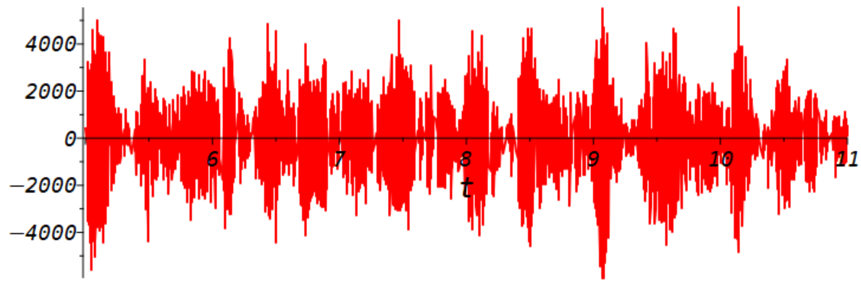
$$K_{o,t,i,m,t,j,n} = K_{o,d,i,m,d,j,n} + K_{o,r,i,m,r,j,n}, \quad i = a,b,c, \quad j = i+1, \cdots, d. \quad (153)$$

Similar to the vector dee-t and ree-t oscillons, six tee oscillons are grouped into three vector tee-t oscillons

$$\boldsymbol{K}_{t,e,e,t,q} = \boldsymbol{K}_{t,e,e,t,q}\left(\boldsymbol{f}_{t,e,e,t,q}\right),\ q = 1,2,3, \tag{154}$$



(a)



(b)

Figure 2. The drik-t oscillons: (a) $-K_{e,d,i,m,r,j,m}(t)$ for $Re = 10^3$, (b) $-K_{e,d,i,m,r,j,m}(t)$ for $Re = 10^5$.

which are formed by three 8-tuples of the turbulent, external, elementary interaction in time:

$$\begin{aligned}
\boldsymbol{f}_{t,e,e,t,1} &= \left\{\boldsymbol{f}_{d,e,e,t,1}, \boldsymbol{f}_{r,e,e,t,1}\right\} = \left\{f_{d,t,2k-1,m,n}, f_{r,t,2k-1,m,n}\right\},\\
\boldsymbol{f}_{t,e,e,t,2} &= \left\{\boldsymbol{f}_{d,e,e,t,2}, \boldsymbol{f}_{r,e,e,t,2}\right\} = \left\{f_{d,t,2k-1,m,n}, f_{r,t,2k-1,m,n}\right\},\\
\boldsymbol{f}_{t,e,e,t,3} &= \left\{\boldsymbol{f}_{d,e,e,t,3}, \boldsymbol{f}_{r,e,e,t,3}\right\} = \left\{f_{d,t,2k,m,n}, \quad f_{r,t,2k,m,n}\right\}.
\end{aligned} \tag{155}$$

Eight-tuple $\boldsymbol{f}_{t,e,e,t,1}$ consists of four deterministic sine waves $f_{d,t,2k-1,m,n}$ with frequencies $\omega_{d,t,k,m,n}$ and four random sine waves $f_{r,t,2k-1,m,n}$ with frequencies $\omega_{r,t,k,m,n}(t)$ for $k = 1,3,5,7$, and each m, n. Eight-tuple $\boldsymbol{f}_{t,e,e,t,2}$ comprises four deterministic sine waves $f_{d,2k-1,m,n}$ with frequencies $\omega_{d,t,k,m,n}$ and four random sine waves $f_{r,t,2k-1,m,n}$ with frequencies $\omega_{r,t,k,m,n}(t)$ for $k = 1,4,6,7$, and each m, n. Eight-tuple $\boldsymbol{f}_{t,e,e,t,3}$ is composed of four deterministic cosine waves $f_{d,t,2k,m,n}$ with frequencies $\omega_{d,t,k,m,n}$ and four random cosine waves $f_{r,t,2k,m,n}$ with frequencies $\omega_{r,t,k,m,n}(t)$ for $k = 1,2,7,8$, and each m, n.

For any frozen $x = x_0, y = y_0, z = z_0$, the 1st vector tee-$t$ oscillon

$$\boldsymbol{K}_{t,e,e,t,1} = \left[K_{o,t,a,m,t,b,n}, K_{o,t,c,m,t,d,n}\right]\left(\boldsymbol{f}_{t,e,e,t,1}\right) \tag{156}$$

is transformed into a list of two 4-f, 4-rf, turbulent oscillons in t, which are produced by 8-tuple $\boldsymbol{f}_{t,e,e,t,1}$ as

$$\begin{aligned}K_{o,t,a,m,t,b,n} = -\frac{\rho_c}{4}\Big[& ez_{d,m} ez_{d,n} \left(Q_{d,m} Q_{d,n} f_{d,t,1,m,n} - Q_{d,m} R_{d,n} f_{d,t,5,m,n} \right. \\ & \left. - R_{d,m} Q_{d,n} f_{d,t,9,m,n} + R_{d,m} R_{d,n} f_{d,t,13,m,n} \right) \\ & + ez_{r,m} ez_{r,n} \left(Q_{r,m} Q_{r,n} f_{r,t,1,m,n} - Q_{r,m} R_{r,n} f_{r,t,5,m,n} \right. \\ & \left. - R_{r,m} Q_{r,n} f_{r,t,9,m,n} + R_{r,m} R_{r,n} f_{r,t,13,m,n} \right)\Big], \\ K_{o,t,c,m,t,d,n} = +\frac{\rho_c}{4}\Big[& ez_{d,m} ez_{d,n} \left(Q_{d,m} Q_{d,n} f_{d,t,1,m,n} + Q_{d,m} R_{d,n} f_{d,t,5,m,n} \right. \\ & \left. + R_{d,m} Q_{d,n} f_{d,t,9,m,n} + R_{d,m} R_{d,n} f_{d,t,13,m,n} \right) \\ & + ez_{r,m} ez_{r,n} \left(Q_{r,m} Q_{r,n} f_{r,t,1,m,n} + Q_{r,m} R_{r,n} f_{r,t,5,m,n} \right. \\ & \left. + R_{r,m} Q_{r,n} f_{r,t,9,m,n} + R_{r,m} R_{r,n} f_{r,t,13,m,n} \right)\Big].\end{aligned} \tag{157}$$

The 2nd vector tee-t oscillon

$$\boldsymbol{K}_{t,e,e,t,2} = \left[K_{o,t,b,m,t,d,n}, K_{o,t,a,m,t,c,n} \right]\left(\boldsymbol{f}_{t,e,e,t,2} \right) \tag{158}$$

is presented by a list of two 4-f, 4-rf, turbulent oscillons in t, which are generated by 8-tuple $\boldsymbol{f}_{t,e,e,t,2}$ since

$$\begin{aligned}K_{o,t,b,m,t,d,n} = +\frac{\rho_c}{4}\Big[& ez_{d,m} ez_{d,n} \left(Q_{d,m} Q_{d,n} f_{d,t,1,m,n} - Q_{d,m} R_{d,n} f_{d,t,7,m,n} \right. \\ & \left. + R_{d,m} Q_{d,n} f_{d,t,11,m,n} - R_{d,m} R_{d,n} f_{d,t,13,m,n} \right) \\ & + ez_{r,m} ez_{r,n} \left(Q_{r,m} Q_{r,n} f_{r,t,1,m,n} - Q_{r,m} R_{r,n} f_{r,t,7,m,n} \right. \\ & \left. + R_{r,m} Q_{r,n} f_{r,t,11,m,n} - R_{r,m} R_{r,n} f_{r,t,13,m,n} \right)\Big], \\ K_{o,t,a,m,t,c,n} = -\frac{\rho_c}{4}\Big[& ez_{d,m} ez_{d,n} \left(Q_{d,m} Q_{d,n} f_{d,t,1,m,n} + Q_{d,m} R_{d,n} f_{d,t,7,m,n} \right. \\ & \left. - R_{d,m} Q_{d,n} f_{d,t,11,m,n} - R_{d,m} R_{d,n} f_{d,t,13,m,n} \right) \\ & + ez_{r,m} ez_{r,n} \left(Q_{r,m} Q_{r,n} f_{r,t,1,m,n} + Q_{r,m} R_{r,n} f_{r,t,7,m,n} \right. \\ & \left. - R_{r,m} Q_{r,n} f_{r,t,11,m,n} - R_{r,m} R_{r,n} f_{r,t,13,m,n} \right)\Big].\end{aligned} \tag{159}$$

The 3rd vector tee-t oscillon

$$\boldsymbol{K}_{t,e,e,t,3} = \left[K_{o,t,a,m,t,d,n}, K_{o,t,b,m,t,c,n} \right]\left(\boldsymbol{f}_{t,e,e,t,3} \right) \tag{160}$$

is visualized by a list of two 4-f, 4-rf, turbulent oscillons in t, which are formed by 8-tuple $\boldsymbol{f}_{t,e,e,t,3}$ because

$$\begin{aligned}K_{o,t,a,m,t,d,n} = +\frac{\rho_c}{4}\Big\{ & ez_{d,m} ez_{d,n} \Big[Q_{d,m} Q_{d,n} \left(f_{d,t,2,m,n} - f_{d,t,4,m,n} \right) \\ & - R_{d,m} R_{d,n} \left(f_{d,t,14,m,n} - f_{d,t,16,m,n} \right)\Big] \\ & + ez_{r,m} ez_{r,n} \Big[Q_{r,m} Q_{r,n} \left(f_{r,t,2,m,n} - f_{r,t,4,m,n} \right) \\ & - R_{r,m} R_{r,n} \left(f_{r,t,14,m,n} - f_{r,t,16,m,n} \right)\Big]\Big\}, \\ K_{o,t,b,m,t,c,n} = +\frac{\rho_c}{4}\Big\{ & ez_{d,m} ez_{d,n} \Big[Q_{d,m} Q_{d,n} \left(f_{d,t,2,m,n} + f_{d,t,4,m,n} \right) \\ & - R_{d,m} R_{d,n} \left(f_{d,t,14,m,n} + f_{d,t,16,m,n} \right)\Big] \\ & + ez_{r,m} ez_{r,n} \Big[Q_{r,m} Q_{r,n} \left(f_{r,t,2,m,n} + f_{r,t,4,m,n} \right) \\ & - R_{r,m} R_{r,n} \left(f_{r,t,14,m,n} + f_{r,t,16,m,n} \right)\Big]\Big\}.\end{aligned} \tag{161}$$

Wave parameters of the vector tee-t oscillons are provided by the corresponding wave parameters of the vector dee-t and ree-t oscillons. Like the ree-t oscillons, the tee-t oscillons are smooth random functions of time with an unbounded period.

5.2. The TEW Oscillons

The turbulent, external, wave oscillons (the tew oscillons for straightforwardness) are constructed as the superposition of the dew oscillons (see (151) of [7]) and rew oscillons (see (183) of [7]) in the following form:

$$K_{w,t,i,m,t,j,n} = K_{w,d,i,m,d,j,n} + K_{w,r,i,m,r,j,n}, \quad i = a,b,c, \quad j = i+1, \cdots, d. \tag{162}$$

Alike the vector dew-t and rew-t oscillons, six tew-t oscillons are combined into three vector tew-t oscillons

$$\boldsymbol{K}_{t,e,w,t,q} = \boldsymbol{K}_{t,e,w,t,q}\left(\boldsymbol{f}_{t,e,e,t,q}\right), \quad q = 1,2,3, \tag{163}$$

which are produced by 8-tuples $\boldsymbol{f}_{t,e,e,t,q}$.

For any frozen $x = x_0, y = y_0, z = z_0$, the 1st vector tew-$t$ oscillon

$$\boldsymbol{K}_{t,e,w,t,1} = \left[K_{w,t,a,m,t,b,n}, K_{w,t,c,m,t,d,n}\right]\left(\boldsymbol{f}_{t,e,e,t,1}\right) \tag{164}$$

is represented by a list of two 4-f, 4-rf, turbulent oscillons in t, which are formed by 8-tuple $\boldsymbol{f}_{t,e,e,t,1}$ in view of

$$\begin{aligned} K_{w,t,a,m,t,b,n} = &+\frac{\rho_c}{4}\Big\{ez_{d,m} ez_{d,n}\Big[\mathrm{N}_{d,m,d,n}\left(Q_{d,m} Q_{d,n} f_{d,t,1,m,n} + R_{d,m} R_{d,n} f_{d,t,13,m,n}\right) \\ &+\Lambda_{d,m,d,n}\left(Q_{d,m} R_{d,n} f_{d,t,5,m,n} + R_{d,m} Q_{d,n} f_{d,t,9,m,n}\right)\Big] \\ &+ ez_{r,m} ez_{r,n}\Big[\mathrm{N}_{r,m,r,n}\left(Q_{r,m} Q_{r,n} f_{r,t,1,m,n} + R_{r,m} R_{r,n} f_{r,t,13,m,n}\right) \\ &+\Lambda_{r,m,r,n}\left(Q_{r,m} R_{r,n} f_{r,t,5,m,n} + R_{r,m} Q_{r,n} f_{r,t,9,m,n}\right)\Big]\Big\}, \end{aligned}$$

$$\begin{aligned} K_{w,t,c,m,t,d,n} = &-\frac{\rho_c}{4}\Big\{ez_{d,m} ez_{d,n}\Big[\mathrm{N}_{d,m,d,n}\left(Q_{d,m} Q_{d,n} f_{d,t,1,m,n} + R_{d,m} R_{d,n} f_{d,t,13,m,n}\right) \\ &-\Lambda_{d,m,d,n}\left(Q_{d,m} R_{d,n} f_{d,t,5,m,n} + R_{d,m} Q_{d,n} f_{d,t,9,m,n}\right)\Big] \\ &+ ez_{r,m} ez_{r,n}\Big[\mathrm{N}_{r,m,r,n}\left(Q_{r,m} Q_{r,n} f_{r,t,1,m,n} + R_{r,m} R_{r,n} f_{r,t,13,m,n}\right) \\ &-\Lambda_{r,m,r,n}\left(Q_{r,m} R_{r,n} f_{r,t,5,m,n} + R_{r,m} Q_{r,n} f_{r,t,9,m,n}\right)\Big]\Big\}. \end{aligned} \tag{165}$$

The 2nd vector tew-t oscillon

$$\boldsymbol{K}_{t,e,w,t,2} = \left[K_{w,t,b,m,t,d,n}, K_{w,t,a,m,t,c,n}\right]\left(\boldsymbol{f}_{t,e,e,t,2}\right) \tag{166}$$

is exposed by a list of two 4-f, 4-rf, turbulent oscillons in t, which are established by 8-tuple $\boldsymbol{f}_{t,e,e,t,2}$ since

$$\begin{aligned} K_{w,t,b,m,t,d,n} = &-\frac{\rho_c}{4}\Big\{ez_{d,m} ez_{d,n}\Big[\mathrm{N}_{d,m,d,n}\left(Q_{d,m} Q_{d,n} f_{d,t,1,m,n} - R_{d,m} R_{d,n} f_{d,t,13,m,n}\right) \\ &+\mathrm{K}_{d,m,d,n}\left(Q_{d,m} R_{d,n} f_{d,t,7,m,n} - R_{d,m} Q_{d,n} f_{d,t,11,m,n}\right)\Big] \\ &+ ez_{r,m} ez_{r,n}\Big[\mathrm{N}_{r,m,r,n}\left(Q_{r,m} Q_{r,n} f_{r,t,1,m,n} - R_{r,m} R_{r,n} f_{r,t,13,m,n}\right) \\ &+\mathrm{K}_{r,m,r,n}\left(Q_{r,m} R_{r,n} f_{r,t,7,m,n} - R_{r,m} Q_{r,n} f_{r,t,11,m,n}\right)\Big]\Big\}, \end{aligned}$$

$$K_{w,t,a,m,t,c,n} = +\frac{\rho_c}{4}\Big\{ez_{d,m}ez_{d,n}\Big[\mathrm{N}_{d,m,d,n}\left(Q_{d,m}\,Q_{d,n}\,f_{d,t,1,m,n} - R_{d,m}\,R_{d,n}\,f_{d,t,13,m,n}\right) \\ -\mathrm{K}_{d,m,d,n}\left(Q_{d,m}\,R_{d,n}\,f_{d,t,7,m,n} - R_{d,m}\,Q_{d,n}\,f_{d,t,11,m,n}\right)\Big] \\ + ez_{r,m}\,ez_{r,n}\Big[\mathrm{N}_{r,m,r,n}\left(Q_{r,m}\,Q_{r,n}\,f_{r,t,1,m,n} - R_{r,m}\,R_{r,n}\,f_{r,t,13,m,n}\right) \\ -\mathrm{K}_{r,m,r,n}\left(Q_{r,m}\,R_{r,n}\,f_{r,t,7,m,n} - R_{r,m}\,Q_{r,n}\,f_{r,t,11,m,n}\right)\Big]\Big\}. \tag{167}$$

The 3rd vector tew-t oscillon

$$\boldsymbol{K}_{t,e,w,t,3} = \left[K_{w,t,a,m,t,d,n} K_{w,t,b,m,t,c,n},\right]\left(\boldsymbol{f}_{t,e,e,t,3}\right) \tag{168}$$

is exhibited by a list of two 4-f, 4-rf, turbulent oscillons in t, which depend on 8-tuple $\boldsymbol{f}_{t,e,e,t,3}$ as

$$K_{w,t,a,m,t,d,n} = -\frac{\rho_c}{4}\Big\{ez_{d,m}ez_{d,n}\Big[\mathrm{N}_{d,m,d,n}\left(Q_{d,m}\,Q_{d,n}\,f_{d,t,2,m,n} - R_{d,m}\,R_{d,n}\,f_{d,t,14,m,n}\right) \\ +\mathrm{M}_{d,m,d,n}\left(Q_{d,m}\,R_{d,n}\,f_{d,t,4,m,n} - R_{d,m}\,Q_{d,n}\,f_{d,t,16,m,n}\right)\Big] \\ + ez_{r,m}\,ez_{r,n}\Big[\mathrm{N}_{r,m,r,n}\left(Q_{r,m}\,Q_{r,n}\,f_{r,t,2,m,n} - R_{r,m}\,R_{r,n}\,f_{r,t,14,m,n}\right) \\ +\mathrm{M}_{r,m,r,n}\left(Q_{r,m}\,R_{r,n}\,f_{r,t,4,m,n} - R_{r,m}\,Q_{r,n}\,f_{r,t,16,m,n}\right)\Big]\Big\},$$

$$K_{w,t,b,m,t,c,n} = -\frac{\rho_c}{4}\Big\{ez_{d,m}ez_{d,n}\Big[\mathrm{N}_{d,m,d,n}\left(Q_{d,m}\,Q_{d,n}\,f_{d,t,2,m,n} - R_{d,m}\,R_{d,n}\,f_{d,t,14,m,n}\right) \\ -\mathrm{M}_{d,m,d,n}\left(Q_{d,m}\,R_{d,n}\,f_{d,t,4,m,n} - R_{d,m}\,Q_{d,n}\,f_{d,t,16,m,n}\right)\Big] \\ + ez_{r,m}\,ez_{r,n}\Big[\mathrm{N}_{r,m,r,n}\left(Q_{r,m}\,Q_{r,n}\,f_{r,t,2,m,n} - R_{r,m}\,R_{r,n}\,f_{r,t,14,m,n}\right) \\ -\mathrm{M}_{r,m,r,n}\left(Q_{r,m}\,R_{r,n}\,f_{r,t,4,m,n} - R_{r,m}\,Q_{r,n}\,f_{r,t,16,m,n}\right)\Big]\Big\}. \tag{169}$$

where

$$\begin{aligned} \mathrm{K}_{d,m,d,n} &= +\kappa_{d,m}\kappa_{d,n} - \lambda_{d,m}\lambda_{d,n} + \mu_{d,m}\mu_{d,n}, \\ \Lambda_{d,m,d,n} &= -\kappa_{d,m}\kappa_{d,n} + \lambda_{d,m}\lambda_{d,n} + \mu_{d,m}\mu_{d,n}, \\ \mathrm{M}_{d,m,d,n} &= +\kappa_{d,m}\kappa_{d,n} + \lambda_{d,m}\lambda_{d,n} + \mu_{d,m}\mu_{d,n}, \\ \mathrm{N}_{d,m,d,n} &= +\kappa_{d,m}\kappa_{d,n} + \lambda_{d,m}\lambda_{d,n} - \mu_{d,m}\mu_{d,n} \end{aligned} \tag{170}$$

and

$$\begin{aligned} \mathrm{K}_{r,m,r,n} &= +\kappa_{r,m}\kappa_{r,n} - \lambda_{r,m}\lambda_{r,n} + \mu_{r,m}\mu_{r,n}, \\ \Lambda_{r,m,r,n} &= -\kappa_{r,m}\kappa_{r,n} + \lambda_{r,m}\lambda_{r,n} + \mu_{r,m}\mu_{r,n}, \\ \mathrm{M}_{r,m,r,n} &= +\kappa_{r,m}\kappa_{r,n} + \lambda_{r,m}\lambda_{r,n} + \mu_{r,m}\mu_{r,n}, \\ \mathrm{N}_{r,m,r,n} &= +\kappa_{r,m}\kappa_{r,n} + \lambda_{r,m}\lambda_{r,n} - \mu_{r,m}\mu_{r,n}. \end{aligned} \tag{171}$$

Wave parameters of the vector tew-t oscillons are provided by the relevant wave parameters of the vector dew-t and rew-t oscillons. Likewise the tee-t oscillons, the tew-t oscillons are smooth random functions of time with an unbounded period.

5.3. The TEG Oscillon

The turbulent, external, group pulson (the teg oscillon for easiness) is composed as the superposition of the deg oscillon (see (152) of [7]) and reg oscillon (see

(184) of [7]), *i.e.*

$$K_{g,t,i,m,t,j,n} = K_{g,d,i,m,d,j,n} + K_{g,r,i,m,r,j,n}. \tag{172}$$

Analogous to the deg-t and reg-t oscillons, the teg-t oscillon

$$K_{g,t,i,m,t,j,n} = K_{g,t,i,m,t,j,n}\left(\boldsymbol{f}_{t,e,g,t}\right), \tag{173}$$

where 12-tuple of the turbulent, external, group interaction in t

$$\boldsymbol{f}_{t,e,g,t} = \left\{\boldsymbol{f}_{d,e,g,t}, \boldsymbol{f}_{r,e,g,t}\right\} = \left\{f_{d,t,2k-1,m,n}, f_{d,t,2l,m,n}, f_{r,t,2k-1,m,n}, f_{r,t,2l,m,n}\right\} \tag{174}$$

includes four deterministic sine waves $f_{d,t,2k-1,m,n}$ with frequencies $\omega_{d,t,k,m,n}$, two deterministic cosine waves $f_{d,t,2l,m,n}$ with frequencies $\omega_{d,t,l,m,n}$, four random sine waves $f_{r,t,2k-1,m,n}$ with frequencies $\omega_{r,t,k,m,n}(t)$, and two random cosine waves $f_{r,t,2l,m,n}$ with frequencies $\omega_{r,t,l,m,n}(t)$, for $k = 3,4,5,6$, $l = 1,7$, and each m, n.

For any frozen $x = x_0, y = y_0, z = z_0$, the teg-$t$ oscillon is displayed by a 6-f, 6-rf, turbulent oscillon in t, which is produced by 12-tuple $\boldsymbol{f}_{t,e,g,t}$ since

$$\begin{aligned} K_{g,t,i,m,t,j,n} = \frac{\rho_c}{2}\Big\{ & ez_{d,m} ez_{d,n}\Big[\Lambda_{d,m,d,n}\left(Q_{d,m} R_{d,n} f_{d,t,5,m,n} + R_{d,m} Q_{d,n} f_{d,t,9,m,n}\right) \\ & -\mathrm{K}_{d,m,d,n}\left(Q_{d,m} R_{d,n} f_{d,t,7,m,n} - R_{d,m} Q_{d,n} f_{d,t,11,m,n}\right) \\ & -\mathrm{N}_{d,m,d,n}\left(Q_{d,m} Q_{d,n} f_{d,t,2,m,n} - R_{d,m} R_{d,n} f_{d,t,14,m,n}\right)\Big] \\ & + ez_{r,m} ez_{r,n}\Big[\Lambda_{r,m,r,n}\left(Q_{r,m} R_{r,n} f_{r,t,5,m,n} + R_{r,m} Q_{r,n} f_{r,t,9,m,n}\right) \\ & -\mathrm{K}_{r,m,r,n}\left(Q_{r,m} R_{r,n} f_{r,t,7,m,n} - R_{r,m} Q_{r,n} f_{r,t,11,m,n}\right) \\ & -\mathrm{N}_{r,m,r,n}\left(Q_{r,m} Q_{r,n} f_{r,t,2,m,n} - R_{r,m} R_{r,n} f_{r,t,14,m,n}\right)\Big]\Big\}. \end{aligned} \tag{175}$$

Wave parameters of the teg-t oscillon are given by the appropriate wave parameters of the deg-t and reg-t oscillons. Parallel to the tew-t oscillons, the teg-t oscillon is a smooth random function of time with an unbounded period.

5.4. The TEK Oscillon

The turbulent, external, kinetic-energy oscillon (the tek oscillon for fastness), which is set as the superposition of the dek and rek oscillons, may be represented as

$$K_{e,t,i,m,t,j,n} = K_{e,d,i,m,d,j,n} + K_{e,r,i,m,r,j,n}. \tag{176}$$

With the help of (77) and (120) of [7], the inhomogeneous Fourier expansion the tek-t oscillon takes the following form:

$$K_{e,t,i,m,t,j,n} = \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} K_{g,t,i,m,t,j,n}. \tag{177}$$

If all deterministic and random frequencies of the tek-t oscillon are distinct, then the tek-t oscillon is visualized as a $3M(M-1)$-f, $3M(M-1)$-rf oscillon. For any frozen $x = x_0, y = y_0, z = z_0$, the tek-$t$ oscillon is converted into the $3M(M-1)$-f, $3M(M-1)$-rf, turbulent oscillon in t, which is produced by $M(M-1)/2$ 12-tuples $\boldsymbol{f}_{t,e,g,t}$ with frequencies $\omega_{d,t,k,m,n}, \omega_{r,t,k,m,n}(t)$ for

$k = 1,3,4,5,6,7$, all *m*, *n*, *Re*, and wave parameters of the teg-*t* oscillons. Like the teg-*t* oscillon, the tek-*t* oscillon is a smooth random function of time with an unbounded period.

If N deterministic frequencies are repeated, then the number of independent modes of the tek-*t* oscillon diminishes to $\left[3M(M-1)-N\right]$-f, $3M(M-1)$-rf due to the frequency resonance.

The dek-*t* and tek-*t* oscillons for $x = x_0, y = y_0, z = z_0, Re = 10^3, Re = 10^5$, and wave parameters (129)-(135) are visualized on $\left[t_0, t_0 + T_{e,d,e}\right]$ in **Figure 3**. Since the range of the 18-rf, random rek-*t* oscillon at $Re = 10^3$ is $[-350, 600]$, the 15-f, 18-rf, turbulent tek-*t* oscillon in **Figure 3(b)** is altered lightly compared with the 15-f, deterministic dek-*t* oscillon in **Figure 3(a)**. The range of the 18-rf, random rek-*t* oscillon at $Re = 10^5$ becomes $[-1850, 1600]$, therefore the range and the shape of the 15-f, 18-rf, turbulent tek-*t* oscillon in **Figure 3(c)** are modified significantly. The dek-*t* oscillon is the 15-f oscillon because $\omega_{d,t,3,1,2} = \omega_{d,t,4,2,3} = 7\pi/3$, $\omega_{d,t,4,1,2} = \omega_{d,t,7,1,2} = \pi$, and $\omega_{d,t,3,2,3} = -\omega_{d,t,6,2,3} = 13\pi/3$.

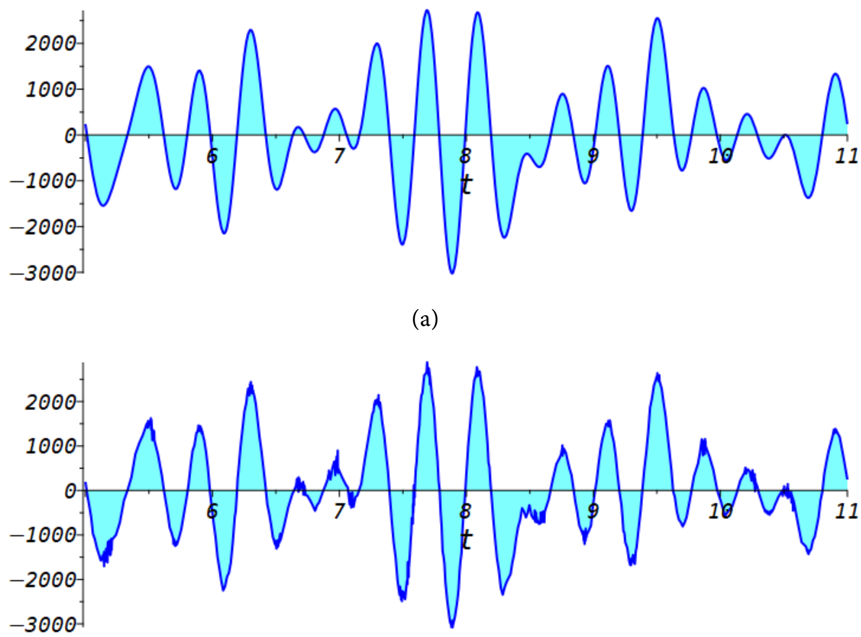
6. Oscillons of Turbulent, Diagonal Interaction

6.1. The TDE Oscillons

The turbulent, diagonal, elementary oscillons (the tde oscillons for clarity) are set as the superposition of the dde oscillons (see (144) of [7]) and rde oscillons (see (176) of [7]) as follows:

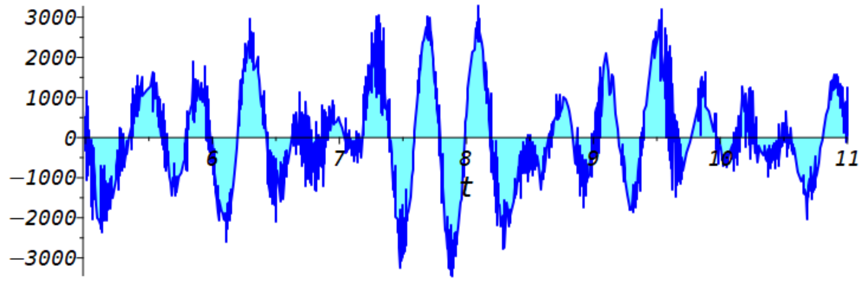
$$K_{o,t,i,m,t,i,n} = K_{o,d,i,m,d,i,n} + K_{o,r,i,m,r,i,n}, \quad i = a,b,c,d. \tag{178}$$

Like the vector dde-*t* and rde-*t* oscillons, four tde-*t* oscillons are grouped in a vector tde-*t* oscillon



(a)

(b)



(c)

Figure 3. The dek-t and tek-t oscillons: (a) $-K_{e,d,i,m,d,j,n}(t)$, (b) $-K_{e,t,i,m,t,j,n}(t)$ for $Re = 10^3$, (c) $-K_{e,t,i,m,t,j,n}(t)$ for $Re = 10^5$.

$$\boldsymbol{K}_{t,d,e,t} = \left[K_{o,t,a,m,t,a,n}, K_{o,t,b,m,t,b,n}, K_{o,t,d,m,t,d,n}, K_{o,t,c,m,t,c,n}\right]\left(\boldsymbol{f}_{t,d,e,t}\right) \tag{179}$$

which is constructed using 16-tuple of the turbulent, diagonal, elementary interaction in t

$$\boldsymbol{f}_{t,d,e,t} = \left\{\boldsymbol{f}_{d,d,e,t}, \boldsymbol{f}_{r,d,e,t}\right\} = \left\{f_{d,t,2k,m,n}, f_{r,t,2k,m,n}\right\}, \tag{180}$$

where 16-tuple $\boldsymbol{f}_{t,d,e,t}$ embraces eight deterministic cosine waves $f_{d,t,2k,m,n}$ with frequencies $\omega_{d,t,k,m,n}$ and eight random cosine waves $f_{r,t,2k,m,n}$ with frequencies $\omega_{r,t,k,m,n}(t)$ for $k = 1, 2, \cdots, 8$ and each m, n.

For any frozen $x = x_0, y = y_0, z = z_0$, the vector tde-$t$ oscillon is represented by a list of four 8-f, 8-rf, turbulent oscillons in t, which depend on16-tuple $\boldsymbol{f}_{t,d,e,t}$ as

$$\begin{aligned} K_{o,t,a,m,t,a,n} &= -\frac{\rho_c}{8} \\ &\times\Big\{ez_{d,m}ez_{d,n}\Big[Q_{d,m}Q_{d,n}\left(f_{d,t,2,m,n} - f_{d,t,4,m,n}\right) - Q_{d,m}R_{d,n}\left(f_{d,t,6,m,n} - f_{d,t,8,m,n}\right) \\ &\qquad - R_{d,m}Q_{d,n}\left(f_{d,t,10,m,n} - f_{d,t,12,m,n}\right) + R_{d,m}R_{d,n}\left(f_{d,t,14,m,n} - f_{d,t,16,m,n}\right)\Big] \\ &\quad + ez_{r,m}ez_{r,n}\Big[Q_{r,m}Q_{r,n}\left(f_{r,t,2,m,n} - f_{r,t,4,m,n}\right) - Q_{r,m}R_{r,n}\left(f_{r,t,6,m,n} - f_{r,t,8,m,n}\right) \\ &\qquad - R_{r,m}Q_{r,n}\left(f_{r,t,10,m,n} - f_{r,t,12,m,n}\right) + R_{r,m}R_{r,n}\left(f_{r,t,14,m,n} - f_{r,t,16,m,n}\right)\Big]\Big\}, \end{aligned}$$

$$\begin{aligned} K_{o,t,b,m,t,b,n} &= +\frac{\rho_c}{8} \\ &\times\Big\{ez_{d,m}ez_{d,n}\Big[Q_{d,m}Q_{d,n}\left(f_{d,t,2,m,n} + f_{d,t,4,m,n}\right) - Q_{d,m}R_{d,n}\left(f_{d,t,6,m,n} + f_{d,t,8,m,n}\right) \\ &\qquad - R_{d,m}Q_{d,n}\left(f_{d,t,10,m,n} + f_{d,t,12,m,n}\right) + R_{d,m}R_{d,n}\left(f_{d,t,14,m,n} + f_{d,t,16,m,n}\right)\Big] \\ &\quad + ez_{r,m}ez_{r,n}\Big[Q_{r,m}Q_{r,n}\left(f_{r,t,2,m,n} + f_{r,t,4,m,n}\right) - Q_{r,m}R_{r,n}\left(f_{r,t,6,m,n} + f_{r,t,8,m,n}\right) \\ &\qquad - R_{r,m}Q_{r,n}\left(f_{r,t,10,m,n} + f_{r,t,12,m,n}\right) + R_{r,m}R_{r,n}\left(f_{r,t,14,m,n} + f_{r,t,16,m,n}\right)\Big]\Big\}, \end{aligned}$$

$$\begin{aligned} K_{o,t,d,m,t,d,n} &= -\frac{\rho_c}{8} \\ &\times\Big\{ez_{d,m}ez_{d,n}\Big[Q_{d,m}Q_{d,n}\left(f_{d,t,2,m,n} - f_{d,t,4,m,n}\right) + Q_{d,m}R_{d,n}\left(f_{d,t,6,m,n} - f_{d,t,8,m,n}\right) \\ &\qquad + R_{d,m}Q_{d,n}\left(f_{d,t,10,m,n} - f_{d,t,12,m,n}\right) + R_{d,m}R_{d,n}\left(f_{d,t,14,m,n} - f_{d,t,16,m,n}\right)\Big] \\ &\quad + ez_{r,m}ez_{r,n}\Big[Q_{r,m}Q_{r,n}\left(f_{r,t,2,m,n} - f_{r,t,4,m,n}\right) + Q_{r,m}R_{r,n}\left(f_{r,t,6,m,n} - f_{r,t,8,m,n}\right) \\ &\qquad + R_{r,m}Q_{r,n}\left(f_{r,t,10,m,n} - f_{r,t,12,m,n}\right) + R_{r,m}R_{r,n}\left(f_{r,t,14,m,n} - f_{r,t,16,m,n}\right)\Big]\Big\}, \end{aligned}$$

$$K_{o,t,c,m,t,c,n} = +\frac{\rho_c}{8}$$
$$\times\{ez_{d,m}ez_{d,n}\left[Q_{d,m}\,Q_{d,n}\left(f_{d,t,2,m,n}+f_{d,t,4,m,n}\right)+Q_{d,m}\,R_{d,n}\left(f_{d,t,6,m,n}+f_{d,t,8,m,n}\right)\right.$$
$$\left.+R_{d,m}\,Q_{d,n}\left(f_{d,t,10,m,n}+f_{d,t,12,m,n}\right)+R_{d,m}\,R_{d,n}\left(f_{d,t,14,m,n}+f_{d,t,16,m,n}\right)\right] \quad (181)$$
$$+ez_{r,m}ez_{r,n}\left[Q_{r,m}\,Q_{r,n}\left(f_{r,t,2,m,n}+f_{r,t,4,m,n}\right)+Q_{r,m}\,R_{r,n}\left(f_{r,t,6,m,n}+f_{r,t,8,m,n}\right)\right.$$
$$\left.+R_{r,m}\,Q_{r,n}\left(f_{r,t,10,m,n}+f_{r,t,12,m,n}\right)+R_{r,m}\,R_{r,n}\left(f_{r,t,14,m,n}+f_{r,t,16,m,n}\right)\right]\}.$$

Wave parameters of the vector tde-t oscillon indicated by similar wave parameters of the vector dde-t and rde-t oscillons. The tde-t oscillons are smooth random functions of time with an unbounded period.

6.2. The TDW Oscillons

The turbulent, diagonal, wave oscillons (the tdw oscillons for clarity) are composed with the help of the superposition of the ddw oscillons (see (146) of [7]) and rdw oscillons (see (178) of [7]) as follows:

$$K_{w,t,i,m,t,i,n} = K_{w,d,i,m,d,i,n} + K_{w,r,i,m,r,i,n},\quad i=a,b,c,d. \quad (182)$$

Parallel to the vector ddw-t and rdw-t oscillons, four tdw-t oscillons are grouped into a vector tdw-t oscillon

$$\boldsymbol{K}_{t,d,w,t} = \left[K_{w,t,a,m,t,a,n}, K_{w,t,b,m,t,b,n}, K_{w,t,d,m,t,d,n}, K_{w,t,c,m,t,c,n}\right]\left(\boldsymbol{f}_{t,d,e,t}\right), \quad (183)$$

which is formed by 16-tuple $\boldsymbol{f}_{t,d,e,t}$.

For any frozen $x=x_0, y=y_0, z=z_0$, the vector tdw-t oscillon is exhibited by a list of four 8-f, 8-rf, turbulent oscillons in t, which are established by 16-tuple $\boldsymbol{f}_{t,d,e,t}$ for the reason that

$$K_{w,t,a,m,t,a,n} = +\frac{\rho_c}{8}\{ez_{d,m}ez_{d,n}\left[Q_{d,m}\,Q_{d,n}\left(\mathrm{N}_{d,m,d,n}f_{d,t,2,m,n}+\mathrm{M}_{d,m,d,n}f_{d,t,4,m,n}\right)\right.$$
$$+Q_{d,m}\,R_{d,n}\left(\Lambda_{d,m,d,n}f_{d,t,6,m,n}-\mathrm{K}_{d,m,d,n}f_{d,t,8,m,n}\right)$$
$$+R_{d,m}\,Q_{d,n}\left(\Lambda_{d,m,d,n}f_{d,t,10,m,n}-\mathrm{K}_{d,m,d,n}f_{d,t,12,m,n}\right)$$
$$\left.+R_{d,m}\,R_{d,n}\left(\mathrm{N}_{d,m,d,n}f_{d,t,14,m,n}+\mathrm{M}_{d,m,d,n}f_{d,t,16,m,n}\right)\right]$$
$$+ez_{r,m}\,ez_{r,n}\left[Q_{r,m}\,Q_{r,n}\left(\mathrm{N}_{r,m,r,n}f_{r,t,2,m,n}+\mathrm{M}_{r,m,r,n}f_{r,t,4,m,n}\right)\right.$$
$$+Q_{r,m}\,R_{r,n}\left(\Lambda_{r,m,r,n}f_{r,t,6,m,n}-\mathrm{K}_{r,m,r,n}f_{r,t,8,m,n}\right)$$
$$+R_{r,m}\,Q_{r,n}\left(\Lambda_{r,m,r,n}f_{r,t,10,m,n}-\mathrm{K}_{r,m,r,n}f_{r,t,12,m,n}\right)$$
$$\left.+R_{r,m}\,R_{r,n}\left(\mathrm{N}_{r,m,r,n}f_{r,t,14,m,n}+\mathrm{M}_{r,m,r,n}f_{r,t,16,m,n}\right)\right]\},$$

$$K_{w,t,b,m,t,b,n} = -\frac{\rho_c}{8}\{ez_{d,m}ez_{d,n}\left[Q_{d,m}\,Q_{d,n}\left(\mathrm{N}_{d,m,d,n}f_{d,t,2,m,n}-\mathrm{M}_{d,m,d,n}f_{d,t,4,m,n}\right)\right.$$
$$+Q_{d,m}\,R_{d,n}\left(\Lambda_{d,m,d,n}f_{d,t,6,m,n}+\mathrm{K}_{d,m,d,n}f_{d,t,8,m,n}\right)$$
$$+R_{d,m}\,Q_{d,n}\left(\Lambda_{d,m,d,n}f_{d,t,10,m,n}+\mathrm{K}_{d,m,d,n}f_{d,t,12,m,n}\right)$$
$$\left.+R_{d,m}\,R_{d,n}\left(\mathrm{N}_{d,m,d,n}f_{d,t,14,m,n}-\mathrm{M}_{d,m,d,n}f_{d,t,16,m,n}\right)\right]$$
$$+ez_{r,m}\,ez_{r,n}\left[Q_{r,m}\,Q_{r,n}\left(\mathrm{N}_{r,m,r,n}f_{r,t,2,m,n}-\mathrm{M}_{r,m,r,n}f_{r,t,4,m,n}\right)\right.$$
$$+Q_{r,m}\,R_{r,n}\left(\Lambda_{r,m,r,n}f_{r,t,6,m,n}+\mathrm{K}_{r,m,r,n}f_{r,t,8,m,n}\right)$$
$$+R_{r,m}\,Q_{r,n}\left(\Lambda_{r,m,r,n}f_{r,t,10,m,n}+\mathrm{K}_{r,m,r,n}f_{r,t,12,m,n}\right)$$
$$\left.+R_{r,m}\,R_{r,n}\left(\mathrm{N}_{r,m,r,n}f_{r,t,14,m,n}-\mathrm{M}_{r,m,r,n}f_{r,t,16,m,n}\right)\right]\},$$

$$\begin{aligned} K_{w,t,d,m,t,d,n} =& +\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{d,n}\Big[Q_{d,m}\,Q_{d,n}\left(\mathrm{N}_{d,m,d,n}\,f_{d,t,2,m,n}+\mathrm{M}_{d,m,d,n}\,f_{d,t,4,m,n}\right)\\ &-Q_{d,m}\,R_{d,n}\left(\Lambda_{d,m,d,n}\,f_{d,t,6,m,n}-\mathrm{K}_{d,m,d,n}\,f_{d,t,8,m,n}\right)\\ &-R_{d,m}\,Q_{d,n}\left(\Lambda_{d,m,d,n}\,f_{d,t,10,m,n}-\mathrm{K}_{d,m,d,n}\,f_{d,t,12,m,n}\right)\\ &+R_{d,m}\,R_{d,n}\left(\mathrm{N}_{d,m,d,n}\,f_{d,t,14,m,n}+\mathrm{M}_{d,m,d,n}\,f_{d,t,16,m,n}\right)\Big]\\ &+ez_{r,m}\,ez_{r,n}\Big[Q_{r,m}\,Q_{r,n}\left(\mathrm{N}_{r,m,r,n}\,f_{r,t,2,m,n}+\mathrm{M}_{r,m,r,n}\,f_{r,t,4,m,n}\right)\\ &-Q_{r,m}\,R_{r,n}\left(\Lambda_{r,m,r,n}\,f_{r,t,6,m,n}-\mathrm{K}_{r,m,r,n}\,f_{r,t,8,m,n}\right)\\ &-R_{r,m}\,Q_{r,n}\left(\Lambda_{r,m,r,n}\,f_{r,t,10,m,n}-\mathrm{K}_{r,m,r,n}\,f_{r,t,12,m,n}\right)\\ &+R_{r,m}\,R_{r,n}\left(\mathrm{N}_{r,m,r,n}\,f_{r,t,14,m,n}+\mathrm{M}_{r,m,r,n}\,f_{r,t,16,m,n}\right)\Big]\Big\},\\ K_{w,t,c,m,t,c,n} =& -\frac{\rho_c}{8}\Big\{ez_{d,m}ez_{d,n}\Big[Q_{d,m}\,Q_{d,n}\left(\mathrm{N}_{d,m,d,n}\,f_{d,t,2,m,n}-\mathrm{M}_{d,m,d,n}\,f_{d,t,4,m,n}\right)\\ &-Q_{d,m}\,R_{d,n}\left(\Lambda_{d,m,d,n}\,f_{d,t,6,m,n}+\mathrm{K}_{d,m,d,n}\,f_{d,t,8,m,n}\right)\\ &-R_{d,m}\,Q_{d,n}\left(\Lambda_{d,m,d,n}\,f_{d,t,10,m,n}+\mathrm{K}_{d,m,d,n}\,f_{d,t,12,m,n}\right)\\ &+R_{d,m}\,R_{d,n}\left(\mathrm{N}_{d,m,d,n}\,f_{d,t,14,m,n}-\mathrm{M}_{d,m,d,n}\,f_{d,t,16,m,n}\right)\Big]\\ &+ez_{r,m}\,ez_{r,n}\Big[Q_{r,m}\,Q_{r,n}\left(\mathrm{N}_{r,m,r,n}\,f_{r,t,2,m,n}-\mathrm{M}_{r,m,r,n}\,f_{r,t,4,m,n}\right)\\ &-Q_{r,m}\,R_{r,n}\left(\Lambda_{r,m,r,n}\,f_{r,t,6,m,n}+\mathrm{K}_{r,m,r,n}\,f_{r,t,8,m,n}\right)\\ &-R_{r,m}\,Q_{r,n}\left(\Lambda_{r,m,r,n}\,f_{r,t,10,m,n}+\mathrm{K}_{r,m,r,n}\,f_{r,t,12,m,n}\right)\\ &+R_{r,m}\,R_{r,n}\left(\mathrm{N}_{r,m,r,n}\,f_{r,t,14,m,n}-\mathrm{M}_{r,m,r,n}\,f_{r,t,16,m,n}\right)\Big]\Big\}. \end{aligned} \tag{184}$$

Wave parameters of the vector tdw-t oscillons are given by relevant wave parameters of the vector ddw-t and rdw-t oscillons. Analogous to the tde-t oscillons, the tdw-t oscillons are smooth random functions of time with an unbounded period.

6.3. The TDG Oscillon

The turbulent, diagonal, group oscillon (the tdg oscillon for simplicity) is set as the superposition of the ddg oscillon (see (147) of [7]) and rdg oscillon (see (179) of [7]), *viz.*

$$K_{g,t,i,m,t,i,n} = K_{g,d,i,m,d,i,n} + K_{g,r,i,m,r,i,n}. \tag{185}$$

Alike the vector ddg-t and rdg-t oscillons, the tdg-t oscillon

$$K_{g,t,i,m,t,i,n} = K_{g,t,i,m,t,i,n}\left(\boldsymbol{f}_{t,d,g,t}\right), \tag{186}$$

where 4-tuple of the turbulent, diagonal, group, interaction in t

$$\boldsymbol{f}_{t,d,g,t} = \left\{\boldsymbol{f}_{d,d,g,t}, \boldsymbol{f}_{r,d,g,t}\right\} = \left\{f_{d,t,2k,m,n}, f_{r,t,2k,m,n}\right\} \tag{187}$$

is composed of two deterministic cosine waves $f_{d,t,2k,m,n}$ with frequencies $\omega_{d,t,k,m,n}$ and two random cosine waves $f_{r,t,2k,m,n}$ with frequencies $\omega_{r,t,k,m,n}(t)$ for $k = 2,8$, and each m, n.

For any frozen $x = x_0, y = y_0, z = z_0$, the tdg-$t$ oscillon is represented by a 2-f, 2-rf, turbulent oscillon in t, which is produced by 4-tuple $\boldsymbol{f}_{t,d,g,t}$ because

$$K_{g,t,i,m,t,i,n} = \frac{\rho_c}{2}\Big[ez_{d,m} ez_{d,n}\, \mathrm{M}_{d,m,d,n}\left(Q_{d,m}\, Q_{d,n}\, f_{d,t,4,m,n} + R_{d,m}\, R_{d,n}\, f_{d,t,16,m,n}\right) + ez_{r,m}\, ez_{r,n}\, \mathrm{M}_{r,m,r,n}\left(Q_{r,m}\, Q_{r,n}\, f_{r,t,4,m,n} + R_{r,m}\, R_{r,n}\, f_{r,t,16,m,n}\right)\Big]. \quad (188)$$

Wave parameters of the tdg-t oscillon are indicated by similar wave parameters of the ddg-t and rdg-t oscillons. Parallel to the tdw-t oscillons, the tdg-t oscillon is a smooth random function of time with an unbounded period.

6.4. The TDK Oscillon

The turbulent, diagonal, kinetic-energy oscillon (the tdk oscillon for terseness), which is established by the superposition of the ddk and rdk oscillons, becomes

$$K_{e,t,i,m,t,i,n} = K_{e,d,i,m,d,i,n} + K_{e,r,i,m,r,i,n}. \quad (189)$$

With the help of (64) and (107) of [7], we obtain the inhomogeneous Fourier form of the tdk-t oscillon

$$K_{e,t,i,m,t,i,n} = \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} K_{g,t,i,m,t,i,n}. \quad (190)$$

If all frequencies of the tdg-t oscillons are distinct, then the tdk-t oscillon is displayed as a $M(M-1)$-f, $M(M-1)$-rf oscillon. For any frozen $x = x_0, y = y_0, z = z_0$, the tdk-$t$ oscillon is converted into the $M(M-1)$-f, $M(M-1)$-rf, turbulent oscillon in t, which is formed by $M(M-1)/2$ 4-tuples $\boldsymbol{f}_{t,d,g,t}$ with frequencies $\omega_{d,t,k,m,n}, \omega_{r,t,k,m,n}(t)$ for $k = 2,8$, all m, n, Re, and wave parameters of the tdg-t oscillons. Like the tdg-t oscillon, the tdk-t oscillon is a smooth random function of time with an unbounded period.

If N deterministic frequencies are repeated, then the number of independent modes of the tdk-t oscillon diminishes to $\left[M(M-1)-N\right]$-f, $M(M-1)$-rf due to the frequency resonance.

The ddk-t and tdk-t oscillons for $x = x_0, y = y_0, z = z_0, Re = 10^3, Re = 10^5$, and wave parameters (129)-(135) are shown on $\left[t_0, t_0 + T_{e,d,e}\right]$ in **Figure 4**. Because the range of the 6-rf, random rdk-t oscillon at $Re = 10^3$ is $[-370, 480]$ the 4-f, 6-rf, turbulent tdk-t oscillon in **Figure 4(b)** is locally modified compared with the 4-f, deterministic ddk-t oscillon in **Figure 4(a)**. Because the range of the 6-rf, random rdk-t oscillon at $Re = 10^5$ increases to $[-1600, 2000]$, the range and the shape of the 4-f, 6-rf, turbulent tdk-t oscillon in **Figure 4(c)** are changed significantly. The ddk-t oscillon becomes the 4-f oscillon in the view of $\omega_{d,t,2,1,2} = \omega_{d,t,2,2,3} = -5\pi/3$ and $\omega_{d,t,8,1,2} = \omega_{d,t,8,2,3} = -\pi/3$.

7. Oscillons of Turbulent, Internal Interaction

7.1. The TIE Oscillons

The turbulent, internal, elementary oscillons (the tie oscillons for pithiness) are defined as the superposition of the die oscillons (see (139) of [7]) and rie oscillons (see (171) of [7]) in the following form:

$$K_{o,t,i,m,t,j,m} = K_{o,d,i,m,d,j,m} + K_{o,r,i,m,r,j,m}, \ i = a,b,c, \ \ j = i+1, \cdots, d. \quad (191)$$

Similar to the vector die-t and rie-t oscillons, six tie-x oscillons are combined in three vector tie-t oscillons

$$\boldsymbol{K}_{t,i,e,t,q} = \boldsymbol{K}_{t,i,e,t,q}\left(\boldsymbol{f}_{t,i,e,t,q}\right),\ q = 1,2,3, \tag{192}$$

which are generated by two 6-tuples and 5-tuple of the turbulent, internal, elementary interaction in t:

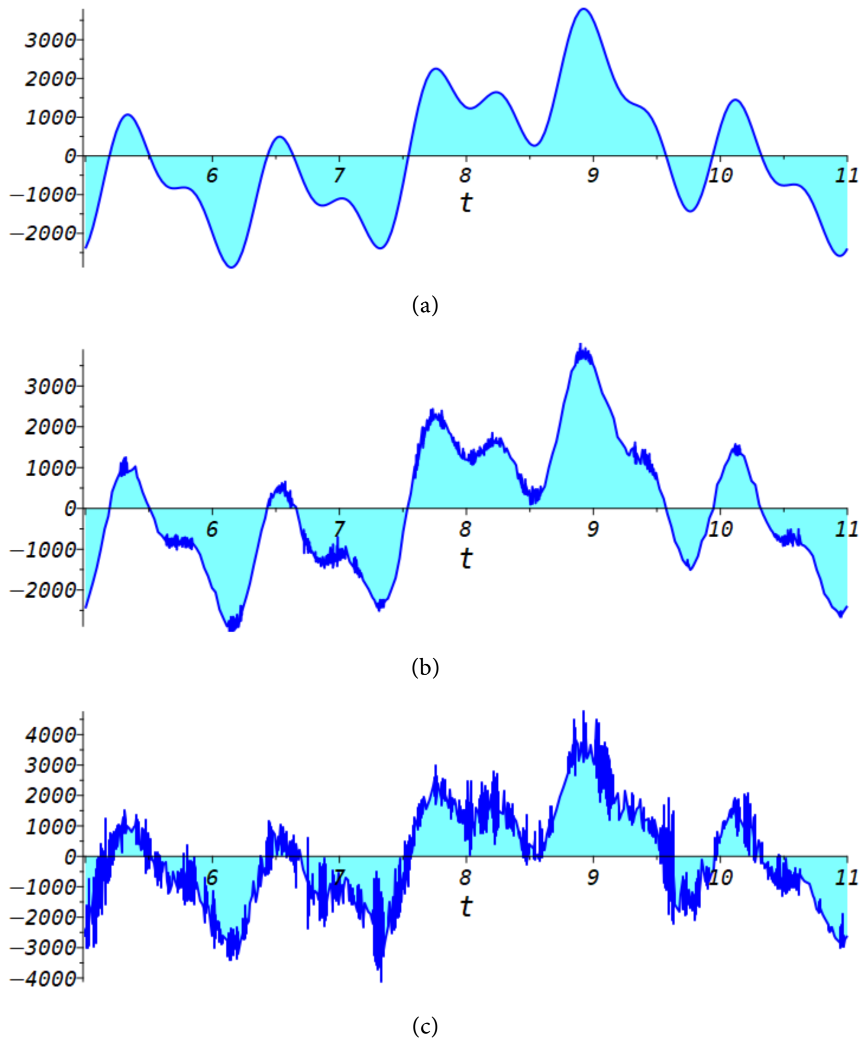


Figure 4. The ddk-t and tdk-t oscillons: (a) $-K_{e,d,i,m,d,i,n}(t)$, (b) $-K_{e,t,i,m,t,i,n}(t)$ for $Re = 10^3$, (c) $-K_{e,t,i,m,t,i,n}(t)$ for $Re = 10^5$.

$$\begin{aligned} \boldsymbol{f}_{t,i,e,t,1} &= \left\{\boldsymbol{f}_{d,i,e,t,1}, \boldsymbol{f}_{r,i,e,t,1}\right\} = \left\{g_{d,t,2k-1,m,m}, g_{r,t,2k-1,m,m}\right\}, \\ \boldsymbol{f}_{t,i,e,t,2} &= \left\{\boldsymbol{f}_{d,i,e,t,2}, \boldsymbol{f}_{r,i,e,t,2}\right\} = \left\{g_{d,t,2k-1,m,m}, g_{r,t,2k-1,m,m}\right\}, \\ \boldsymbol{f}_{t,i,e,t,3} &= \left\{\boldsymbol{f}_{d,i,e,t,3}, \boldsymbol{f}_{r,i,e,t,3}\right\} = \left\{1, g_{d,t,2k,m,m}, g_{r,t,2k,m,m}\right\}. \end{aligned} \tag{193}$$

Six-tuple $\boldsymbol{f}_{t,i,e,t,1}$ consists of three deterministic sine waves $g_{d,t,2k-1,m,m}$ with frequencies $\omega_{d,t,k,m,m}$ and three random sine waves $g_{r,t,2k-1,m,m}$ with frequencies $\omega_{r,t,k,m,m}(t)$ for $k = 1,2,4$ and each m. Six-tuple $\boldsymbol{f}_{t,i,e,t,2}$ comprises three deterministic sine waves $g_{d,t,2k-1,m,m}$ with frequencies $\omega_{d,t,k,m,m}$ and three random sine waves $g_{r,t,2k-1,m,m}$ with frequencies $\omega_{r,t,k,m,m}(t)$ for $k=1,3,4$ and each m. Five-tuple $\boldsymbol{f}_{t,i,e,t,3}$ is composed of the unit pulson, two deterministic cosine waves $g_{d,t,2k,m,m}$ with frequencies $\omega_{d,t,k,m,m}$ and two random cosine waves $g_{r,t,2k,m,m}$ with frequencies $\omega_{r,t,k,m,m}(t)$ for $k=1,4$ and each m.

For any frozen $x=x_0, y=y_0, z=z_0$, the 1st vector tie-t oscillon

$$\boldsymbol{K}_{t,i,e,t,1}=\left[K_{o,t,a,m,t,b,m},K_{o,t,c,m,t,d,m}\right]\left(\boldsymbol{f}_{t,i,e,t,1}\right) \tag{194}$$

is reduced to a list of two 3-f, 3-rf, turbulent oscillons in t, which are generated by 6-tuple $\boldsymbol{f}_{t,i,e,t,1}$ as

$$\begin{aligned} K_{o,t,a,m,t,b,m} &= -\frac{\rho_c}{8}\Big[ez_{d,m}^2\left(Q_{d,m}^2 g_{d,t,1,m,m}-2Q_{d,m}R_{d,m}g_{d,t,3,m,m}+R_{d,m}^2 g_{d,t,7,m,m}\right)\\ &\quad+ez_{r,m}^2\left(Q_{r,m}^2 g_{r,t,1,m,m}-2Q_{r,m}R_{r,m}g_{r,t,3,m,m}+R_{r,m}^2 g_{r,t,7,m,m}\right)\Big],\\ K_{o,t,c,m,t,d,m} &= +\frac{\rho_c}{8}\Big[ez_{d,m}^2\left(Q_{d,m}^2 g_{d,t,1,m,m}+2Q_{d,m}R_{d,m}g_{d,t,3,m,m}+R_{d,m}^2 g_{d,t,7,m,m}\right)\\ &\quad+ez_{r,m}^2\left(Q_{r,m}^2 g_{r,t,1,m,m}+2Q_{r,m}R_{r,m}g_{r,t,3,m,m}+R_{r,m}^2 g_{r,t,7,m,m}\right)\Big]. \end{aligned} \tag{195}$$

The 2nd vector tie-t oscillon

$$\boldsymbol{K}_{t,i,e,t,2}=\left[K_{o,t,b,m,t,d,m},K_{o,t,a,m,t,c,m}\right]\left(\boldsymbol{f}_{t,i,e,t,2}\right) \tag{196}$$

is presented by a list of two 3-f, 3-rf, turbulent oscillons in t, which are generated by 6-tuple $\boldsymbol{f}_{t,i,e,t,2}$ since

$$\begin{aligned} K_{o,t,b,m,t,d,m} &= +\frac{\rho_c}{8}\Big[ez_{d,m}^2\left(Q_{d,m}^2 g_{d,t,1,m,m}-2Q_{d,m}R_{d,m}g_{d,t,5,m,m}-R_{d,m}^2 g_{d,t,7,m,m}\right)\\ &\quad+ez_{r,m}^2\left(Q_{r,m}^2 g_{r,t,1,m,m}-2Q_{r,m}R_{r,m}g_{r,t,5,m,m}-R_{r,m}^2 g_{r,t,7,m,m}\right)\Big],\\ K_{o,t,a,m,t,c,m} &= -\frac{\rho_c}{8}\Big[ez_{d,m}^2\left(Q_{d,m}^2 g_{d,t,1,m,m}+2Q_{d,m}R_{d,m}g_{d,t,5,m,m}-R_{d,m}^2 g_{d,t,7,m,m}\right)\\ &\quad+ez_{r,m}^2\left(Q_{r,m}^2 g_{r,t,1,m,m}+2Q_{r,m}R_{r,m}g_{r,t,5,m,m}-R_{r,m}^2 g_{r,t,7,m,m}\right)\Big]. \end{aligned} \tag{197}$$

The 3rd vector tie-t oscillon

$$\boldsymbol{K}_{t,i,e,t,3}=\left[K_{o,t,a,m,t,d,m},K_{o,t,b,m,t,c,m}\right]\left(\boldsymbol{f}_{t,i,e,t,3}\right) \tag{198}$$

is visualized by a list of two 2-f, 2-rf, turbulent oscillons in t, which are formed by 5-tuple $\boldsymbol{f}_{t,i,e,t,3}$ because

$$\begin{aligned} K_{o,t,a,m,t,d,m} &= -\frac{\rho_c}{8}\Big[ez_{d,m}^2\left(Q_{d,m}^2-R_{d,m}^2-Q_{d,m}^2 g_{d,t,2,m,m}+R_{d,m}^2 g_{d,t,8,m,m}\right)\\ &\quad+ez_{r,m}^2\left(Q_{r,m}^2-R_{r,m}^2-Q_{r,m}^2 g_{r,t,2,m,m}+R_{r,m}^2 g_{r,t,8,m,m}\right)\Big],\\ K_{o,t,b,m,t,c,m} &= +\frac{\rho_c}{8}\Big[ez_{d,m}^2\left(Q_{d,m}^2-R_{d,m}^2+Q_{d,m}^2 g_{d,t,2,m,m}-R_{d,m}^2 g_{d,t,8,m,m}\right)\\ &\quad+ez_{r,m}^2\left(Q_{r,m}^2-R_{r,m}^2+Q_{r,m}^2 g_{r,t,2,m,m}-R_{r,m}^2 g_{r,t,8,m,m}\right)\Big]. \end{aligned} \tag{199}$$

Wave parameters of the vector tie-t oscillons are given by correspondent parameters of the vector die-t and rie-t oscillons. The tie-t oscillons are smooth random functions of time with an unbounded period.

7.2. The TIW Oscillons

The turbulent, internal, wave oscillons (the tiw oscillons for quickness) are constructed as the superposition of the diw oscillons (see (141) of [7]) and riw oscillons (see (173) of [7]) as follows:

$$\begin{aligned} K_{w,t,a,m,t,b,m} &= + K_{w,d,a,m,d,b,m} + K_{w,r,a,m,r,b,m} \\ = + K_{w,t,c,m,t,d,m} &= + K_{w,d,c,m,d,d,m} + K_{w,r,c,m,r,d,m}, \\ K_{w,t,b,m,t,d,m} &= + K_{w,d,b,m,d,d,m} + K_{w,r,b,m,r,d,m} \\ = + K_{w,t,a,m,t,c,m} &= + K_{w,d,a,m,d,c,m} + K_{w,r,a,m,r,c,m}, \\ K_{w,t,a,m,t,d,m} &= + K_{w,d,a,m,d,d,m} + K_{w,r,a,m,r,d,m} \\ = - K_{w,t,b,m,t,c,m} &= - K_{w,d,b,m,d,c,m} - K_{w,r,b,m,r,c,m}. \end{aligned} \tag{200}$$

Alike the vector diw-t and riw-t oscillons, six tiw-t oscillons are combined into three vector tiw-t oscillons

$$\boldsymbol{K}_{t,i,w,t,q} = \boldsymbol{K}_{t,i,w,t,q}\left(\boldsymbol{f}_{t,i,w,t,q}\right), \quad q = 1, 2, 3, \tag{201}$$

which are produced by two 2-tuples and 1-tuple of the turbulent, internal, wave interaction in t:

$$\begin{aligned} \boldsymbol{f}_{t,i,w,t,1} &= \left\{\boldsymbol{f}_{d,i,w,t,1}, \boldsymbol{f}_{r,i,w,t,1}\right\} = \left\{g_{d,t,3,m,m}, g_{r,t,3,m,m}\right\}, \\ \boldsymbol{f}_{t,i,w,t,2} &= \left\{\boldsymbol{f}_{d,i,w,t,2}, \boldsymbol{f}_{r,i,w,t,2}\right\} = \left\{g_{d,t,5,m,m}, g_{r,t,5,m,m}\right\}, \\ \boldsymbol{f}_{t,i,w,t,3} &= \left\{\boldsymbol{f}_{d,i,w,t,3}, \boldsymbol{f}_{r,i,w,t,3}\right\} = \{1\}. \end{aligned} \tag{202}$$

Two-tuple $\boldsymbol{f}_{t,i,w,t,1}$ is composed of deterministic sine wave $g_{d,t,3,m,m}$ with frequency $\omega_{d,t,2,m,m}$ and random sine wave $g_{r,t,3,m,m}$ with frequency $\omega_{r,t,2,m,m}(t)$ for each m. Two-tuple $\boldsymbol{f}_{t,i,w,t,2}$ is constructed of deterministic sine wave $g_{d,t,5,m,m}$ with frequency $\omega_{d,t,3,m,m}$ and random sine wave $g_{r,t,5,m,m}$ with frequency $\omega_{r,t,3,m,m}(t)$ for each m. One-tuple $\boldsymbol{f}_{t,i,w,t,3}$ includes the unit pulson.

For any frozen $x = x_0, y = y_0, z = z_0$, the 1st vector tiw-$t$ oscillon

$$\boldsymbol{K}_{t,i,w,t,1} = \left[K_{w,t,a,m,t,b,m}, K_{w,t,c,m,t,d,m}\right]\left(\boldsymbol{f}_{t,i,w,t,1}\right) \tag{203}$$

is represented by a list of two 1-f, 1-rf, turbulent oscillons in t, which are formed by 2-tuple $\boldsymbol{f}_{t,i,w,t,1}$ in view of

$$\begin{aligned} &K_{w,t,a,m,t,b,m} = +K_{w,t,c,m,t,d,m} \\ &= +\frac{\rho_c}{2}\left(ez_{d,m}^2\lambda_{d,m}^2 Q_{d,m} R_{d,m} g_{d,t,3,m,m} + ez_{r,m}^2\lambda_{r,m}^2 Q_{r,m} R_{r,m} g_{r,t,3,m,m}\right). \end{aligned} \tag{204}$$

The 2nd vector tiw-t oscillon

$$\boldsymbol{K}_{t,i,w,t,2} = \left[K_{w,t,b,m,t,d,m}, K_{w,t,a,m,t,c,m}\right]\left(\boldsymbol{f}_{t,i,w,t,2}\right) \tag{205}$$

is exposed by a list of two 1-f, 1-rf, turbulent oscillons in t, which are established by 2-tuple $\boldsymbol{f}_{t,i,w,t,2}$ since

$$\begin{aligned} &K_{w,t,b,m,t,d,m} = +K_{w,t,a,m,t,c,m} \\ &= -\frac{\rho_c}{2}\left(ez_{d,m}^2\kappa_{d,m}^2 Q_{d,m} R_{d,m} g_{d,t,5,m,m} + ez_{r,m}^2\kappa_{r,m}^2 Q_{r,m} R_{r,m} g_{r,t,5,m,m}\right). \end{aligned} \tag{206}$$

The 3rd vector tiw-t oscillon

$$\boldsymbol{K}_{t,i,w,t,3} = \left[K_{o,t,a,m,t,d,m}, K_{o,t,b,m,t,c,m} \right] \left(\boldsymbol{f}_{t,i,w,t,3} \right) \tag{207}$$

is exhibited by a list of two 0-f, 4-rf, turbulent pulsons in t, which are set by 1-tuple $\boldsymbol{f}_{t,i,w,t,3}$ with an amplitude depending on $\rho_c, \mu_{d,m}, \mu_{r,m}, Av_{d,m}, Bv_{d,m}, Cv_{d,m}, Dv_{d,m}, z_0, t$ via $ez_{d,m}(z_0), ez_{r,m}(z_0), Av_{r,m}(t), Bv_{r,m}(t), Cv_{r,m}(t), Dv_{r,m}(t)$ for each m as

$$\begin{aligned} K_{w,t,a,m,t,d,m} = -K_{w,t,b,m,t,c,m} = +\rho_c \Big[& ez_{d,m}^2 \mu_{d,m}^2 \left(Av_{d,m} Dv_{d,m} - Bv_{d,m} Cv_{d,m} \right) \\ & + ez_{r,m}^2 \mu_{r,m}^2 \left(Av_{r,m} Dv_{r,m} - Bv_{r,m} Cv_{r,m} \right) \Big]. \end{aligned} \tag{208}$$

Wave parameters of the vector tiw-t oscillons are provided by relevant wave parameters of the vector diw-t and riw-t oscillons. Analogous to the tie-t oscillons, the tiw-t oscillons are smooth random functions of time with an unbounded period.

7.3. The TIG Oscillon

The turbulent, internal, group pulson (the tig oscillon for clarity) is constructed as the superposition of the dig oscillon (see (142) of [7]) and the rig oscillon (see (174) of [7]), *i.e.*

$$K_{g,t,i,m,t,j,m} = K_{g,d,i,m,d,j,m} + K_{g,r,i,m,r,j,m}. \tag{209}$$

Summation of the dig-t and rig-t oscillons yields the inhomogeneous Fourier expansion of the tig-t oscillon in the following form:

$$K_{g,t,i,m,t,j,m} = K_{g,t,i,m,t,j,m} \left(\boldsymbol{f}_{t,i,g,t} \right), \tag{210}$$

where 4-tuple of the turbulent, internal, group interaction in t

$$\boldsymbol{f}_{t,i,g,t} = \left\{ \boldsymbol{f}_{d,i,g,t}, \boldsymbol{f}_{r,i,g,t} \right\} = \left\{ g_{d,t,2k-1,m,m}, g_{r,t,2k-1,m,m} \right\} \tag{211}$$

is composed of deterministic sine waves $g_{d,t,2k-1,m,m}$ with frequencies $\omega_{d,t,k,m,m}$ and two random sine waves $g_{r,t,2k-1,m,m}$ with frequencies $\omega_{r,t,k,m,m}(t)$ for $k = 2,3$, and each m.

For any frozen $x = x_0, y = y_0, z = z_0$, the tig-$t$ oscillon is displayed by a 2-f, 2-rf, turbulent oscillon in t, which is generated by 4-tuple $\boldsymbol{f}_{t,i,g,t}$ since

$$\begin{aligned} K_{g,t,i,m,t,j,m} = \rho_c \Big[& ez_{d,m}^2 Q_{d,m} R_{d,m} \left(\lambda_{d,m}^2 g_{d,t,3,m,m} - \kappa_{d,m}^2 g_{d,t,5,m,m} \right) \\ & + ez_{r,m}^2 Q_{r,m} R_{r,m} \left(\lambda_{r,m}^2 g_{r,t,3,m,m} - \kappa_{r,m}^2 g_{r,t,5,m,m} \right) \Big]. \end{aligned} \tag{212}$$

Wave parameters of the tig-t oscillon are specified by appropriate parameters of the dig-t and rig-t oscillons. Parallel to the tiw-t oscillons, the tig-t oscillon is a smooth random function of time with an unbounded period.

7.4. The TIK Oscillon

The turbulent, internal, kinetic-energy oscillon (the tik oscillon for shortness), which is defined as the superposition of the dik-t and rik-t oscillons, takes the following form:

$$K_{e,t,i,m,t,j,m} = K_{e,d,i,m,d,j,m} + K_{e,r,i,m,r,j,m}. \tag{213}$$

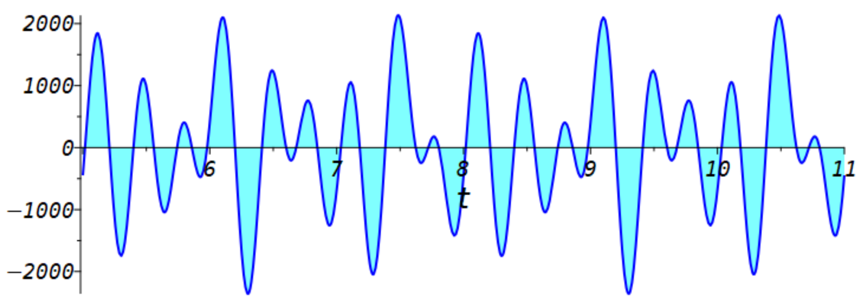
With the help of (72) and (115) of [7], the inhomogeneous Fourier expansion of the tik-t oscillon becomes

$$K_{e,t,i,m,t,j,m} = \sum_{m=1}^{M} K_{g,t,i,m,t,j,m}. \tag{214}$$

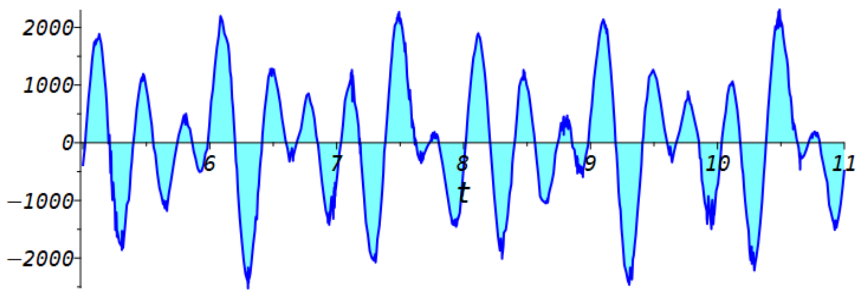
If all frequencies of the tig-t oscillons are distinct, then the tik-t oscillon is visualized as a $2M$-f, $2M$-rf, turbulent oscillon. For any frozen $x = x_0, y = y_0, z = z_0$, the tik-$t$ oscillon is represented by the $2M$-f, $2M$-rf, turbulent oscillon in t, which is produced by M 4-tuples $\boldsymbol{f}_{t,i,g,t}$ with frequencies $\omega_{d,t,k,m,n}, \omega_{r,t,k,m,n}(t)$ for $k = 2,3$, all m, Re, and wave parameters of the tig-t oscillons. Like the tig-t oscillon, the tik-t oscillon is a smooth random function of time with an unbounded period.

If N determinisric frequencies are repeated, then the number of independent modes of the tik-t oscillon diminishes to $[2M - N]$-f, $2M$-rf because of the frequency resonance.

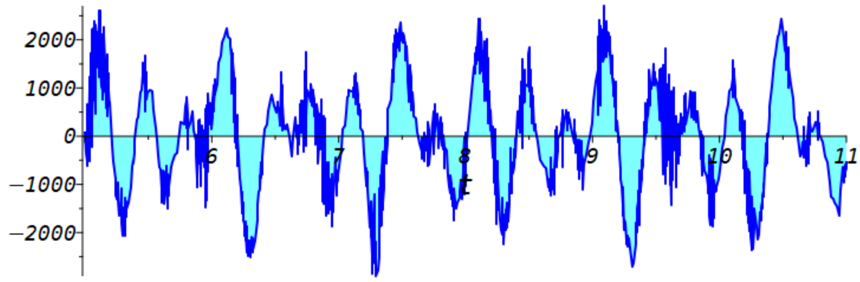
The dik-t and tik-t oscillons for $x = x_0, y = y_0, z = z_0, Re = 10^3, Re = 10^5$, and wave parameters (129)-(135) are presented on $[t_0, t_0 + T_{e,d,e}]$ in **Figure 5**. The 5-f, 6-rf, turbulent tik-t oscillon in **Figure 5(b)** changes insignificantly compared with the 5-f, deterministic dik-t oscillon in **Figure 5(a)** because the range of the 6-rf, random rik-t oscillon at $Re = 10^3$ is $[-280, 275]$. The shape of the 5-f, 6-rf, turbulent tik-t oscillon in **Figure 5(c)** is modified significantly because the range of the 6-rf, random rik-t oscillon at $Re = 10^5$ becomes $[-1425, 1350]$. The dik-t oscillon is the 5-f oscillon since $\omega_{d,t,2,2,2} = \omega_{d,t,3,3,3} = 4\pi$.



(a)



(b)



(c)

Figure 5. The dik-t and tik-t oscillons: (a) $-K_{e,d,i,m,d,j,m}(t)$, (b) $-K_{e,t,i,m,t,j,m}(t)$ for $Re=10^3$, (c) $-K_{e,t,i,m,t,j,m}(t)$ for $Re=10^5$.

8. Turbulent and Cumulative Pulsons

8.1. The TE Pulsons

The turbulent, elementary pulsons (the te-pulsons for briefness) are set as the superposition of the de pulsons (see (134) of [7]) and the re pulsons (see (166) of [7]) as follows:

$$K_{p,t,i,m,t,i,m}=K_{p,d,i,m,d,i,m}+K_{p,r,i,m,r,i,m},\ i=a,b,c,d. \tag{215}$$

Like the vector de-t and re-t pulsons, four te-t pulsons are grouped into a vector te-t pulson

$$\boldsymbol{K}_{t,e,t}=\left[K_{p,t,a,m,t,a,m},K_{p,t,b,m,t,b,m},K_{p,t,d,m,t,d,m},K_{p,t,c,m,t,c,m}\right]\left(\boldsymbol{f}_{t,e,t}\right), \tag{216}$$

which is constructed using 9-tuple of the turbulent, elementary pulsons varying in t

$$\boldsymbol{f}_{t,e,t}=\left\{\boldsymbol{f}_{d,e,t},\boldsymbol{f}_{r,e,t}\right\}=\left\{1,g_{d,t,2k,m,m},g_{r,t,2k,m,m}\right\}. \tag{217}$$

Nine-tuple $\boldsymbol{f}_{t,e,t}$ embraces of the unit pulson, four deterministic cosine waves $g_{d,t,2k,m,m}$ with frequencies $\omega_{d,t,k,m,m}$ and four random cosine waves $g_{r,t,2k,m,m}$ with frequencies $\omega_{r,t,k,m,m}(t)$ for $k=1,2,3,4$ and each m.

For any frozen $x=x_0,y=y_0,z=z_0$, the vector te-t pulson is represented by a list of four 4-f, 4-rf, turbulent, supercritical pulsons in t, which depend on 9-tuple $\boldsymbol{f}_{t,e,t}$ seeing that

$$\begin{aligned}K_{p,t,a,m,t,a,m}=\frac{\rho_c}{16}\Big\{&ez_{d,m}^2\Big[Q_{d,m}^2+R_{d,m}^2-Q_{d,m}^2g_{d,t,2,m,m}\\&\quad+2Q_{d,m}R_{d,m}\left(g_{d,t,4,m,m}-g_{d,t,6,m,m}\right)-R_{d,m}^2g_{d,t,8,m,m}\Big]\\&+ez_{r,m}^2\Big[Q_{r,m}^2+R_{r,m}^2-Q_{r,m}^2g_{r,t,2,m,m}\\&\quad+2Q_{r,m}R_{r,m}\left(g_{r,t,4,m,m}-g_{r,t,6,m,m}\right)-R_{r,m}^2g_{r,t,8,m,m}\Big]\Big\},\end{aligned}$$

$$\begin{aligned}K_{p,t,b,m,t,b,m}=\frac{\rho_c}{16}\Big\{&ez_{d,m}^2\Big[Q_{d,m}^2+R_{d,m}^2+Q_{d,m}^2g_{d,t,2,m,m}\\&\quad-2Q_{d,m}R_{d,m}\left(g_{d,t,4,m,m}+g_{d,t,6,m,m}\right)+R_{d,m}^2g_{d,t,8,m,m}\Big]\\&+ez_{r,m}^2\Big[Q_{r,m}^2+R_{r,m}^2+Q_{r,m}^2g_{r,t,2,m,m}\\&\quad-2Q_{r,m}R_{r,m}\left(g_{r,t,4,m,m}+g_{r,t,6,m,m}\right)+R_{r,m}^2g_{r,t,8,m,m}\Big]\Big\},\end{aligned}$$

$$K_{p,t,d,m,t,d,m} = \frac{\rho_c}{16}\Big\{ez_{d,m}^2\Big[Q_{d,m}^2 + R_{d,m}^2 - Q_{d,m}^2 g_{d,t,2,m,m} - 2Q_{d,m}R_{d,m}\left(g_{d,t,4,m,m} - g_{d,t,6,m,m}\right) - R_{d,m}^2 g_{d,t,8,m,m}\Big] + ez_{r,m}^2\Big[Q_{r,m}^2 + R_{r,m}^2 - Q_{r,m}^2 g_{r,t,2,m,m} - 2Q_{r,m}R_{r,m}\left(g_{r,t,4,m,m} - g_{r,t,6,m,m}\right) - R_{r,m}^2 g_{r,t,8,m,m}\Big]\Big\},$$

$$K_{p,t,c,m,t,c,m} = \frac{\rho_c}{16}\Big\{ez_{d,m}^2\Big[Q_{d,m}^2 + R_{d,m}^2 + Q_{d,m}^2 g_{d,t,2,m,m} + 2Q_{d,m}R_{d,m}\left(g_{d,t,4,m,m} + g_{d,t,6,m,m}\right) + R_{d,m}^2 g_{d,t,8,m,m}\Big] + ez_{r,m}^2\Big[Q_{r,m}^2 + R_{r,m}^2 + Q_{r,m}^2 g_{r,t,2,m,m} + 2Q_{r,m}R_{r,m}\left(g_{r,t,4,m,m} + g_{r,t,6,m,m}\right) + R_{r,m}^2 g_{r,t,8,m,m}\Big]\Big\}. \quad (218)$$

Wave parameters of vector te-t pulson are indicated by similar parameters of the vector de-t and re-t pulsons. Analogous to re-t pulsons, the te-t pulsons are smooth random functions of time with an unbounded period.

8.2. The TW Pulsons

The turbulent, wave pulsons (the tw pulsons for conciseness) are composed with the help of the superposition of the dw (see (136) of [7]) and rw (see (168) of [7]) pulsons in the following form:

$$K_{w,t,i,m,t,i,m} = K_{w,d,i,m,d,i,m} + K_{w,r,i,m,r,i,m}, \quad i = a,b,c,d. \quad (219)$$

Similar to the vector dw-t and rw-t pulsons, four tw-t pulsons are assembled into a vector tw-t pulson

$$\boldsymbol{K}_{t,w,t} = \left[K_{w,t,a,m,t,a,m}, K_{w,t,b,m,t,b,m}, K_{w,t,d,m,t,d,m}, K_{w,t,c,m,t,c,m}\right]\left(\boldsymbol{f}_{t,w,t}\right), \quad (220)$$

which is formed by 5-tuple of the turbulent, wave pulsons varying in t

$$\boldsymbol{f}_{t,w,t} = \left\{\boldsymbol{f}_{d,w,t}, \boldsymbol{f}_{r,w,t}\right\} = \left\{1, g_{d,t,2k,m,m}, g_{r,t,2k,m,m}\right\}. \quad (221)$$

Five-tuple $\boldsymbol{f}_{t,w,t}$ consists of the unit pulson, two deterministic cosine waves $g_{d,t,2k,m,m}$ with frequencies $\omega_{d,t,k,m,m}$ and two random cosine waves $g_{r,t,2k,m,m}$ with frequencies $\omega_{r,t,k,m,m}(t)$ for $k = 2,3$ and each m.

For any frozen $x = x_0, y = y_0, z = z_0$, the vector tw-$t$ pulson is exhibited by a list of four 2-f, 2-rf, turbulent, supercritical pulsons in t, which are established by 5-tuple $\boldsymbol{f}_{t,w,t}$ for the reason that

$$K_{w,t,a,m,t,a,m} = \frac{\rho_c}{8}\Big\{ez_{d,m}^2\Big[\mu_{d,m}^2\left(Q_{d,m}^2 + R_{d,m}^2\right) + 2Q_{d,m}R_{d,m}\left(\lambda_{d,m}^2 g_{d,t,4,m,m} - \kappa_{d,m}^2 g_{d,t,6,m,m}\right)\Big] + \Big\{ez_{r,m}^2\Big[\mu_{r,m}^2\left(Q_{r,m}^2 + R_{r,m}^2\right) + 2Q_{r,m}R_{r,m}\left(\lambda_{r,m}^2 g_{r,t,4,m,m} - \kappa_{r,m}^2 g_{r,t,6,m,m}\right)\Big]\Big\},$$

$$K_{w,t,b,m,t,b,m} = \frac{\rho_c}{8}\Big\{ez_{d,m}^2\Big[\mu_{d,m}^2\left(Q_{d,m}^2 + R_{d,m}^2\right) - 2Q_{d,m}R_{d,m}\left(\lambda_{d,m}^2 g_{d,t,4,m,m} + \kappa_{d,m}^2 g_{d,t,6,m,m}\right)\Big] + \Big\{ez_{r,m}^2\Big[\mu_{r,m}^2\left(Q_{r,m}^2 + R_{r,m}^2\right) - 2Q_{r,m}R_{r,m}\left(\lambda_{r,m}^2 g_{r,t,4,m,m} + \kappa_{r,m}^2 g_{r,t,6,m,m}\right)\Big]\Big\},$$

$$K_{w,t,d,m,t,d,m} =$$
$$\frac{\rho_c}{8}\left\{ez_{d,m}^2\left[\mu_{d,m}^2\left(Q_{d,m}^2+R_{d,m}^2\right)-2Q_{d,m}R_{d,m}\left(\lambda_{d,m}^2 g_{d,t,4,m,m}-\kappa_{d,m}^2 g_{d,t,6,m,m}\right)\right]\right.$$
$$\left.+\left\{ez_{r,m}^2\left[\mu_{r,m}^2\left(Q_{r,m}^2+R_{r,m}^2\right)-2Q_{r,m}R_{r,m}\left(\lambda_{r,m}^2 g_{r,t,4,m,m}-\kappa_{r,m}^2 g_{r,t,6,m,m}\right)\right]\right\}\right\},$$
$$K_{w,t,c,m,t,c,m} =$$
$$\frac{\rho_c}{8}\left\{ez_{d,m}^2\left[\mu_{d,m}^2\left(Q_{d,m}^2+R_{d,m}^2\right)+2Q_{d,m}R_{d,m}\left(\lambda_{d,m}^2 g_{d,t,4,m,m}+\kappa_{d,m}^2 g_{d,t,6,m,m}\right)\right]\right. \quad (222)$$
$$\left.+\left\{ez_{r,m}^2\left[\mu_{r,m}^2\left(Q_{r,m}^2+R_{r,m}^2\right)+2Q_{r,m}R_{r,m}\left(\lambda_{r,m}^2 g_{r,t,4,m,m}+\kappa_{r,m}^2 g_{r,t,6,m,m}\right)\right]\right\}\right\}.$$

Wave parameters of vector tw-t pulson are given by relevant wave parameters of the vector dw-t and rw-t pulsons. Alike the rw-t pulsons, the tw-t pulsons are smooth random functions of time with an unbounded period.

8.3. The TG Pulson

The turbulent, group pulson (the tg pulson for brevity) is set as the superposition of the dg (see (137) of [7]) pulson and the rg pulson (see (169) of [7]), *viz.*

$$K_{g,t,i,m,t,i,m} = K_{g,d,i,m,d,i,m} + K_{g,r,i,m,r,i,m}. \quad (223)$$

Summation of the dg-t and rg-t pulsons yields the tg-t pulson in the following form:

$$K_{g,t,i,m,t,i,m} = K_{g,t,i,m,t,i,m}\left(\boldsymbol{f}_{t,g,t}\right), \quad (224)$$

where 1-tuple of the turbulent, group pulson, which is constant in t,

$$\boldsymbol{f}_{t,g,t} = \left\{\boldsymbol{f}_{d,g,t}, \boldsymbol{f}_{r,g,t}\right\} = \{1\} \quad (225)$$

consists of the unit pulson.

For any frozen $z = z_0$ and all x, y the tg-t pulson is represented by the 0-f, 4-rf, turbulent, supercritical pulson in t, which is produced by 1-tuple $\boldsymbol{f}_{t,g,t}$ with the amplitude depending on $\rho_c, \mu_{d,m}, \mu_{r,m}, Av_{d,m}, Bv_{d,m}, Cv_{d,m}, Dv_{d,m}, z_0, t$ via $ez_{d,m}(z_0), ez_{r,m}(z_0), Av_{r,m}(t), Bv_{r,m}(t), Cv_{r,m}(t), Dv_{r,m}(t)$ for each m because

$$K_{g,t,i,m,t,i,m} = \rho_c\left[ez_{d,m}^2\mu_{d,m}^2\left(Av_{d,m}^2+Bv_{d,m}^2+Cv_{d,m}^2+Dv_{d,m}^2\right)\right.$$
$$\left.+\,ez_{r,m}^2\mu_{r,m}^2\left(Av_{r,m}^2+Bv_{r,m}^2+Cv_{r,m}^2+Dv_{r,m}^2\right)\right]. \quad (226)$$

Analogous to the te-t and tw-t pulsons, the tg-t pulson is a smooth random function of time with an unbounded period.

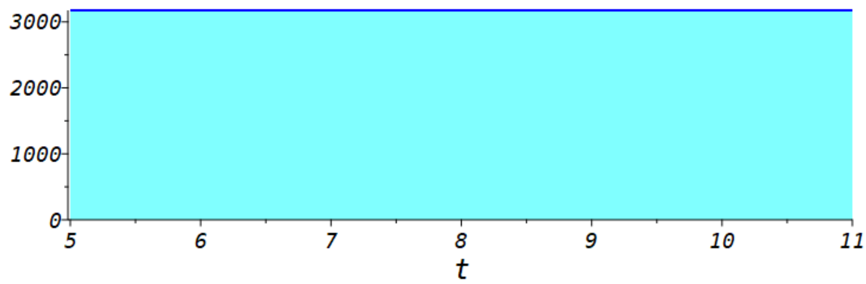
8.4. The TK Pulson

The turbulent, kinetic-energy pulson (the tk pulson for easiness) is written as the superposition of the dk pulson (see (58) of [7]) and the rk pulson (see (101) of [7]) by

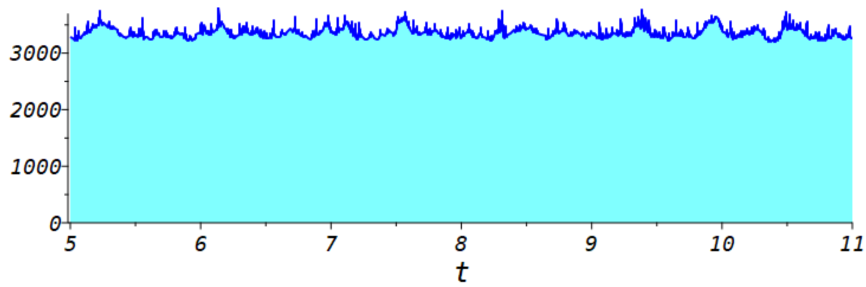
$$K_{e,t,i,m,t,i,m} = K_{e,d,i,m,d,i,m} + K_{e,r,i,m,r,i,m} = \sum_{m=1}^{M} K_{g,t,i,m,t,i,m}. \quad (227)$$

For any frozen $z = z_0$ and all x, y the tk-t pulson also becomes a 0-f, 4M-rf, turbulent, supercritical, pulson in t, which is formed by 1-tuple $\boldsymbol{f}_{t,g,t}$ with the same wave parameters as the tg-t pulson for all m, Re. The tk-t pulson is a smooth random function of time with an unbounded period, as well.

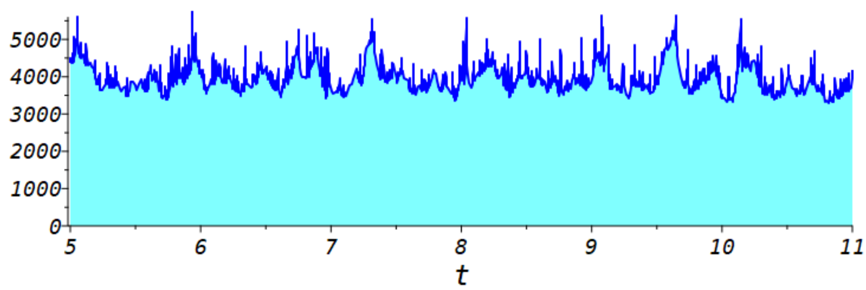
The dk-t and tk-t pulsons for $x = x_0, y = y_0, z = z_0, Re = 10^3, Re = 10^5$, and wave parameters (129)-(135) are presented on $[t_0, t_0 + T_{e,d,e}]$ in Figure 6. The 0-f, 12-rf, turbulent, supercritical tk-t pulson in Figure 6(b) at $Re = 10^3$ becomes non-constant pulson compared with the constant 0-f, deterministic, supercritical dk-t oscillon in Figure 6(a) because of smooth random functions of time $Av_{r,m}(t), Bv_{r,m}(t), Cv_{r,m}(t), Dv_{r,m}(t)$. Both the range and the shape of the 0-f, 12-rf, turbulent, supercritical tk-t pulson in Figure 6(c) are altered significantly compared with Figure 6(a) since the range of the 12-rf, random, supercritical rk-t pulson at $Re = 10^5$ becomes $[0, 2650]$.



(a)



(b)



(c)

Figure 6. The dk-t and tk-t pulsons: (a) $-K_{e,d,i,m,d,i,m}$, (b) $-K_{e,t,i,m,t,i,m}(t)$ for $Re = 10^3$, (c) $-K_{e,t,i,m,t,i,m}(t)$ for $Re = 10^5$.

8.5. The DCK Pulson

The deterministic, cumulative, kinetic-energy pulson in t (the dck-t pulson for simplicity), which is the superposition of the dk-t pulson and the dik-t, ddk-t, and dek-t oscillons,

$$\begin{aligned} K_{e,d,d} &= K_{e,d,i,m,d,i,m} + K_{e,d,i,m,d,j,m} + K_{e,d,i,m,d,i,n} + K_{e,d,i,m,d,j,n} \\ &= K_{e,d,d}\left(\boldsymbol{f}_{d,c,i,t}, \boldsymbol{f}_{d,c,e,t}\right) \end{aligned} \tag{228}$$

is formed by 3-tuple of the deterministic, cumulative, internal interaction in t

$$\boldsymbol{f}_{d,c,i,t} = \left\{\boldsymbol{f}_{d,g,t}, \boldsymbol{f}_{d,i,g,t}\right\} = \left\{1, g_{d,t,2k-1,m,m}\right\} \tag{229}$$

and 8-tuple of the deterministic, cumulative, external interaction in t

$$\boldsymbol{f}_{d,c,e,t} = \left\{\boldsymbol{f}_{d,d,g,t}, \boldsymbol{f}_{d,e,g,t}\right\} = \left\{f_{d,t,2k-1,m,n}, f_{d,t,2l,m,n}\right\}. \tag{230}$$

Three-tuple $\boldsymbol{f}_{d,c,i,t}$ consists of the unit pulson and two sine waves $g_{d,t,2k-1,m,m}$ with frequencies $\omega_{d,t,k,m,m}$ for $k = 2,3$ and each m. Eight-tuple $\boldsymbol{f}_{d,c,e,t}$ comprises four sine waves $f_{d,t,2k-1,m,n}$ with frequencies $\omega_{d,t,k,m,n}$ and four cosine waves $f_{d,t,2l,m,n}$ with frequency $\omega_{d,t,l,m,n}$ for $k = 3,4,5,6$, $l = 1,2,7,8$, and each m, n.

If all frequencies of the dck-t pulson are distinct, then the dck-t pulson is visualized as a $2M(2M-1)$-f pulson. For any frozen $x = x_0, y = y_0, z = z_0$, the dck-$t$ pulson is represented by the $2M(2M-1)$-f, deterministic, supercritical pulson in t, which is produced by M 3-tuples $\boldsymbol{f}_{d,c,i,t}$ with frequencies $\omega_{d,t,k,m,m}$ for $k = 2,3$ and all m, Re and $M(M-1)/2$ 8-tuples $\boldsymbol{f}_{d,c,e,t}$ with frequencies $\omega_{d,t,k,m,n}$ for $k = 1,2,\cdots,8$ and all m, n, Re because

$$\begin{aligned} K_{e,d,d} = \rho_c \Bigg\{ &\sum_{m=1}^{M} ez_{d,m}^2 \Big[\mu_{d,m}^2 \left(Av_{d,m}^2 + Bv_{d,m}^2 + Cv_{d,m}^2 + Dv_{d,m}^2\right) \\ &\qquad + Q_{d,m} R_{d,m} \left(\lambda_{d,m}^2 g_{d,t,3,m,m} - \kappa_{d,m}^2 g_{d,t,5,m,m}\right)\Big] \\ &+ \frac{1}{2}\sum_{m=1}^{M-1}\sum_{n=m+1}^{M} ez_{d,m} ez_{d,n} \Big[Q_{d,m} R_{d,n} \left(\Lambda_{d,m,d,n} f_{d,t,5,m,n} - \mathrm{K}_{d,m,d,n} f_{d,t,7,m,n}\right) \\ &\qquad + R_{d,m} Q_{d,n} \left(\Lambda_{d,m,d,n} f_{d,t,9,m,n} + \mathrm{K}_{d,m,d,n} f_{d,t,11,m,n}\right) \\ &\qquad - Q_{d,m} Q_{d,n} \left(\mathrm{N}_{d,m,d,n} f_{d,t,2,m,n} - \mathrm{M}_{d,m,d,n} f_{d,t,4,m,n}\right) \\ &\qquad + R_{d,m} R_{d,n} \left(\mathrm{N}_{d,m,d,n} f_{d,t,14,m,n} + \mathrm{M}_{d,m,d,n} f_{d,t,16,m,n}\right)\Big]\Bigg\}. \end{aligned} \tag{231}$$

If N frequencies are repeated, then the number of independent modes of the dck-t pulson diminishes to $2M(2M-1) - N$ due to the frequency resonance.

A period of the dck-t pulson is given by

$$T_{e,d,t} = \mathrm{LCM}\left(T_{e,d,i}, T_{e,d,d}, T_{e,d,e}\right) = k_{e,d,i} T_{e,d,i} = k_{e,d,d} T_{e,d,d} = k_{e,d,e} T_{e,d,e}, \tag{232}$$

where $k_{e,d,i}, k_{e,d,d}, k_{e,d,e}$ are integers.

The global period of the deterministic internal interaction

$$\begin{aligned} &T_{e,d,i} = \mathrm{LCM}\left(T_{w,d,m,m}\right) = k_{w,d,m,m} T_{w,d,m,m}, \\ &T_{w,d,m,m} = \mathrm{LCM}\left(T_{d,l,m,m}\right) = k_{w,d,l,m,m} T_{d,l,m,m}, \quad T_{d,l,m,m} = \frac{2\pi}{\omega_{d,t,l,m,m}}, \end{aligned} \tag{233}$$

where $k_{w,d,m,m}, k_{w,d,l,m,m}$ are integers, $l = 2,3$.

The global period of the deterministic diagonal interaction

$$T_{e,d,d}=\mathrm{LCM}\left(T_{g,d,m,n}\right)=k_{e,d,m,n}T_{g,d,m,n},$$
$$T_{g,d,m,n}=\mathrm{LCM}\left(T_{d,l,m,n}\right)=k_{g,d,l,m,n}T_{d,l,m,n},\quad T_{d,l,m,n}=\frac{2\pi}{\omega_{d,t,l,m,n}},\tag{234}$$

where $k_{e,d,m,n},k_{g,d,l,m,n}$ are integers, $l=2,8$.

The global period of the deterministic external interaction

$$T_{e,d,e}=\mathrm{LCM}\left(T_{g,e,m,n}\right)=k_{e,e,m,n}T_{g,e,m,n},$$
$$T_{g,e,m,n}=\mathrm{LCM}\left(T_{d,l,m,n}\right)=k_{g,e,l,m,n}T_{d,l,m,n},\quad T_{d,l,m,n}=\frac{2\pi}{\omega_{d,t,l,m,n}},\tag{235}$$

where $k_{e,e,m,n},k_{g,e,l,m,n}$ are integers, $l=1,3,4,5,6,7$.

Combining the average of the dk-t pulson and the dik-t, ddk-t, dek-t oscillons, we find the average of the dck-t pulson over $T_{e,d,t}$

$$\frac{1}{T_{e,d,t}}\int_0^{T_{e,d,t}}K_{e,d,d}\mathrm{d}x=\rho_c\sum_{m=1}^{M}ez_{d,m}^2\mu_{d.m}^2\left(Av_{d,m}^2+Bv_{d,m}^2+Cv_{d,m}^2+Dv_{d,m}^2\right).\tag{236}$$

Since

$$K_{e,d,d}\left(x_0,y_0,z_0,t\right)=\frac{\rho_c}{2}\left(u_{d,x}^2+u_{d,y}^2+u_{d,z}^2\right)\left(x_0,y_0,z_0,t\right),\tag{237}$$

where $u_{d,x},u_{d,y},u_{d,z}$ are x-, y-, z-components of deterministic velocity $\boldsymbol{u}_d$, the dck-t pulson remains positive for all times and transfers a positive amount of the kinetic energy in time.

8.6. The RCK Pulson

The random, cumulative, kinetic-energy pulson in t (the rck-t pulson for fastness), which is constructed as the superposition of the rk-t pulson and the rik-t, rdk-t, rek-t oscillons,

$$K_{e,r,r}=K_{e,r,i,m,r,i,m}+K_{e,r,i,m,r,j,m}+K_{e,r,i,m,r,i,n}+K_{e,r,i,m,r,j,n}$$
$$=K_{e,r,r}\left(\boldsymbol{f}_{r,c,i,t},\boldsymbol{f}_{r,c,e,t}\right)\tag{238}$$

is produced by 3-tuple of the random, cumulative, internal interaction in time

$$\boldsymbol{f}_{r,c,i,t}=\left\{\boldsymbol{f}_{r,g,t},\boldsymbol{f}_{r,i,g,t}\right\}=\left\{1,g_{r,t,2k-1,m,m}\right\}\tag{239}$$

and 8-tuple of the random, cumulative, external interaction along in time

$$\boldsymbol{f}_{r,c,e,t}=\left\{\boldsymbol{f}_{r,d,g,t},\boldsymbol{f}_{r,e,g,t}\right\}=\left\{f_{r,t,2k-1,m,n},f_{r,t,2l,m,n}\right\}.\tag{240}$$

Three-tuple $\boldsymbol{f}_{r,c,i,t}$ is composed of the unit pulson and two sine waves $g_{r,t,2k-1,m,m}$ with frequencies $\omega_{r,t,k,m,m}(t)$ for $k=2,3$ and each m. Eight-tuple $\boldsymbol{f}_{r,c,e,t}$ includes four sine waves $f_{r,t,2k-1,m,n}$ with frequencies $\omega_{r,t,k,m,n}(t)$ and four cosine waves $f_{r,t,2l,m,n}$ with frequencies $\omega_{r,t,l,m,n}(t)$ for $k=3,4,5,6$, $l=1,2,7,8$, and each m, n.

As all frequencies of the rck-t pulson are distinct smooth random functions of time, the rck-t pulson is exposed as a $2M(2M-1)$-rf pulson. For any frozen $x=x_0,y=y_0$, $z=z_0$, the rck-t pulson then exhibited by the $2M(2M-1)$-rf, random, supercritical pulson in *t*, which is established by M 3-tuples $\boldsymbol{f}_{r,c,i,t}$ with frequencies $\omega_{r,t,k,m,m}(t)$ for $k=2,3$ and all *m*, *Re* and $M(M-1)/2$ 8-tuples $\boldsymbol{f}_{r,c,e,t}$ with frequencies $\omega_{r,t,k,m,n}(t)$ for $k=1,2,\cdots,8$ and all *m*, *n*, *Re* since

$$\begin{aligned} K_{e,r,r} = \rho_c \Bigg\{ & \sum_{m=1}^{M} ez_{r,m}^2 \Big[\mu_{r,m}^2 \left(Av_{r,m}^2 + Bv_{r,m}^2 + Cv_{r,m}^2 + Dv_{r,m}^2 \right) \\ & \qquad + Q_{r,m} R_{r,m} \left(\lambda_{r,m}^2 g_{r,t,3,m,m} - \kappa_{r,m}^2 g_{r,t,5,m,m} \right) \Big] \\ & + \frac{1}{2} \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} ez_{r,m} ez_{r,n} \Big[Q_{r,m} R_{r,n} \left(\Lambda_{r,m,r,n} f_{r,t,5,m,n} - \mathrm{K}_{r,m,r,n} f_{r,t,7,m,n} \right) \\ & \qquad + R_{r,m} Q_{r,n} \left(\Lambda_{r,m,r,n} f_{r,t,9,m,n} + \mathrm{K}_{r,m,r,n} f_{r,t,11,m,n} \right) \\ & \qquad - Q_{r,m} Q_{r,n} \left(\mathrm{N}_{r,m,r,n} f_{r,t,2,m,n} - \mathrm{M}_{r,m,r,n} f_{r,t,4,m,n} \right) \\ & \qquad + R_{r,m} R_{r,n} \left(\mathrm{N}_{r,m,r,n} f_{r,t,14,m,n} + \mathrm{M}_{r,m,r,n} f_{r,t,16,m,n} \right) \Big] \Bigg\}. \end{aligned} \tag{241}$$

The rck-t pulson is a smooth random function of time with an unbounded period. As

$$K_{e,r,r}(x_0, y_0, z_0, t) = \frac{\rho_c}{2}\left(u_{r,x}^2 + u_{r,y}^2 + u_{r,z}^2\right)(x_0, y_0, z_0, t), \tag{242}$$

where $u_{r,x}, u_{r,y}, u_{r,z}$ are *x*-, *y*-, *z*-components of random velocity $\boldsymbol{u}_r$, the rck-*t* pulson is positively defined for all times and also transfers a positive amount of the kinetic energy in time.

8.7. The TCK Pulson

The turbulent, cumulative, kinetic-energy pulson in *t* (the tck-*t* pulson for swiftness), which is the superposition of the dck-*t*, rck-*t* pulsons and the drik-*t*, drek-*t* oscillons (see (1) and (201) of [7]),

$$\begin{aligned} K_{e,t} &= K_{e,d,d} + K_{e,r,r} + K_{e,d,i,m,r,j,m} + K_{e,d,i,m,r,j,n} \\ &= K_{e,t}\left\{ \boldsymbol{f}_{t,c,i,t}, \boldsymbol{f}_{t,c,e,t} \right\} \end{aligned} \tag{243}$$

is produced by 13-tuple of the turbulent, cumulative, internal interaction in t

$$\begin{aligned} \boldsymbol{f}_{t,c,i,t} &= \left\{ \boldsymbol{f}_{d,c,i,t}, \boldsymbol{f}_{r,c,i,t}, \boldsymbol{f}_{d,r,i,g,t} \right\} \\ &= \left\{ 1, g_{d,t,2q-1,m,m}, g_{r,t,2q-1,m,m}, g_{d,r,t,2k-1,m,m}, g_{d,r,t,2l,m,m} \right\} \end{aligned} \tag{244}$$

and 32-tuple of the turbulent, cumulative, external interaction in t

$$\begin{aligned} \boldsymbol{f}_{t,c,e,t} &= \left\{ \boldsymbol{f}_{d,c,e,t}, \boldsymbol{f}_{r,c,e,t}, \boldsymbol{f}_{d,r,e,g,t} \right\} \\ &= \left\{ f_{d,t,2k-1,m,n}, \ f_{d,t,2l,m,n}, \ f_{r,t,2k-1,m,n}, \ f_{r,t,2l,m,n}, \right. \\ &\qquad \left. f_{d,r,t,2k-1,m,n}, f_{d,r,t,2l,m,n}, f_{r,d,t,2k-1,m,n}, f_{r,d,t,2l,m,n} \right\}. \end{aligned} \tag{245}$$

Thirteen-tuple $\boldsymbol{f}_{t,c,i,t}$ embraces the unit pulson, two deterministic sine waves $g_{d,t,2q-1,m,m}$ with frequencies $\omega_{d,t,q,m,m}$, two random sine waves $g_{r,t,2q-1,m,m}$ with frequencies $\omega_{r,t,q,m,m}(t)$, four deterministic-random sine waves $g_{d,r,t,2k-1,m,m}$ with frequencies $\omega_{d,r,t,k,m,m}(t)$, and four deterministic-random cosine waves $g_{d,r,t,2l,m,m}$ with frequencies $\omega_{d,r,t,l,m,m}(t)$ for $q=2,3$, $k=3,4,5,6$, $l=1,2,7,8$, and each *m*.

Thirty-two-tuple $\boldsymbol{f}_{t,c,e,t}$ is constructed of four deterministic sine waves $f_{d,t,2k-1,m,n}$ with frequencies $\omega_{d,t,k,m,n}$, four deterministic cosine waves $f_{d,t,2l,m,n}$ with frequencies $\omega_{d,t,l,m,n}$, four random sine waves $f_{r,t,2k-1,m,n}$ with frequencies $\omega_{r,t,k,m,n}(t)$, four random cosine waves $f_{r,t,2l,m,n}$ with frequencies $\omega_{r,t,l,m,n}(t)$, four deterministic-random sine waves $f_{d,r,t,2k-1,m,n}$ with frequencies $\omega_{d,r,t,k,m,n}(t)$, four deterministic-random cosine waves $f_{d,r,t,2l,m,n}$ with frequencies $\omega_{d,r,t,l,m,n}(t)$, four random-deterministic sine waves $f_{r,d,t,2k-1,m,n}$ with frequencies $\omega_{r,d,t,k,m,n}(t)$, and four random-deterministic cosine waves $f_{r,d,t,2l,m,n}$ with frequencies $\omega_{r,d,t,l,m,n}(t)$ for $k=3,4,5,6$, $l=1,2,7,8$, and each *m*, *n*.

If all deterministic frequencies of the tck-*t* pulson are distinct, then the tck-*t* pulson is displayed as a $2M(2M-1)$-f, $2M(6M-1)$-rf pulson. For any frozen $x=x_0$, $y=y_0, z=z_0$, the tck-*t* pulson is given by the $2M(2M-1)$-f, $2M(6M-1)$-rf, turbulent, supercritical pulson in *t*, which depends on M 13-tuples $\boldsymbol{f}_{t,c,i,t}$ with frequencies $\omega_{d,t,q,m,m}, \omega_{r,t,q,m,m}(t), \omega_{d,r,t,k,m,m}(t)$ for $q=2,3$, $k=1,2,\cdots,8$, and all *m*, *Re* and $M(M-1)/2$ 32-tuples $\boldsymbol{f}_{t,c,e,t}$ with frequencies $\omega_{d,t,k,m,n}, \omega_{r,t,k,m,n}(t), \omega_{d,r,t,k,m,n}(t)$, $\omega_{r,d,t,k,m,n}(t)$ for $k=1,2,\cdots,8$ and all *m*, *n*, *Re* seeing that

$$
\begin{aligned}
K_{e,t} = \rho_c \Bigg(\sum_{m=1}^{M} \Big\{ & ez_{d,m}^2 \Big[\mu_{d,m}^2 \left(Av_{d,m}^2 + Bv_{d,m}^2 + Cv_{d,m}^2 + Dv_{d,m}^2 \right) \\
& + Q_{d,m} R_{d,m} \left(\lambda_{d,m}^2 g_{d,t,3,m,m} - \kappa_{d,m}^2 g_{d,t,5,m,m} \right) \Big] \\
& + ez_{r,m}^2 \Big[\mu_{r,m}^2 \left(Av_{r,m}^2 + Bv_{r,m}^2 + Cv_{r,m}^2 + Dv_{r,m}^2 \right) \\
& + Q_{r,m} R_{r,m} \left(\lambda_{r,m}^2 g_{r,t,3,m,m} - \kappa_{r,m}^2 g_{r,t,5,m,m} \right) \Big] \\
& + \frac{1}{2} ez_{d,m} ez_{r,m} \Big[Q_{d,m} R_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,5,m,m} - \mathrm{K}_{d,m,r,m} g_{d,r,t,7,m,m} \right) \\
& + R_{d,m} Q_{r,m} \left(\Lambda_{d,m,r,m} g_{d,r,t,9,m,m} + \mathrm{K}_{d,m,r,m} g_{d,r,t,11,m,m} \right) \\
& - Q_{d,m} Q_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,2,m,m} - \mathrm{M}_{d,m,r,m} g_{d,r,t,4,m,m} \right) \\
& + R_{d,m} R_{r,m} \left(\mathrm{N}_{d,m,r,m} g_{d,r,t,14,m,m} + \mathrm{M}_{d,m,r,m} g_{d,r,t,16,m,m} \right) \Big] \Big\} \\
+ \frac{1}{2} \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} \Big\{ & ez_{d,m} ez_{d,n} \Big[Q_{d,m} R_{d,n} \left(\Lambda_{d,m,d,n} f_{d,t,5,m,n} - \mathrm{K}_{d,m,d,n} f_{d,t,7,m,n} \right) \\
& + R_{d,m} Q_{d,n} \left(\Lambda_{d,m,d,n} f_{d,t,9,m,n} + \mathrm{K}_{d,m,d,n} f_{d,t,11,m,n} \right) \\
& - Q_{d,m} Q_{d,n} \left(\mathrm{N}_{d,m,d,n} f_{d,t,2,m,n} - \mathrm{M}_{d,m,d,n} f_{d,t,4,m,n} \right) \\
& + R_{d,m} R_{d,n} \left(\mathrm{N}_{d,m,d,n} f_{d,t,14,m,n} + \mathrm{M}_{d,m,d,n} f_{d,t,16,m,n} \right) \Big] \\
& + ez_{r,m} ez_{r,n} \Big[Q_{r,m} R_{r,n} \left(\Lambda_{r,m,r,n} f_{r,t,5,m,n} - \mathrm{K}_{r,m,r,n} f_{r,t,7,m,n} \right) \\
& + R_{r,m} Q_{r,n} \left(\Lambda_{r,m,r,n} f_{r,t,9,m,n} + \mathrm{K}_{r,m,r,n} f_{r,t,11,m,n} \right) \\
& - Q_{r,m} Q_{r,n} \left(\mathrm{N}_{r,m,r,n} f_{r,t,2,m,n} - \mathrm{M}_{r,m,r,n} f_{r,t,4,m,n} \right) \\
& + R_{r,m} R_{r,n} \left(\mathrm{N}_{r,m,r,n} f_{r,t,14,m,n} + \mathrm{M}_{r,m,r,n} f_{r,t,16,m,n} \right) \Big] \\
& + ez_{d,m} ez_{r,n} \Big[Q_{d,m} R_{r,n} \left(\Lambda_{d,m,r,n} f_{d,r,t,5,m,n} - \mathrm{K}_{d,m,r,n} f_{d,r,t,7,m,n} \right)
\end{aligned} \tag{246}
$$

$$
\begin{aligned}
&+R_{d,m}\ Q_{r,n}\left(\Lambda_{d,m,r,n} f_{d,r,t,9,m,n}\ +\mathrm{K}_{d,m,r,n} f_{d,r,t,11,m,n}\right)\\
&-Q_{d,m}\ Q_{r,n}\left(\mathrm{N}_{d,m,r,n} f_{d,r,t,2,m,n}\ -\mathrm{M}_{d,m,r,n} f_{d,r,t,4,m,n}\right)\\
&+R_{d,m}\ R_{r,n}\left(\mathrm{N}_{d,m,r,n} f_{d,r,t,14,m,n}+\mathrm{M}_{d,m,r,n} f_{d,r,t,16,m,n}\right)\Big]\\
+\ ez_{r,m}ez_{d,n}\Big[&Q_{r,m}\ R_{d,n}\left(\Lambda_{r,m,d,n} f_{r,d,t,5,m,n}\ -\mathrm{K}_{r,m,d,n} f_{r,d,t,7,m,n}\right)\\
&+R_{r,m}\ Q_{d,n}\left(\Lambda_{r,m,d,n} f_{r,d,t,9,m,n}\ +\mathrm{K}_{r,m,d,n} f_{r,d,t,11,m,n}\right)\\
&-Q_{r,m}\ Q_{d,n}\left(\mathrm{N}_{r,m,d,n} f_{r,d,t,2,m,n}\ -\mathrm{M}_{r,m,d,n} f_{r,d,t,4,m,n}\right)\\
&+R_{r,m}\ R_{d,n}\left(\mathrm{N}_{r,m,d,n} f_{r,d,t,14,m,n}+\mathrm{M}_{r,m,d,n} f_{r,d,t,16,m,n}\right)\Big]\Big\}.
\end{aligned}
$$

If N deterministic frequencies are repeated, then the number of independent modes of the tck-t pulson diminishes to $\left[2M\left(2M-1\right)-N\right]$-f, $2M\left(6M-1\right)$-rf due to the frequency resonance. Analogous to the rck-t pulson, the tck-t oscillon is a smooth random function of time with an unbounded period.

Because

$$
K_{e,t}\left(x_0,y_0,z_0,t\right)=\frac{\rho_c}{2}\left(u_{t,x}^2+u_{t,y}^2+u_{t,z}^2\right)\left(x_0,y_0,z_0,t\right), \tag{247}
$$

where $u_{t,x},u_{t,y},u_{t,z}$ are *x*-, *y*-, *z*-components of turbulent velocity $\boldsymbol{u}_t$, the tck-t pulson remains positive for all times and transfers a positive amount of the kinetic energy in time, as well.

The dck-t, rck-t, and tck-t pulsons for $x=x_0,y=y_0,z=z_0,Re=10^3,Re=10^5$, and wave parameters (129)-(135) are compared on $\left[t_0,t_0+T_{e,d,e}\right]$ in **Figure 7**. Despite the fact that the range of the 30-rf, random, supercritical rck-t pulson in **Figure 7(b)** is $\left[0,1520\right]$, the 23-f, 102-rf, turbulent, supercritical tck-t pulson at $Re=10^3$ in **Figure 7(c)** changes significantly compared with the 23-f, deterministic, supercritical dck-t pulson in **Figure 7(a)** due to range $\left[-3000,3300\right]$ of the 48-rf, deterministic-random, random-deterministic drek-t oscillon in **Figure 1(a)** and range $\left[-2900,2600\right]$ of the 24-rf, deterministic-random drik-t oscillon in **Figure 2(a)**. The shape and the range of the 23-f, 102-rf, turbulent, supercritical tck-t pulson at $Re=10^5$ in **Figure 7(e)** becomes unrecognizable in comparison with the 23-f, deterministic, supercritical dck-t pulson in **Figure 7(a)** because of the comparable ranges $\left[0,6000\right]$ of the 30-rf, random, supercritical rck-t pulson in **Figure 7(d)**, $\left[-4800,7000\right]$ of the 48-rf, deterministic-random, random-deterministic drek-t oscillon in **Figure 1(b)**, and $\left[-5900,5400\right]$ of the 24-rf, deterministic-random drik-t oscillon in **Figure 2(b)**. There are 23 of 30 independent deterministic frequencies since deterministic frequencies $\pi,\pi/3,4\pi,8\pi/3,13\pi/3$ are repeated and deterministic frequency $5\pi/3$ is repeated twice.

9. Conclusions

The method of inhomogeneous Fourier expansions, which was originally developed for deterministic *x*-, *y*-, *t*-eigenfunctions in [6], [9] and generalized in [5] at the deterministic-random, random-deterministic, random, external, and internal

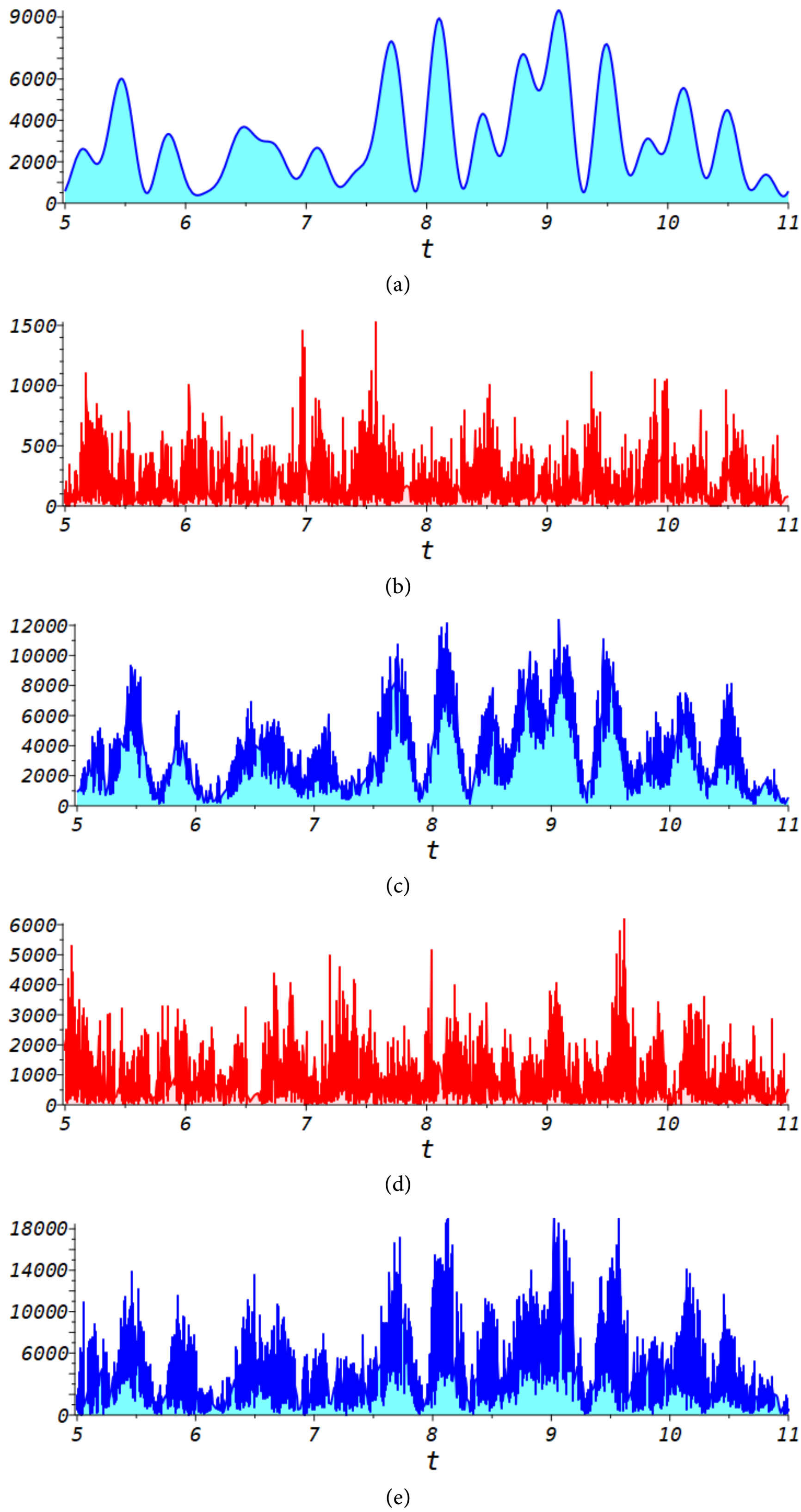


Figure 7. The dck-*t*, rck-*t*, and tck-*t* pulsons: (a) $-K_{e,d,d}(t)$, (b) $-K_{e,r,r}(t)$ for $Re = 10^3$, (c) $-K_{e,t}(t)$ for $Re = 10^3$, (d) $-K_{e,r,r}(t)$ for $Re = 10^5$, (e) $-K_{e,t}(t)$ for $Re = 10^5$.

x-eigenfunctions. In the current paper, we have successfully extended the method on the deterministic-random, random-deterministic, random, external, and internal t-eigenfunctions that include smooth random functions of time. Theoretical quantization [7] in experimental DDSD, DRSD, RDSD, and RRSD structures has been confirmed, analyzed, and visualized in the present work using experimental quantization in the developed t-eigenfunctions.

It was shown that similar to spatial quantization [5], the exact solution for temporal quantization may be grouped into the vector, deterministic-random, elementary, external oscillons, the vector, random-deterministic, elementary, external oscillons, the vector, deterministic-random, random-deterministic, wave, external oscillons, the vector, deterministic-random, elementary, internal oscillons, and the vector, deterministic-random, wave, internal oscillons with eight components. The vector, turbulent, elementary and wave, external oscillons and the vector, turbulent, elementary and wave, internal oscillons have two components. The vector, turbulent, elementary and wave, diagonal oscillons and the vector, turbulent, elementary and wave pulsons include four components.

The vector and scalar oscillons and pulsons depend on 1-, 2-, 3-, 4-, 5-, 6-, 8-, 9-, 12-, 13-, 16-, and 32-tuples of the correspondent t-eigenfunctions. Namely, the vector, deterministic-random, elementary, external oscillons on two 8-tuples $\boldsymbol{f}_{d,r,e,e,t,q}$, the vector, random-deterministic, elementary, external oscillons on two 8-tuples $\boldsymbol{f}_{r,d,e,e,t,q}$, the vector, deterministic-random, random-deterministic, wave, external oscillons on two 16-tuples $\boldsymbol{f}_{d,r,e,w,t,q}$, and the vector, deterministic-random, elementary and wave, internal oscillons on two 8-tuples $\boldsymbol{f}_{d,r,i,e,t,q}$. The vector, turbulent, elementary and wave, external oscillons on three 8-tuples $\boldsymbol{f}_{t,e,e,t,q}$, the vector, turbulent, elementary and wave, diagonal oscillons on 16-tuple $\boldsymbol{f}_{t,d,e,t}$, the vector, turbulent, elementary, internal oscillons on two 6-tuples $\boldsymbol{f}_{t,i,e,t,q}$, and 5-tuple $\boldsymbol{f}_{t,i,e,t,3}$, the vector, turbulent, wave, internal oscillons on two 2-tuples $\boldsymbol{f}_{t,i,w,t,q}$, and 1-tuple $\boldsymbol{f}_{t,i,w,t,3}$, the vector, turbulent, elementary pulson on 9-tuple $\boldsymbol{f}_{t,e,t}$, and the vector, turbulent, wave pulson on 5-tuple $\boldsymbol{f}_{t,w,t}$.

Independent random parameters (130) of the deterministic-random, random-deterministic, and turbulent oscillons and pulsons have been computed using the random oscillatory cn-noise [8] for all m, $Re = 10^3$, and $Re = 10^5$. Despite the fact that the deterministic oscillons and pulsons do not depend on the Reynolds number, empirical scales (133) produce a strong dependence of the quantized oscillons and pulsons on Re. So, the 23-f, 102-rf, turbulent, supercritical tck-t pulson at $Re = 10^3$ in **Figure 7(c)** modifies considerably in comparison with the 23-f, deterministic, supercritical dck-t pulson in **Figure 7(a)** due to the 48-rf, deterministic-random, random-deterministic drek-t oscillon and the 24-rf, deterministic-random drik-t oscillon. The 23-f, 102-rf, turbulent, supercritical tck-t pulson at $Re = 10^5$ in **Figure 7(e)** becomes unrecognizable compared with the 23-f, deterministic, supercritical dck-t pulson because of the 30-rf, random, supercritical rck-t pulson, the 48-rf, deterministic-random, random-deterministic drek-t oscillon, and the 24-rf, deterministic-random drik-t oscillon.

From the mathematical point of view, components of the vector quantized oscillons and pulsons are invariant structures, which are constructed on the correspondent tuples of temporal eigenfunctions. Wave oscillons and pulsons, which are dot products of various DVK and RVK structures, always become more sophisticated than elementary oscillons and pulsons, which are expressed via correspondent DDSD, DRSD, RDSD, and RRSD structures. Compared with wave oscillons and pulsons, group oscillons and pulsons become simplified due to symmetry and identity resonances. Kinetic-energy oscillons and pulsons again become more complicated than group oscillons and pulsons since they are composed of M internal group oscillons and pulsons and $M(M-1)/2$ external group oscillons and pulsons. The maximal number $4M(4M-1)$ of independent modes in *t*-eigenfunctions, which equals to 132 for $M=3$, of the $2M(M-1)$-f, $2M(6M-1)$-rf, turbulent, supercritical tck-*t* pulson considerably exceeds the maximal number $4M^2$ of independent modes in *x*-eigenfunctions, which equals to 36 for $M=3$, of the $4M^2$-w, turbulent, supercritical tck-*x* pulson [5].

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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