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Experimental Quantization of Exact Wave Turbulence I: Spatial Quantization

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Abstract

In previous articles, the exact solutions for deterministic chaos, stochastic chaos, and wave turbulence have been developed in terms of exponential oscillons and pulsons, which are governed by the nonstationary three-dimensional Navier-Stokes equations. We have later considered theoretical quantization of the deterministic chaos in invariant structures and experimental quantization in spatial and temporal eigenfunctions with the help of inhomogeneous Fourier expansions. The study of exact wave turbulence was also continued with the theoretical quantization of stochastic chaos and wave turbulence. The current paper proceeds with experimental quantization of the stochastic chaos and the wave turbulence in spatial *x*-eigenfunctions. The method of inhomogeneous Fourier expansions in the deterministic eigenfunctions has been extended to deterministic-random, random-deterministic, random, external, and internal eigenfunctions. The previous results on theoretical quantization in invariant structures have been confirmed, analyzed, and visualized in this work using experimental quantization in the novel eigenfunctions. Arguments of exact solutions for quantized oscillons and pulsons are given by 1-, 2-, 3-, 4-, 5-, 6-, 8-, 12-, 15-, 16, and 32-tuples of the spatial eigenfunctions. The exact solutions are grouped into the vector, deterministic-random, external oscillons, the vector, random-deterministic, external oscillons, the vector, deterministic-random, internal oscillons, the vector, turbulent, external oscillons, the vector, turbulent, diagonal oscillons, the vector, turbulent, internal, oscillons, and the vector, turbulent pulsons. We compute independent random parameters with the help of the random model of oscillatory cn-noise. Computation is performed by experimental and theoretical programming in Maple. The obtained results demonstrate a strong dependence of the quantized oscillons and pulsons on the Reynolds number.

Keywords

Exact Solutions, Navier-Stokes Equations, Vector Deterministic-Random External Oscillon, Vector Random-Deterministic External Oscillon, Vector Deterministic-Random Internal Oscillon, Vector Turbulent External Oscillon, Vector Turbulent Diagonal Oscillon, Vector Turbulent Internal Oscillon, Vector Turbulent Pulson, 1-Tuple, 2-Tuple, 3-Tuple, 4-Tuple, 5-Tuple, 6-Tuple, 8-Tuple, 12-Tuple, 15-Tuple, 16-Tuple, 32-Tuple of Spatial Eigenfunctions

1. Introduction

The exact solution for deterministic chaos of exponential oscillons and pulsons governed by the nonstationary, three-dimensional (3-d) Navier-Stokes equations has been developed by the method of decomposition in invariant structures (DIS) [1]. Using the Helmholtz decomposition, we decompose the Dirichlet problem for the Navier-Stokes equations into the Archimedean, Stokes, and Navier problems. A cascade differential algebra is developed in [2] for four families of invariant structures: deterministic scalar kinematic (DSK) structures, deterministic vector kinematic (DVK) structures, deterministic scalar dynamic (DSD) structures, and deterministic vector dynamic (DVD) structures. Scalar and vector variables of the Stokes problem are decomposed into the theoretical DSK and DVK structures, respectively. Scalar and vector variables of the Navier problem are expanded into the theoretical DSD and DVD structures, correspondingly.

The exact solution for stochastic chaos of random exponential oscillons and pulsons controlled by the nonstationary, 3-d Navier-Stokes equations has been considered in [3]. Differential algebra is constructed for six families of random invariant structures: random scalar kinematic (RSK) structures, time-complementary random scalar kinematic (RSK_t) structures, random vector kinematic (RVK) structures, time-complementary random vector kinematic (RVK_t) structures, random scalar dynamic (RSD) structures, and random vector dynamic (RVD) structures. We expand the random Dirichlet problem for the Navier-Stokes equations into the Archimedean, the random Stokes, and the random Navier problems. Scalar and vector solutions of the random Stokes problem are represented via the theoretical RSK, RSK_t, RVK, and RVK_t structures. Scalar and vector solutions of the random Navier problem are computed using the theoretical RSD and RVD structures.

The exact wave turbulence of exponential oscillons and pulsons is treated in [4] by developing eight families of invariant structures: deterministic-deterministic scalar dynamic (DDSD) structures, deterministic-random scalar dynamic (DRSD) structures, random-deterministic scalar dynamic (RDSD) structures, random-random scalar dynamic (RRSD) structures, deterministic-deterministic vector dynamic (DDVD) structures, deterministic-random vector dynamic (DRVD) structures, random-deterministic vector dynamic (RDVD) structures, and random-random vector dynamic (RRVD) structures in theoretical and experimental representations. The deterministic Stokes problem, the random Stokes problem, and the turbulent Stokes problem, which are subjected to the Dirichlet boundary conditions and conditions at infinities, are solved in the theoretical DSK, DVK, RSK, RSK_b, RVK, and RVK_t structures. The turbulent Navier problem is tackled and justified in the theoretical DDVD, DRVD, RDVD, RRVD, DDSD, DRSD, RDSD, and RRSD structures.

Theoretical quantization of the deterministic chaos in the elementary, wave, group, kinetic-energy pulsons, in the elementary, wave, group, kinetic-energy, diagonal oscillons, in the elementary, wave, group, kinetic-energy, internal oscillons, and in the elementary, wave, group, kinetic-energy, external oscillons was developed via the experimental DSD structures in [5] [6]. Three families of x-eigenfunctions, y-eigenfunctions, and t-eigenfunctions were defined to treat experimental quantization and to study topology, periodicity, local properties, integral properties, and shapes of the quantized deterministic oscillons and pulsons.

Theoretical quantization of the stochastic chaos in the random, elementary, wave, group, kinetic-energy pulsons, in the random, elementary, wave, group, kinetic-energy, diagonal oscillons, in the random, elementary, wave, group, kinetic-energy, internal oscillons, and in the random, elementary, wave, group, kinetic-energy, external oscillons was considered in terms of the experimental RSD structures in [3]. Theoretical quantization of the wave turbulence, which complements theoretical quantization of the deterministic chaos and the stochastic chaos by the deterministic-random, elementary, wave, group, kinetic-energy, internal oscillons, by the deterministic-random and random-deterministic, elementary, external oscillons, and by the deterministic-random, random-deterministic, wave, group, kinetic-energy, external oscillons, was treated in [7] using the experimental DDSD, DRSD, RDSD, and RRSD structures.

The objective of this paper is to complete theoretical quantization of the wave turbulence by experimental quantization in x-eigenfunctions of the deterministic-random, external, and internal interaction, the random-deterministic, external interaction, the random, external, and internal interaction, and the deterministic, external, and internal interaction and to explore topology, periodicity, integral properties, and visualizations of the quantized, deterministic-random, random-deterministic, and turbulent oscillons and pulsons.

The contents of the current paper are as follows. The deterministic-random, random-deterministic, and random x-eigenfunctions are introduced in Section 2. Section 3 deals with oscillons of the deterministic-random, external interaction. Oscillons of the deterministic-random, internal interaction are considered in Section 4. Oscillons of the turbulent, external interaction are computed in Section 5. Oscillons of the turbulent, diagonal interaction and the turbulent, internal interaction are studied in Sections 6 and 7, respectively. Turbulent and cumulative pulsons are treated in Section 8. Section 9 contains a brief discussion of the main results and further developments.

2. Eigenfunctions of Inhomogeneous Fourier Expansions

2.1. Deterministic and Random Eigenfunctions of the Turbulent Velocity Potential

The deterministic, velocity-potential, elementary oscillons (the dpe oscillons for brevity, see (126) of [7])

$$K_{o,d,a,m} = a_{d,m},\ K_{o,d,b,m} = b_{d,m},\ K_{o,d,c,m} = c_{d,m},\ K_{o,d,d,m} = d_{d,m} \tag{1}$$

are defined via the experimental DSK (eDSK) structures

$$\begin{aligned} a_{d,m} &= +Av_{d,m}\,sse_{d,m} + Bv_{d,m}\,cse_{d,m} + Cv_{d,m}\,sce_{d,m} + Dv_{d,m}\,cce_{d,m}, \\ b_{d,m} &= -Bv_{d,m}\,sse_{d,m} + Av_{d,m}\,cse_{d,m} - Dv_{d,m}\,sce_{d,m} + Cv_{d,m}\,cce_{d,m}, \\ c_{d,m} &= -Cv_{d,m}\,sse_{d,m} - Dv_{d,m}\,cse_{d,m} + Av_{d,m}\,sce_{d,m} + Bv_{d,m}\,cce_{d,m}, \\ d_{d,m} &= +Dv_{d,m}\,sse_{d,m} - Cv_{d,m}\,cse_{d,m} - Bv_{d,m}\,sce_{d,m} + Av_{d,m}\,cce_{d,m}, \end{aligned} \tag{2}$$

where $m = 1, 2, \cdots, M$ is an index of oscillons, $\left[Av_{d,m}, Bv_{d,m}, Cv_{d,m}, Dv_{d,m}\right]$ are deterministic functional amplitudes of a deterministic harmonic variable $v_d\left(x, y, z, t\right)$, $\left[sse_{d,m}, cse_{d,m}, sce_{d,m}, cce_{d,m}\right]\left(X_{d,m}, Y_{d,m}, z\right)$ are three-variables (3-v) eDSK functions, explicitly,

$$\begin{aligned} sse_{d,m} &= sx_{d,m}\,sy_{d,m}\,ez_{d,m}, \quad & cse_{d,m} &= cx_{d,m}\,sy_{d,m}\,ez_{d,m}, \\ sce_{d,m} &= sx_{d,m}\,cy_{d,m}\,ez_{d,m}, \quad & cce_{d,m} &= cx_{d,m}\,cy_{d,m}\,ez_{d,m}, \end{aligned} \tag{3}$$

$\left[sx_{d,m}, cx_{d,m}\right]\left(X_{d,m}\right), \left[sy_{d,m}, cy_{d,m}\right]\left(Y_{d,m}\right), ez_{d,m}\left(z\right)$ are one-variable (1-v) eDSK functions, which are calculated by

$$\begin{aligned} sx_{d,m} &= \sin\left(\kappa_{d,m} X_{d,m}\right), \qquad & cx_{d,m} &= \cos\left(\kappa_{d,m} X_{d,m}\right), \\ sy_{d,m} &= \sin\left(\lambda_{d,m} Y_{d,m}\right), \qquad & cy_{d,m} &= \cos\left(\lambda_{d,m} Y_{d,m}\right), \end{aligned} \tag{4}$$

$$ez_{d,m} = \exp\left(\left(-1\right)^{\eta} \mu_{d,m}\, z\right),\ \mu_{d,m} = \sqrt{\kappa_{d,m}^2 + \lambda_{d,m}^2}, \tag{5}$$

$X_{d,m}\left(x, t\right), Y_{d,m}\left(y, t\right)$ are deterministic propagation variables defined by

$$X_{d,m} = x - U_{d,m} t + X_{d,m,0},\ Y_{d,m} = y - V_{d,m} t + Y_{d,m,0}. \tag{6}$$

In the above definition of the dpe oscillons, $\left(x, y, z\right)$ is the Cartesian coordinate of a motionless frame of reference, t is time, $\left(X_{d,m}, Y_{d,m}, z\right)$ is the Cartesian coordinate of the mth frame of reference moving with the dpe oscillon, $\left(U_{d,m}, V_{d,m}, 0\right)$ is a celerity of propagation of the mth dpe oscillon, $\left(X_{d,m,0}, Y_{d,m,0}\right)$ is a reference value of $\left(X_{d,m}, Y_{d,m}\right)$ at $t = 0, x = 0, y = 0$, $\kappa_{d,m}, \lambda_{d,m}, \mu_{d,m}$ are deterministic wavenumbers of the mth dpe oscillon in the x-, y-, z-directions, and a sign parameter $\eta = 0$ for $z < 0$ and $\eta = 1$ for $z > 0$.

Following [5] [6], we express the 3-v eDSK functions in terms of the 1-v eDSK functions and collect $\left[sx_{d,m}, cx_{d,m}, ez_{d,m}\right]$ to obtain the dpe-x oscillons propagating in the x-direction via deterministic trigonometric polynomials

$$\begin{aligned} K_{o,d,a,m} &= \left(+Ay_{d,m}\, sx_{d,m} + Cy_{d,m}\, cx_{d,m}\right) ez_{d,m}, \\ K_{o,d,b,m} &= \left(-Cy_{d,m}\, sx_{d,m} + Ay_{d,m}\, cx_{d,m}\right) ez_{d,m}, \\ K_{o,d,c,m} &= \left(+By_{d,m}\, sx_{d,m} + Dy_{d,m}\, cx_{d,m}\right) ez_{d,m}, \\ K_{o,d,d,m} &= \left(-Dy_{d,m}\, sx_{d,m} + By_{d,m}\, cx_{d,m}\right) ez_{d,m}, \end{aligned} \tag{7}$$

where deterministic coefficients

$$\begin{aligned} Ay_{d,m} &= + Av_{d,m}\, sy_{d,m} + Cv_{d,m}\, cy_{d,m}, \quad By_{d,m} = -Cv_{d,m}\, sy_{d,m} + Av_{d,m}\, cy_{d,m}, \\ Cy_{d,m} &= +Bv_{d,m}\, sy_{d,m} + Dv_{d,m}\, cy_{d,m}, \quad Dy_{d,m} = -Dv_{d,m}\, sy_{d,m} + Bv_{d,m}\, cy_{d,m} \end{aligned} \tag{8}$$

depend on (y,t).

We then define four deterministic trigonometric functions in the y-direction

$$\begin{aligned} \sin\left(\kappa_{d,m}\, \alpha_{d,y,m}\right) &= \frac{Cy_{d,m}}{Q_{d,y,m}}, \quad \cos\left(\kappa_{d,m}\, \alpha_{d,y,m}\right) = \frac{Ay_{d,m}}{Q_{d,y,m}}, \\ \sin\left(\kappa_{d,m}\, \beta_{d,y,m}\right) &= \frac{Dy_{d,m}}{R_{d,y,m}}, \quad \cos\left(\kappa_{d,m}\, \beta_{d,y,m}\right) = \frac{By_{d,m}}{R_{d,y,m}}, \end{aligned} \tag{9}$$

deterministic X-shifts, depending on (y,t),

$$\alpha_{d,y,m} = \frac{1}{\kappa_{d,m}} \arcsin \frac{Cy_{d,m}}{Q_{d,y,m}}, \quad \beta_{d,y,m} = \frac{1}{\kappa_{d,m}} \arcsin \frac{Dy_{d,m}}{R_{d,y,m}}, \tag{10}$$

and deterministic trigonometric amplitudes

$$Q_{d,y,m} = \sqrt{Ay_{d,m}^2 + Cy_{d,m}^2}, \quad R_{d,y,m} = \sqrt{By_{d,m}^2 + Dy_{d,m}^2}. \tag{11}$$

Substituting the deterministic trigonometric functions in the deterministic trigonometric polynomials and combining terms yields the inhomogeneous Fourier form of the dpe-x oscillons in the mth moving frame (see (67) of [5])

$$\begin{aligned} K_{o,d,a,m} &= ez_{d,m}\, Q_{d,y,m} \sin\left(\kappa_{d,m}\left(X_{d,m} + \alpha_{d,y,m}\right)\right), \\ K_{o,d,b,m} &= ez_{d,m}\, Q_{d,y,m} \cos\left(\kappa_{d,m}\left(X_{d,m} + \alpha_{d,y,m}\right)\right), \\ K_{o,d,c,m} &= ez_{d,m}\, R_{d,y,m} \sin\left(\kappa_{d,m}\left(X_{d,m} + \beta_{d,y,m}\right)\right), \\ K_{o,d,d,m} &= ez_{d,m}\, R_{d,y,m} \cos\left(\kappa_{d,m}\left(X_{d,m} + \beta_{d,y,m}\right)\right). \end{aligned} \tag{12}$$

Eventually, four dpe-x oscillons may be grouped into two vector dpe-x oscillons

$$\boldsymbol{K}_{d,p,e,x,q} = \boldsymbol{K}_{d,p,e,x,q}\left(\boldsymbol{f}_{d,p,e,x,q}\right), \quad q = 1,2, \tag{13}$$

which are formed by two 2-tuples of the dpe-x oscillons:

$$\boldsymbol{f}_{d,p,e,x,1} = \left\{ f_{d,x,1,m}, f_{d,x,2,m} \right\}, \quad \boldsymbol{f}_{d,p,e,x,2} = \left\{ f_{d,x,3,m}, f_{d,x,4,m} \right\}. \tag{14}$$

Two-tuple $\boldsymbol{f}_{d,p,e,x,1}$ consists of sine wave $f_{d,x,1,m}$ and cosine wave $f_{d,x,2,m}$ with wavenumber $\kappa_{d,m}$ for each m. Two-tuple $\boldsymbol{f}_{d,p,e,x,2}$ comprises sine wave $f_{d,x,3,m}$ and cosine wave $f_{d,x,4,m}$ with wavenumber $\kappa_{d,m}$ for each m.

Here, deterministic eigenfunctions of the dpe-x oscillons in the motionless frame

$$\begin{aligned} f_{d,x,1,m} &= \sin\left(\mathrm{A}_{d,x,1,m}\right), \quad f_{d,x,2,m} = \cos\left(\mathrm{A}_{d,x,1,m}\right), \\ f_{d,x,3,m} &= \sin\left(\mathrm{A}_{d,x,2,m}\right), \quad f_{d,x,4,m} = \cos\left(\mathrm{A}_{d,x,2,m}\right) \end{aligned} \tag{15}$$

depend on two arguments

$$\mathrm{A}_{d,x,1,m} = \kappa_{d,m}\left(x + s_{d,y,\alpha,m}\right),\ \mathrm{A}_{d,x,2,m} = \kappa_{d,m}\left(x + s_{d,y,\beta,m}\right), \tag{16}$$

where

$$s_{d,y,\alpha,m} = X_{d,m,0} - U_{d,m}t + \alpha_{d,y,m},\ s_{d,y,\beta,m} = X_{d,m,0} - U_{d,m}t + \beta_{d,y,m} \tag{17}$$

are deterministic *x*-shifts, which depend on (y,t).

For any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector dpe-*x* oscillon

$$\boldsymbol{K}_{d,p,e,x,1} = \left[K_{o,d,a,m}, K_{o,d,b,m}\right]\left(\boldsymbol{f}_{d,p,e,x,1}\right) \tag{18}$$

is reduced to a list of two one-wavenumber (1-w), deterministic oscillons in *x*, which are produced by 2-tuple $\boldsymbol{f}_{d,p,e,x,1}$ as

$$K_{o,d,a,m} = ez_{d,m}\,Q_{d,y,m}\,f_{d,x,1,m},\ K_{o,d,b,m} = ez_{d,m}\,Q_{d,y,m}\,f_{d,x,2,m}. \tag{19}$$

The 2nd vector dpe-*x* oscillon

$$\boldsymbol{K}_{d,p,e,x,2} = \left[K_{o,d,c,m}, K_{o,d,d,m}\right]\left(\boldsymbol{f}_{d,p,e,x,2}\right) \tag{20}$$

is also transformed into a list of two 1-w, deterministic oscillons in *x*, which are generated by 2-tuple $\boldsymbol{f}_{d,p,e,x,2}$ since

$$K_{o,d,c,m} = ez_{d,m}\,R_{d,y,m}\,f_{d,x,3,m},\ K_{o,d,d,m} = ez_{d,m}\,R_{d,y,m}\,f_{d,x,4,m}. \tag{21}$$

For all vector dpe-*x* oscillons, amplitudes of complementary eigenfunctions vanish, while amplitudes of eigenfunctions depend on z_0, y_0, t_0 via $ez_{d,m}, Q_{d,y,m}, R_{d,y,m}$ and *x*-shifts are determined by y_0, t_0 through $s_{d,y,\alpha,m}, s_{d,y,\beta,m}$.

The *x*-period of the dpe-*x* oscillons

$$L_{d,x,m} = \frac{2\pi}{\kappa_{d,m}}. \tag{22}$$

Since the average of the dpe-*x* oscillons over $L_{d,x,m}$ vanishes for each *m*, *i.e.*

$$\frac{1}{L_{d,x,m}}\int_0^{L_{d,x,m}} K_{o,d,i,m}\mathrm{d}x = 0,\ i = [a,b,c,d], \tag{23}$$

the dpe-*x* oscillons are neutral.

The random, velocity-potential, elementary oscillons (the rpe oscillons for briefness, see (156) of [7])

$$K_{o,r,a,m} = a_{r,m},\ K_{o,r,b,m} = b_{r,m},\ K_{o,r,c,m} = c_{r,m},\ K_{o,r,d,m} = d_{r,m} \tag{24}$$

are represented via the experimental RSK (eRSK) structures

$$\begin{aligned} a_{r,m} &= +Av_{r,m}\,sse_{r,m} + Bv_{r,m}\,cse_{r,m} + Cv_{r,m}\,sce_{r,m} + Dv_{r,m}\,cce_{r,m}, \\ b_{r,m} &= -Bv_{r,m}\,sse_{r,m} + Av_{r,m}\,cse_{r,m} - Dv_{r,m}\,sce_{r,m} + Cv_{r,m}\,cce_{r,m}, \\ c_{d,m} &= -Cv_{r,m}\,sse_{r,m} - Dv_{r,m}\,cse_{r,m} + Av_{r,m}\,sce_{r,m} + Bv_{r,m}\,cce_{r,m}, \\ d_{r,m} &= +Dv_{r,m}\,sse_{r,m} - Cv_{r,m}\,cse_{r,m} - Bv_{r,m}\,sce_{r,m} + Av_{r,m}\,cce_{r,m}, \end{aligned} \tag{25}$$

where $\left[Av_{r,m}, Bv_{r,m}, Cv_{r,m}, Dv_{r,m}\right](t)$ are random functional amplitudes of a random harmonic variable $v_r(x,y,z,t)$, $\left[sse_{r,m}, cse_{r,m}, sce_{r,m}, cce_{r,m}\right]\left(X_{r,m}, Y_{r,m}, z\right)$

are 3-v eRSK functions, namely,

$$\begin{aligned} sse_{r,m} &= sx_{r,m}\, sy_{r,m}\, ez_{r,m}, \quad & cse_{r,m} &= cx_{r,m}\, sy_{r,m}\, ez_{r,m}, \\ sce_{r,m} &= sx_{r,m}\, cy_{r,m}\, ez_{r,m}, \quad & cce_{r,m} &= cx_{r,m}\, cy_{r,m}\, ez_{r,m}, \end{aligned} \tag{26}$$

$\left[sx_{r,m}, cx_{r,m}\right]\left(X_{r,m}\right), \left[sy_{r,m}, cy_{r,m}\right]\left(Y_{r,m}\right), ez_{r,m}(z)$ are 1-v eRSK functions, which are computed by

$$\begin{aligned} sx_{r,m} &= \sin\left(\kappa_{r,m} X_{r,m}\right), \quad & cx_{r,m} &= \cos\left(\kappa_{r,m} X_{r,m}\right), \\ sy_{r,m} &= \sin\left(\lambda_{r,m} Y_{r,m}\right), \quad & cy_{r,m} &= \cos\left(\lambda_{r,m} Y_{r,m}\right), \end{aligned} \tag{27}$$

$$ez_{r,m} = \exp\left(\left(-1\right)^{\eta} \mu_{r,m}\, z\right), \quad \mu_{r,m} = \sqrt{\kappa_{r,m}^2 + \lambda_{r,m}^2}, \tag{28}$$

$X_{r,m}(x,t), Y_{r,m}(y,t)$ are random propagation variables:

$$X_{r,m} = x - U_{r,m} t + X_{r,m,0}, \quad Y_{r,m} = y - V_{r,m} t + Y_{r,m,0}. \tag{29}$$

In the above definition of the rpe oscillons, $\left(X_{r,m}, Y_{r,m}, z\right)$ is the Cartesian coordinate of the mth frame of reference moving with the rpe oscillon, $\left(U_{r,m}, V_{r,m}, 0\right)(t)$ is a celerity of propagation of the mth rpe oscillon, $\left(X_{r,m,0}, Y_{r,m,0}\right)(t)$ is a reference value of $\left(X_{r,m}, Y_{r,m}\right)$ at $t = 0, x = 0, y = 0$, and $\kappa_{r,m}, \lambda_{r,m}, \mu_{r,m}$ are random wavenumbers of the mth rpe oscillon in the x-, y-, z-directions. Wave parameters

$$U_{r,m} = U_{r,m}(t), \; V_{r,m} = V_{r,m}(t), \; X_{r,m,0} = X_{r,m,0}(t), \; Y_{r,m,0} = Y_{r,m,0}(t) \tag{30}$$

together with functional amplitudes

$$Av_{r,m} = Av_{r,m}(t), \; Bv_{r,m} = Bv_{r,m}(t), \; Cv_{r,m} = Cv_{r,m}(t), \; Dv_{r,m} = Dv_{r,m}(t) \tag{31}$$

are smooth random functions of time from C^{∞}. Wavenumbers $\kappa_{r,m}, \lambda_{r,m}, \mu_{r,m}$ are random parameters.

Substitution of the 3-v eRSK functions in terms of the 1-v eRSK functions and collection of $\left[sx_{r,m}, cx_{r,m}, ez_{r,m}\right]$ yield the rpe-x oscillons in terms of random trigonometric polynomials

$$\begin{aligned} K_{o,r,a,m} &= \left(+Ay_{r,m}\, sx_{r,m} + Cy_{r,m}\, cx_{r,m}\right) ez_{r,m}, \\ K_{o,r,b,m} &= \left(-Cy_{r,m}\, sx_{r,m} + Ay_{r,m}\, cx_{r,m}\right) ez_{r,m}, \\ K_{o,r,c,m} &= \left(+By_{r,m}\, sx_{r,m} + Dy_{r,m}\, cx_{r,m}\right) ez_{r,m}, \\ K_{o,r,d,m} &= \left(-Dy_{r,m}\, sx_{r,m} + By_{r,m}\, cx_{r,m}\right) ez_{r,m}, \end{aligned} \tag{32}$$

where random coefficients

$$\begin{aligned} Ay_{r,m} &= +Av_{r,m}\, sy_{r,m} + Cv_{r,m}\, cy_{r,m}, \quad & By_{r,m} &= -Cv_{r,m}\, sy_{r,m} + Av_{r,m}\, cy_{r,m}, \\ Cy_{r,m} &= +Bv_{r,m}\, sy_{r,m} + Dv_{r,m}\, cy_{r,m}, \quad & Dy_{r,m} &= -Dv_{r,m}\, sy_{r,m} + Bv_{r,m}\, cy_{r,m} \end{aligned} \tag{33}$$

depend on (y,t).

We then introduce four random trigonometric functions in the y-direction

$$\begin{aligned} \sin\left(\kappa_{r,m}\, \alpha_{r,y,m}\right) &= \frac{Cy_{r,m}}{Q_{r,y,m}}, \quad & \cos\left(\kappa_{r,m}\, \alpha_{r,y,m}\right) &= \frac{Ay_{r,m}}{Q_{r,y,m}}, \\ \sin\left(\kappa_{r,m}\, \beta_{r,y,m}\right) &= \frac{Dy_{r,m}}{R_{r,y,m}}, \quad & \cos\left(\kappa_{r,m}\, \beta_{r,y,m}\right) &= \frac{By_{r,m}}{R_{r,y,m}}, \end{aligned} \tag{34}$$

random *X*-shifts, depending on (y,t),

$$\alpha_{r,y,m} = \frac{1}{\kappa_{r,m}} \arcsin \frac{Cy_{r,m}}{Q_{r,y,m}}, \quad \beta_{r,y,m} = \frac{1}{\kappa_{r,m}} \arcsin \frac{Dy_{r,m}}{R_{r,y,m}}, \tag{35}$$

and random trigonometric amplitudes

$$Q_{r,y,m} = \sqrt{Ay_{r,m}^2 + Cy_{r,m}^2}, \quad R_{r,y,m} = \sqrt{By_{r,m}^2 + Dy_{r,m}^2}. \tag{36}$$

Substitution of the random trigonometric functions in the random trigonometric polynomials gives the inhomogeneous Fourier form of the rpe-*x* oscillons in the *m*th moving frame

$$\begin{aligned} K_{o,r,a,m} &= ez_{r,m}\, Q_{r,y,m} \sin\left(\kappa_{r,m}\left(X_{r,m} + \alpha_{r,y,m}\right)\right), \\ K_{o,r,b,m} &= ez_{r,m}\, Q_{r,y,m} \cos\left(\kappa_{r,m}\left(X_{r,m} + \alpha_{r,y,m}\right)\right), \\ K_{o,r,c,m} &= ez_{r,m}\, R_{r,y,m} \sin\left(\kappa_{r,m}\left(X_{r,m} + \beta_{r,y,m}\right)\right), \\ K_{o,r,d,m} &= ez_{r,m}\, R_{r,y,m} \cos\left(\kappa_{r,m}\left(X_{r,m} + \beta_{r,y,m}\right)\right). \end{aligned} \tag{37}$$

Similar to the dpe-*x* oscillons, four rpe-*x* oscillons may be combined into two vector rpe-*x* oscillons

$$\boldsymbol{K}_{r,p,e,x,q} = \boldsymbol{K}_{r,p,e,x,q}\left(\boldsymbol{f}_{r,p,e,x,q}\right), \quad q = 1, 2, \tag{38}$$

which are produced by two 2-tuples of the rpe-*x* oscillons:

$$\boldsymbol{f}_{r,p,e,x,1} = \left\{ f_{r,x,1,m}, f_{r,x,2,m} \right\}, \quad \boldsymbol{f}_{r,p,e,x,2} = \left\{ f_{r,x,3,m}, f_{r,x,4,m} \right\}. \tag{39}$$

Two-tuple $\boldsymbol{f}_{r,p,e,x,1}$ is composed of sine wave $f_{r,x,1,m}$ and cosine wave $f_{r,x,2,m}$ with wavenumber $\kappa_{r,m}$ for each *m*. Two-tuple $\boldsymbol{f}_{r,p,e,x,2}$ includes sine wave $f_{r,x,3,m}$ and cosine wave $f_{r,x,4,m}$ with wavenumber $\kappa_{r,m}$ for each *m*.

So, random eigenfunctions of the rpe-*x* oscillons in the motionless frame

$$\begin{aligned} f_{r,x,1,m} &= \sin\left(\mathrm{A}_{r,x,1,m}\right), \quad f_{r,x,2,m} = \cos\left(\mathrm{A}_{r,x,1,m}\right), \\ f_{r,x,3,m} &= \sin\left(\mathrm{A}_{r,x,2,m}\right), \quad f_{r,x,4,m} = \cos\left(\mathrm{A}_{r,x,2,m}\right) \end{aligned} \tag{40}$$

depend on two arguments

$$\mathrm{A}_{r,x,1,m} = \kappa_{r,m}\left(x + s_{r,y,\alpha,m}\right), \quad \mathrm{A}_{r,x,2,m} = \kappa_{r,m}\left(x + s_{r,y,\beta,m}\right), \tag{41}$$

where

$$s_{r,y,\alpha,m} = X_{r,m,0} - U_{r,m} t + \alpha_{r,y,m}, \quad s_{r,y,\beta,m} = X_{r,m,0} - U_{r,m} t + \beta_{r,y,m} \tag{42}$$

are random *x*-shifts, which depend on (y,t).

For any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector rpe-*x* oscillon

$$\boldsymbol{K}_{r,p,e,x,1} = \left[K_{o,r,a,m}, K_{o,r,b,m}\right]\left(\boldsymbol{f}_{r,p,e,x,1}\right) \tag{43}$$

is presented by a list of two 1-w, random oscillons in *x*, which are established by 2-tuple $\boldsymbol{f}_{r,p,e,x,1}$ as

$$K_{o,r,a,m} = ez_{r,m}\, Q_{r,y,m}\, f_{r,x,1,m}, \quad K_{o,r,b,m} = ez_{r,m}\, Q_{r,y,m}\, f_{r,x,2,m}. \tag{44}$$

The 2nd vector dpe-*x* oscillon

$$\boldsymbol{K}_{r,p,e,x,2} = \left[K_{o,r,c,m}, K_{o,r,d,m}\right]\left(\boldsymbol{f}_{r,p,e,x,2}\right) \tag{45}$$

is visualized by a list of two 1-w, random, neutral oscillons in *x*, which are formed by 2-tuple $\boldsymbol{f}_{r,p,e,x,2}$ since

$$K_{o,r,c,m} = ez_{r,m} R_{r,y,m} f_{r,x,3,m}, \quad K_{o,r,d,m} = ez_{r,m} R_{r,y,m} f_{r,x,4,m}. \tag{46}$$

For all vector rpe-x oscillons, amplitudes of complementary eigenfunctions vanish, while amplitudes of eigenfunctions are governed by z_0, y_0, t_0 through $ez_{r,m}$, $Q_{r,y,m}, R_{r,y,m}$ and *x*-shifts are influenced by y_0, t_0 via $s_{r,y,\alpha,m}, s_{r,y,\beta,m}$.

The *x*-period of the rpe-x oscillons

$$L_{r,x,m} = \frac{2\pi}{\kappa_{r,m}}. \tag{47}$$

Because the average of the rpe-x oscillons over $L_{r,x,m}$ vanishes for each *m*, *i.e.*

$$\frac{1}{L_{r,x,m}} \int_0^{L_{r,x,m}} K_{o,r,i,m} \mathrm{d}x = 0, \; i = [a,b,c,d], \tag{48}$$

the rpe-x oscillons are neutral, as well.

2.2. Eigenfunctions of Deterministic-Random, External Interaction

Consider the *m*th and *n*th families of deterministic eigenfunctions of the dpe-x oscillons

$$\begin{aligned} f_{d,x,1,m} &= \sin\left(\mathrm{A}_{d,x,1,m}\right), & f_{d,x,2,m} &= \cos\left(\mathrm{A}_{d,x,1,m}\right), \\ f_{d,x,3,m} &= \sin\left(\mathrm{A}_{d,x,2,m}\right), & f_{d,x,4,m} &= \cos\left(\mathrm{A}_{d,x,2,m}\right), \\ f_{d,x,1,n} &= \sin\left(\mathrm{A}_{d,x,1,n}\right), & f_{d,x,2,n} &= \cos\left(\mathrm{A}_{d,x,1,n}\right), \\ f_{d,x,3,n} &= \sin\left(\mathrm{A}_{d,x,2,n}\right), & f_{d,x,4,n} &= \cos\left(\mathrm{A}_{d,x,2,n}\right) \end{aligned} \tag{49}$$

of four arguments

$$\begin{aligned} \mathrm{A}_{d,x,1,m} &= \kappa_{d,m}\left(x + s_{d,y,\alpha,m}\right), & \mathrm{A}_{d,x,2,m} &= \kappa_{d,m}\left(x + s_{d,y,\beta,m}\right), \\ \mathrm{A}_{d,x,1,n} &= \kappa_{d,n}\left(x + s_{d,y,\alpha,n}\right), & \mathrm{A}_{d,x,2,n} &= \kappa_{d,n}\left(x + s_{d,y,\beta,n}\right) \end{aligned} \tag{50}$$

with two deterministic wavenumbers $\kappa_{d,m}$ and $\kappa_{d,n}$ for $m = 1, 2, \cdots, M$ and $n = 1, 2, \cdots, M$.

We also consider the *m*th and *n*th families of random eigenfunctions of the rpe-x oscillons

$$\begin{aligned} f_{r,x,1,m} &= \sin\left(\mathrm{A}_{r,x,1,m}\right), & f_{r,x,2,m} &= \cos\left(\mathrm{A}_{r,x,1,m}\right), \\ f_{r,x,3,m} &= \sin\left(\mathrm{A}_{r,x,2,m}\right), & f_{r,x,4,m} &= \cos\left(\mathrm{A}_{r,x,2,m}\right), \\ f_{r,x,1,n} &= \sin\left(\mathrm{A}_{r,x,1,n}\right), & f_{r,x,2,n} &= \cos\left(\mathrm{A}_{r,x,1,n}\right), \\ f_{r,x,3,n} &= \sin\left(\mathrm{A}_{r,x,2,n}\right), & f_{r,x,4,n} &= \cos\left(\mathrm{A}_{r,x,2,n}\right) \end{aligned} \tag{51}$$

of four arguments

$$\begin{aligned} \mathrm{A}_{r,x,1,m} &= \kappa_{r,m}\left(x + s_{r,y,\alpha,m}\right), & \mathrm{A}_{r,x,2,m} &= \kappa_{r,m}\left(x + s_{r,y,\beta,m}\right), \\ \mathrm{A}_{r,x,1,n} &= \kappa_{r,n}\left(x + s_{r,y,\alpha,n}\right), & \mathrm{A}_{r,x,2,n} &= \kappa_{r,n}\left(x + s_{r,y,\beta,n}\right) \end{aligned} \tag{52}$$

with two random wavenumbers $\kappa_{r,m}$ and $\kappa_{r,n}$.

Matrix $\boldsymbol{f}_{d,x,i,m,r,x,j,n}$ of the deterministic-random, external interaction $(n \neq m)$ in the *x*-direction between $f_{d,x,i,m}$ and $f_{r,x,j,n}$ with $i = 1,2,3,4$ and $j = 1,2,3,4$ (between scaled dpe-*x* and rpe-*x* oscillons) takes the following form:

$$\boldsymbol{f}_{d,x,i,m,r,x,j,n} = \begin{bmatrix} f_{d,x,1,m} f_{r,x,1,n} & f_{d,x,1,m} f_{r,x,2,n} & f_{d,x,1,m} f_{r,x,3,n} & f_{d,x,1,m} f_{r,x,4,n} \\ f_{d,x,2,m} f_{r,x,1,n} & f_{d,x,2,m} f_{r,x,2,n} & f_{d,x,2,m} f_{r,x,3,n} & f_{d,x,2,m} f_{r,x,4,n} \\ f_{d,x,3,m} f_{r,x,1,n} & f_{d,x,3,m} f_{r,x,2,n} & f_{d,x,3,m} f_{r,x,3,n} & f_{d,x,3,m} f_{r,x,4,n} \\ f_{d,x,4,m} f_{r,x,1,n} & f_{d,x,4,m} f_{r,x,2,n} & f_{d,x,4,m} f_{r,x,3,n} & f_{d,x,4,m} f_{r,x,4,n} \end{bmatrix}. \quad (53)$$

Because of the trigonometric identities for products of sine and cosine (the product identities), there are 16 deterministic-random, external eigenfunctions $f_{d,r,x,l,m,n}$, namely,

$$\begin{aligned} f_{d,r,x,1,m,n} &= \sin\left(\mathrm{A}_{d,r,x,1,m,n}\right), & f_{d,r,x,2,m,n} &= \cos\left(\mathrm{A}_{d,r,x,1,m,n}\right), \\ f_{d,r,x,3,m,n} &= \sin\left(\mathrm{A}_{d,r,x,2,m,n}\right), & f_{d,r,x,4,m,n} &= \cos\left(\mathrm{A}_{d,r,x,2,m,n}\right), \\ f_{d,r,x,5,m,n} &= \sin\left(\mathrm{A}_{d,r,x,3,m,n}\right), & f_{d,r,x,6,m,n} &= \cos\left(\mathrm{A}_{d,r,x,3,m,n}\right), \\ f_{d,r,x,7,m,n} &= \sin\left(\mathrm{A}_{d,r,x,4,m,n}\right), & f_{d,r,x,8,m,n} &= \cos\left(\mathrm{A}_{d,r,x,4,m,n}\right), \\ f_{d,r,x,9,m,n} &= \sin\left(\mathrm{A}_{d,r,x,5,m,n}\right), & f_{d,r,x,10,m,n} &= \cos\left(\mathrm{A}_{d,r,x,5,m,n}\right), \\ f_{d,r,x,11,m,n} &= \sin\left(\mathrm{A}_{d,r,x,6,m,n}\right), & f_{d,r,x,12,m,n} &= \cos\left(\mathrm{A}_{d,r,x,6,m,n}\right), \\ f_{d,r,x,13,m,n} &= \sin\left(\mathrm{A}_{d,r,x,7,m,n}\right), & f_{d,r,x,14,m,n} &= \cos\left(\mathrm{A}_{d,r,x,7,m,n}\right), \\ f_{d,r,x,15,m,n} &= \sin\left(\mathrm{A}_{d,r,x,8,m,n}\right), & f_{d,r,x,16,m,n} &= \cos\left(\mathrm{A}_{d,r,x,8,m,n}\right) \end{aligned} \quad (54)$$

of eight deterministic-random, external arguments $\mathrm{A}_{d,r,x,k,m,n}$, explicitly,

$$\begin{aligned} \mathrm{A}_{d,r,x,1,m,n} &= \kappa_{d,r,1,m,n} x + \kappa_{d,m} s_{d,y,\alpha,m} + \kappa_{r,n} s_{r,y,\alpha,n}, \\ \mathrm{A}_{d,r,x,2,m,n} &= \kappa_{d,r,2,m,n} x + \kappa_{d,m} s_{d,y,\alpha,m} - \kappa_{r,n} s_{r,y,\alpha,n}, \\ \mathrm{A}_{d,r,x,3,m,n} &= \kappa_{d,r,1,m,n} x + \kappa_{d,m} s_{d,y,\alpha,m} + \kappa_{r,n} s_{r,y,\beta,n}, \\ \mathrm{A}_{d,r,x,4,m,n} &= \kappa_{d,r,2,m,n} x + \kappa_{d,m} s_{d,y,\alpha,m} - \kappa_{r,n} s_{r,y,\beta,n}, \\ \mathrm{A}_{d,r,x,5,m,n} &= \kappa_{d,r,1,m,n} x + \kappa_{d,m} s_{d,y,\beta,m} + \kappa_{r,n} s_{r,y,\alpha,n}, \\ \mathrm{A}_{d,r,x,6,m,n} &= \kappa_{d,r,2,m,n} x + \kappa_{d,m} s_{d,y,\beta,m} - \kappa_{r,n} s_{r,y,\alpha,n}, \\ \mathrm{A}_{d,r,x,7,m,n} &= \kappa_{d,r,1,m,n} x + \kappa_{d,m} s_{d,y,\beta,m} + \kappa_{r,n} s_{r,y,\beta,n}, \\ \mathrm{A}_{d,r,x,8,m,n} &= \kappa_{d,r,2,m,n} x + \kappa_{d,m} s_{d,y,\beta,m} - \kappa_{r,n} s_{r,y,\beta,n} \end{aligned} \quad (55)$$

with two deterministic-random, external wavenumbers

$$\kappa_{d,r,1,m,n} = \kappa_{d,m} + \kappa_{r,n}, \quad \kappa_{d,r,2,m,n} = \kappa_{d,m} - \kappa_{r,n}. \quad (56)$$

Deterministic-random, external eigenfunctions $f_{d,r,x,l,m,n}$ include sine and cosine waves of all combinations of deterministic-random, external *x*-wave-numbers and *x*-shifts.

Computing matrix $\boldsymbol{f}_{d,x,i,m,r,x,j,n}$ yields the following Fourier expansions in $f_{d,r,x,l,m,n}$:

$$\begin{aligned} 2 f_{d,x,1,m} f_{r,x,1,n} &= - f_{d,r,x,2,m,n} + f_{d,r,x,4,m,n}, \\ 2 f_{d,x,1,m} f_{r,x,2,n} &= + f_{d,r,x,1,m,n} + f_{d,r,x,3,m,n}, \\ 2 f_{d,x,1,m} f_{r,x,3,n} &= - f_{d,r,x,6,m,n} + f_{d,r,x,8,m,n}, \\ 2 f_{d,x,1,m} f_{r,x,4,n} &= + f_{d,r,x,5,m,n} + f_{d,r,x,7,m,n}, \end{aligned}$$

$$\begin{aligned}
2\,f_{d,x,2,m}\,f_{r,x,1,n} &= +\,f_{d,r,x,1,m,n} - f_{d,r,x,3,m,n},\\
2\,f_{d,x,2,m}\,f_{r,x,2,n} &= +\,f_{d,r,x,2,m,n} + f_{d,r,x,4,m,n},\\
2\,f_{d,x,2,m}\,f_{r,x,3,n} &= +\,f_{d,r,x,5,m,n} - f_{d,r,x,7,m,n},\\
2\,f_{d,x,2,m}\,f_{r,x,4,n} &= +\,f_{d,r,x,6,m,n} + f_{d,r,x,8,m,n},\\
2\,f_{d,x,3,m}\,f_{r,x,1,n} &= -\,f_{d,r,x,10,m,n} + f_{d,r,x,12,m,n},\\
2\,f_{d,x,3,m}\,f_{r,x,2,n} &= +\,f_{d,r,x,9,m,n} + f_{d,r,x,11,m,n},\\
2\,f_{d,x,3,m}\,f_{r,x,3,n} &= -\,f_{d,r,x,14,m,n} + f_{d,r,x,16,m,n},\\
2\,f_{d,x,3,m}\,f_{r,x,4,n} &= +\,f_{d,r,x,13,m,n} + f_{d,r,x,15,m,n},\\
2\,f_{d,x,4,m}\,f_{r,x,1,n} &= +\,f_{d,r,x,9,m,n} - f_{d,r,x,11,m,n},\\
2\,f_{d,x,4,m}\,f_{r,x,2,n} &= +\,f_{d,r,x,10,m,n} + f_{d,r,x,12,m,n},\\
2\,f_{d,x,4,m}\,f_{r,x,3,n} &= +\,f_{d,r,x,13,m,n} - f_{d,r,x,15,m,n},\\
2\,f_{d,x,4,m}\,f_{r,x,4,n} &= +\,f_{d,r,x,14,m,n} + f_{d,r,x,16,m,n}.
\end{aligned} \tag{57}$$

2.3. Eigenfunctions of Random-Deterministic, External Interaction

Matrix $\boldsymbol{f}_{r,x,i,m,d,x,j,n}$ of the random-deterministic, external interaction $(n \neq m)$ in the x-direction between $f_{r,x,i,m}$ and $f_{d,x,j,n}$ (between the scaled rpe-x and dpe-x oscillons) becomes

$$\boldsymbol{f}_{r,x,i,m,d,x,j,n} = \begin{bmatrix} f_{r,x,1,m}\,f_{d,x,1,n} & f_{r,x,1,m}\,f_{d,x,2,n} & f_{r,x,1,m}\,f_{d,x,3,n} & f_{r,x,1,m}\,f_{d,x,4,n} \\ f_{r,x,2,m}\,f_{d,x,1,n} & f_{r,x,2,m}\,f_{d,x,2,n} & f_{r,x,2,m}\,f_{d,x,3,n} & f_{r,x,2,m}\,f_{d,x,4,n} \\ f_{r,x,3,m}\,f_{d,x,1,n} & f_{r,x,3,m}\,f_{d,x,2,n} & f_{r,x,3,m}\,f_{d,x,3,n} & f_{r,x,3,m}\,f_{d,x,4,n} \\ f_{r,x,4,m}\,f_{d,x,1,n} & f_{r,x,4,m}\,f_{d,x,2,n} & f_{r,x,4,m}\,f_{d,x,3,n} & f_{r,x,4,m}\,f_{d,x,4,n} \end{bmatrix}. \tag{58}$$

In view of the product identities, there are also 16 random-deterministic, external eigenfunctions $f_{r,d,x,l,m,n}$, specifically,

$$\begin{aligned}
f_{r,d,x,1,m,n} &= \sin\left(\mathrm{A}_{r,d,x,1,m,n}\right), & f_{r,d,x,2,m,n} &= \cos\left(\mathrm{A}_{r,d,x,1,m,n}\right),\\
f_{r,d,x,3,m,n} &= \sin\left(\mathrm{A}_{r,d,x,2,m,n}\right), & f_{r,d,x,4,m,n} &= \cos\left(\mathrm{A}_{r,d,x,2,m,n}\right),\\
f_{r,d,x,5,m,n} &= \sin\left(\mathrm{A}_{r,d,x,3,m,n}\right), & f_{r,d,x,6,m,n} &= \cos\left(\mathrm{A}_{r,d,x,3,m,n}\right),\\
f_{r,d,x,7,m,n} &= \sin\left(\mathrm{A}_{r,d,x,4,m,n}\right), & f_{r,d,x,8,m,n} &= \cos\left(\mathrm{A}_{r,d,x,4,m,n}\right),\\
f_{r,d,x,9,m,n} &= \sin\left(\mathrm{A}_{r,d,x,5,m,n}\right), & f_{r,d,x,10,m,n} &= \cos\left(\mathrm{A}_{r,d,x,5,m,n}\right),\\
f_{r,d,x,11,m,n} &= \sin\left(\mathrm{A}_{r,d,x,6,m,n}\right), & f_{r,d,x,12,m,n} &= \cos\left(\mathrm{A}_{r,d,x,6,m,n}\right),\\
f_{r,d,x,13,m,n} &= \sin\left(\mathrm{A}_{r,d,x,7,m,n}\right), & f_{r,d,x,14,m,n} &= \cos\left(\mathrm{A}_{r,d,x,7,m,n}\right),\\
f_{r,d,x,15,m,n} &= \sin\left(\mathrm{A}_{r,d,x,8,m,n}\right), & f_{r,d,x,16,m,n} &= \cos\left(\mathrm{A}_{r,d,x,8,m,n}\right)
\end{aligned} \tag{59}$$

of eight random-deterministic, external arguments $\mathrm{A}_{r,d,x,k,m,n}$, namely,

$$\begin{aligned}
\mathrm{A}_{r,d,x,1,m,n} &= \kappa_{r,d,1,m,n}x + \kappa_{r,m}s_{r,y,\alpha,m} + \kappa_{d,n}s_{d,y,\alpha,n},\\
\mathrm{A}_{r,d,x,2,m,n} &= \kappa_{r,d,2,m,n}x + \kappa_{r,m}s_{r,y,\alpha,m} - \kappa_{d,n}s_{d,y,\alpha,n},\\
\mathrm{A}_{r,d,x,3,m,n} &= \kappa_{r,d,1,m,n}x + \kappa_{r,m}s_{r,y,\alpha,m} + \kappa_{d,n}s_{d,y,\beta,n},\\
\mathrm{A}_{r,d,x,4,m,n} &= \kappa_{r,d,2,m,n}x + \kappa_{r,m}s_{r,y,\alpha,m} - \kappa_{d,n}s_{d,y,\beta,n},\\
\mathrm{A}_{r,d,x,5,m,n} &= \kappa_{r,d,1,m,n}x + \kappa_{r,m}s_{r,y,\beta,m} + \kappa_{d,n}s_{d,y,\alpha,n},\\
\mathrm{A}_{r,d,x,6,m,n} &= \kappa_{r,d,2,m,n}x + \kappa_{r,m}s_{r,y,\beta,m} - \kappa_{d,n}s_{d,y,\alpha,n},
\end{aligned}$$

$$\begin{aligned} \mathrm{A}_{r,d,x,7,m,n} &= \kappa_{r,d,1,m,n} x + \kappa_{r,m} s_{r,y,\beta,m} + \kappa_{d,n} s_{d,y,\beta,n}, \\ \mathrm{A}_{r,d,x,8,m,n} &= \kappa_{r,d,2,m,n} x + \kappa_{r,m} s_{r,y,\beta,m} - \kappa_{d,n} s_{d,y,\beta,n} \end{aligned} \tag{60}$$

with two random-deterministic, external wavenumbers

$$\kappa_{r,d,1,m,n} = \kappa_{r,m} + \kappa_{d,n}, \ \kappa_{r,d,r,2,m,n} = \kappa_{r,m} - \kappa_{d,n}. \tag{61}$$

Random-deterministic, external eigenfunctions $f_{r,d,x,l,m,n}$ include sine and cosine waves of all combinations of random-deterministic, external *x*-wavenumbers and *x*-shifts.

We compute matrix $\boldsymbol{f}_{r,x,i,m,d,x,j,n}$ in terms of $f_{r,d,x,l,m,n}$ as follows:

$$\begin{aligned} 2 f_{r,x,1,m} f_{d,x,1,n} &= - f_{r,d,x,2,m,n} + f_{r,d,x,4,m,n}, \\ 2 f_{r,x,1,m} f_{d,x,2,n} &= + f_{r,d,x,1,m,n} + f_{r,d,x,3,m,n}, \\ 2 f_{r,x,1,m} f_{d,x,3,n} &= - f_{r,d,x,6,m,n} + f_{r,d,x,8,m,n}, \\ 2 f_{r,x,1,m} f_{d,x,4,n} &= + f_{r,d,x,5,m,n} + f_{r,d,x,7,m,n}, \\ 2 f_{r,x,2,m} f_{d,x,1,n} &= + f_{r,d,x,1,m,n} - f_{r,d,x,3,m,n}, \\ 2 f_{r,x,2,m} f_{d,x,2,n} &= + f_{r,d,x,2,m,n} + f_{r,d,x,4,m,n}, \\ 2 f_{r,x,2,m} f_{d,x,3,n} &= + f_{r,d,x,5,m,n} - f_{r,d,x,7,m,n}, \\ 2 f_{r,x,2,m} f_{d,x,4,n} &= + f_{r,d,x,6,m,n} + f_{r,d,x,8,m,n}, \\ 2 f_{r,x,3,m} f_{d,x,1,n} &= - f_{r,d,x,10,m,n} + f_{r,d,x,12,m,n}, \\ 2 f_{r,x,3,m} f_{d,x,2,n} &= + f_{r,d,x,9,m,n} + f_{r,d,x,11,m,n}, \\ 2 f_{r,x,3,m} f_{d,x,3,n} &= - f_{r,d,x,14,m,n} + f_{r,d,x,16,m,n}, \\ 2 f_{r,x,3,m} f_{d,x,4,n} &= + f_{r,d,x,13,m,n} + f_{r,d,x,15,m,n}, \\ 2 f_{r,x,4,m} f_{d,x,1,n} &= + f_{r,d,x,9,m,n} - f_{r,d,x,11,m,n}, \\ 2 f_{r,x,4,m} f_{d,x,2,n} &= + f_{r,d,x,10,m,n} + f_{r,d,x,12,m,n}, \\ 2 f_{r,x,4,m} f_{d,x,3,n} &= + f_{r,d,x,13,m,n} - f_{r,d,x,15,m,n}, \\ 2 f_{r,x,4,m} f_{d,x,4,n} &= + f_{r,d,x,14,m,n} + f_{r,d,x,16,m,n}. \end{aligned} \tag{62}$$

It is a tedious, but a straightforward matter to show that deterministic-random, external eigenfunctions $f_{d,r,x,l,m,n}$ and random-deterministic, external eigenfunctions $f_{r,d,x,l,m,n}$ coincide up to the changing of index *l* and sign of eigenfunctions. Consequently, inhomogeneous Fourier expansions in either $f_{d,r,x,l,m,n}$ or $f_{r,d,x,l,m,n}$ are structurally invariant since they will differ only in order and signs of functional amplitudes.

2.4. Eigenfunctions of Deterministic-Random, Internal Interaction

If $n = m$, then two wavenumbers $\kappa_{d,r,1,m,n}$ and $\kappa_{d,r,2,m,n}$ of $f_{d,r,x,l,m,n}$ are transformed into two wavenumbers of the deterministic-random, internal interaction in the *x*-direction

$$\kappa_{d,r,1,m,m} = \kappa_{d,m} + \kappa_{r,m}, \ \kappa_{d,r,2,m,m} = \kappa_{d,m} - \kappa_{r,m}. \tag{63}$$

Therefore, we introduce the following eight deterministic-random internal arguments $\mathrm{B}_{d,r,x,k,m,m}$, explicitly,

$$
\begin{aligned}
\mathrm{B}_{d,r,x,1,m,m} &= \kappa_{d,r,1,m,m}x + \kappa_{d,m}s_{d,y,\alpha,m} + \kappa_{r,m}s_{r,y,\alpha,m},\\
\mathrm{B}_{d,r,x,2,m,m} &= \kappa_{d,r,2,m,m}x + \kappa_{d,m}s_{d,y,\alpha,m} - \kappa_{r,m}s_{r,y,\alpha,m},\\
\mathrm{B}_{d,r,x,3,m,m} &= \kappa_{d,r,1,m,m}x + \kappa_{d,m}s_{d,y,\alpha,m} + \kappa_{r,m}s_{r,y,\beta,m},\\
\mathrm{B}_{d,r,x,4,m,m} &= \kappa_{d,r,2,m,m}x + \kappa_{d,m}s_{d,y,\alpha,m} - \kappa_{r,m}s_{r,y,\beta,m},\\
\mathrm{B}_{d,r,x,5,m,m} &= \kappa_{d,r,1,m,m}x + \kappa_{d,m}s_{d,y,\beta,m} + \kappa_{r,m}s_{r,y,\alpha,m},\\
\mathrm{B}_{d,r,x,6,m,m} &= \kappa_{d,r,2,m,m}x + \kappa_{d,m}s_{d,y,\beta,m} - \kappa_{r,m}s_{r,y,\alpha,m},\\
\mathrm{B}_{d,r,x,7,m,m} &= \kappa_{d,r,1,m,m}x + \kappa_{d,m}s_{d,y,\beta,m} + \kappa_{r,m}s_{r,y,\beta,m},\\
\mathrm{B}_{d,r,x,8,m,m} &= \kappa_{d,r,2,m,m}x + \kappa_{d,m}s_{d,y,\beta,m} - \kappa_{r,m}s_{r,y,\beta,m}.
\end{aligned}
\tag{64}
$$

Consequently, 16 deterministic-random, internal eigenfunctions $g_{d,r,x,l,m,m}$ are defined as follows:

$$
\begin{aligned}
g_{d,r,x,1,m,m} &= \sin\left(\mathrm{B}_{d,r,x,1,m,m}\right), & g_{d,r,x,2,m,m} &= \cos\left(\mathrm{B}_{d,r,x,1,m,m}\right),\\
g_{d,r,x,3,m,m} &= \sin\left(\mathrm{B}_{d,r,x,2,m,m}\right), & g_{d,r,x,4,m,m} &= \cos\left(\mathrm{B}_{d,r,x,2,m,m}\right),\\
g_{d,r,x,5,m,m} &= \sin\left(\mathrm{B}_{d,r,x,3,m,m}\right), & g_{d,r,x,6,m,m} &= \cos\left(\mathrm{B}_{d,r,x,3,m,m}\right),\\
g_{d,r,x,7,m,m} &= \sin\left(\mathrm{B}_{d,r,x,4,m,m}\right), & g_{d,r,x,8,m,m} &= \cos\left(\mathrm{B}_{d,r,x,4,m,m}\right),\\
g_{d,r,x,9,m,m} &= \sin\left(\mathrm{B}_{d,r,x,5,m,m}\right), & g_{d,r,x,10,m,m} &= \cos\left(\mathrm{B}_{d,r,x,5,m,m}\right),\\
g_{d,r,x,11,m,m} &= \sin\left(\mathrm{B}_{d,r,x,6,m,m}\right), & g_{d,r,x,12,m,m} &= \cos\left(\mathrm{B}_{d,r,x,6,m,m}\right),\\
g_{d,r,x,13,m,m} &= \sin\left(\mathrm{B}_{d,r,x,7,m,m}\right), & g_{d,r,x,14,m,m} &= \cos\left(\mathrm{B}_{d,r,x,7,m,m}\right),\\
g_{d,r,x,15,m,m} &= \sin\left(\mathrm{B}_{d,r,x,8,m,m}\right), & g_{d,r,x,16,m,m} &= \cos\left(\mathrm{B}_{d,r,x,8,m,m}\right).
\end{aligned}
\tag{65}
$$

Matrix $\boldsymbol{f}_{d,x,i,m,r,x,j,m}$ of the deterministic-random, internal $(n=m)$ in the *x*-direction between $f_{d,x,i,m}$ and $f_{r,x,j,m}$ is not a symmetrical matrix, specifically,

$$
\boldsymbol{f}_{d,x,i,m,r,x,j,m} = \begin{bmatrix}
f_{d,x,1,m}f_{r,x,1,m} & f_{d,x,1,m}f_{r,x,2,m} & f_{d,x,1,m}f_{r,x,3,m} & f_{d,x,1,m}f_{r,x,4,m}\\
f_{d,x,2,m}f_{r,x,1,m} & f_{d,x,2,m}f_{r,x,2,m} & f_{d,x,2,m}f_{r,x,3,m} & f_{d,x,2,m}f_{r,x,4,m}\\
f_{d,x,3,m}f_{r,x,1,m} & f_{d,x,3,m}f_{r,x,2,m} & f_{d,x,3,m}f_{r,x,3,m} & f_{d,x,3,m}f_{r,x,4,m}\\
f_{d,x,4,m}f_{r,x,1,m} & f_{d,x,4,m}f_{r,x,2,m} & f_{d,x,4,m}f_{r,x,3,m} & f_{d,x,4,m}f_{r,x,4,m}
\end{bmatrix}. \tag{66}
$$

Computation of matrix $\boldsymbol{f}_{d,x,i,m,r,x,j,m}$ via $g_{d,r,x,l,m,m}$ returns

$$
\begin{aligned}
2f_{d,x,1,m}f_{r,x,1,m} &= -g_{d,r,x,2,m,m} + g_{d,r,x,4,m,m},\\
2f_{d,x,1,m}f_{r,x,2,m} &= +g_{d,r,x,1,m,m} + g_{d,r,x,3,m,m},\\
2f_{d,x,1,m}f_{r,x,3,m} &= -g_{d,r,x,6,m,m} + g_{d,r,x,8,m,m},\\
2f_{d,x,1,m}f_{r,x,4,m} &= +g_{d,r,x,5,m,m} + g_{d,r,x,7,m,m},\\
2f_{d,x,2,m}f_{r,x,1,m} &= +g_{d,r,x,1,m,m} - g_{d,r,x,3,m,m},\\
2f_{d,x,2,m}f_{r,x,2,m} &= +g_{d,r,x,2,m,m} + g_{d,r,x,4,m,m},\\
2f_{d,x,2,m}f_{r,x,3,m} &= +g_{d,r,x,5,m,m} - g_{d,r,x,7,m,m},\\
2f_{d,x,2,m}f_{r,x,4,m} &= +g_{d,r,x,6,m,m} + g_{d,r,x,8,m,m},\\
2f_{d,x,3,m}f_{r,x,1,m} &= -g_{d,r,x,10,m,m} + g_{d,r,x,12,m,m},\\
2f_{d,x,3,m}f_{r,x,2,m} &= +g_{d,r,x,9,m,m} + g_{d,r,x,11,m,m},\\
2f_{d,x,3,m}f_{r,x,3,m} &= -g_{d,r,x,14,m,m} + g_{d,r,x,16,m,m},\\
2f_{d,x,3,m}f_{r,x,4,m} &= +g_{d,r,x,13,m,m} + g_{d,r,x,15,m,m},
\end{aligned}
$$

$$
\begin{aligned}
2 f_{d,x,4,m} f_{r,x,1,m} &= +g_{d,r,x,9,m,m} - g_{d,r,x,11,m,m}, \\
2 f_{d,x,4,m} f_{r,x,2,m} &= +g_{d,r,x,10,m,m} + g_{d,r,x,12,m,m}, \\
2 f_{d,x,4,m} f_{r,x,3,m} &= +g_{d,r,x,13,m,m} - g_{d,r,x,15,m,m}, \\
2 f_{d,x,4,m} f_{r,x,4,m} &= +g_{d,r,x,14,m,m} + g_{d,r,x,16,m,m}.
\end{aligned}
\tag{67}
$$

It is also straightforward to show that deterministic-random, internal eigenfunctions $g_{d,r,x,l,m,m}$ and random-deterministic, internal eigenfunctions $g_{r,d,x,l,m,m}$ coincide up to the changing of index *l* and sign of eigenfunctions. Therefore, inhomogeneous Fourier decompositions in either $g_{d,r,x,l,m,m}$ or $g_{r,d,x,l,m,m}$ are structurally invariant since they will differ only in order and signs of functional amplitudes.

2.5. Eigenfunctions of Random, External Interaction

Matrix $\boldsymbol{f}_{r,x,i,m,r,x,j,n}$ of the random, external interaction $(n \neq m)$ in the *x*-direction between $f_{r,x,i,m}$ and $f_{r,x,j,n}$ (between scaled rpe-*x* oscillons) may be written as follows:

$$
\boldsymbol{f}_{r,x,i,m,r,x,j,n} = \begin{bmatrix}
f_{r,x,1,m} f_{r,x,1,n} & f_{r,x,1,m} f_{r,x,2,n} & f_{r,x,1,m} f_{r,x,3,n} & f_{r,x,1,m} f_{r,x,4,n} \\
f_{r,x,2,m} f_{r,x,1,n} & f_{r,x,2,m} f_{r,x,2,n} & f_{r,x,2,m} f_{r,x,3,n} & f_{r,x,2,m} f_{r,x,4,n} \\
f_{r,x,3,m} f_{r,x,1,n} & f_{r,x,3,m} f_{r,x,2,n} & f_{r,x,3,m} f_{r,x,3,n} & f_{r,x,3,m} f_{r,x,4,n} \\
f_{r,x,4,m} f_{r,x,1,n} & f_{r,x,4,m} f_{r,x,2,n} & f_{r,x,4,m} f_{r,x,3,n} & f_{r,x,4,m} f_{r,x,4,n}
\end{bmatrix}.
\tag{68}
$$

In agreement with the product identities, there are 16 random, external eigenfunctions $f_{r,x,l,m,n}$, specifically,

$$
\begin{aligned}
f_{r,x,1,m,n} &= \sin\left(\mathrm{A}_{r,x,1,m,n}\right), & f_{r,x,2,m,n} &= \cos\left(\mathrm{A}_{r,x,1,m,n}\right), \\
f_{r,x,3,m,n} &= \sin\left(\mathrm{A}_{r,x,2,m,n}\right), & f_{r,x,4,m,n} &= \cos\left(\mathrm{A}_{r,x,2,m,n}\right), \\
f_{r,x,5,m,n} &= \sin\left(\mathrm{A}_{r,x,3,m,n}\right), & f_{r,x,6,m,n} &= \cos\left(\mathrm{A}_{r,x,3,m,n}\right), \\
f_{r,x,7,m,n} &= \sin\left(\mathrm{A}_{r,x,4,m,n}\right), & f_{r,x,8,m,n} &= \cos\left(\mathrm{A}_{r,x,4,m,n}\right), \\
f_{r,x,9,m,n} &= \sin\left(\mathrm{A}_{r,x,5,m,n}\right), & f_{r,x,10,m,n} &= \cos\left(\mathrm{A}_{r,x,5,m,n}\right), \\
f_{r,x,11,m,n} &= \sin\left(\mathrm{A}_{r,x,6,m,n}\right), & f_{r,x,12,m,n} &= \cos\left(\mathrm{A}_{r,x,6,m,n}\right), \\
f_{r,x,13,m,n} &= \sin\left(\mathrm{A}_{r,x,7,m,n}\right), & f_{r,x,14,m,n} &= \cos\left(\mathrm{A}_{r,x,7,m,n}\right), \\
f_{r,x,15,m,n} &= \sin\left(\mathrm{A}_{r,x,8,m,n}\right), & f_{r,x,16,m,n} &= \cos\left(\mathrm{A}_{r,x,8,m,n}\right)
\end{aligned}
\tag{69}
$$

of eight random, external arguments $\mathrm{A}_{r,x,l,m,n}$, explicitly,

$$
\begin{aligned}
\mathrm{A}_{r,x,1,m,n} &= \kappa_{r,1,m,n} x + \kappa_{r,m} s_{r,y,\alpha,m} + \kappa_{r,n} s_{r,y,\alpha,n}, \\
\mathrm{A}_{r,x,2,m,n} &= \kappa_{r,2,m,n} x + \kappa_{r,m} s_{r,y,\alpha,m} - \kappa_{r,n} s_{r,y,\alpha,n}, \\
\mathrm{A}_{r,x,3,m,n} &= \kappa_{r,1,m,n} x + \kappa_{r,m} s_{r,y,\alpha,m} + \kappa_{r,n} s_{r,y,\beta,n}, \\
\mathrm{A}_{r,x,4,m,n} &= \kappa_{r,2,m,n} x + \kappa_{r,m} s_{r,y,\alpha,m} - \kappa_{r,n} s_{r,y,\beta,n}, \\
\mathrm{A}_{r,x,5,m,n} &= \kappa_{r,1,m,n} x + \kappa_{r,m} s_{r,y,\beta,m} + \kappa_{r,n} s_{r,y,\alpha,n}, \\
\mathrm{A}_{r,x,6,m,n} &= \kappa_{r,2,m,n} x + \kappa_{r,m} s_{r,y,\beta,m} - \kappa_{r,n} s_{r,y,\alpha,n}, \\
\mathrm{A}_{r,x,7,m,n} &= \kappa_{r,1,m,n} x + \kappa_{r,m} s_{r,y,\beta,m} + \kappa_{r,n} s_{r,y,\beta,n}, \\
\mathrm{A}_{r,x,8,m,n} &= \kappa_{r,2,m,n} x + \kappa_{r,m} s_{r,y,\beta,m} - \kappa_{r,n} s_{r,y,\beta,n},
\end{aligned}
\tag{70}
$$

which include two random, external wavenumbers

$$\kappa_{r,1,m,n} = \kappa_{r,m} + \kappa_{r,n}, \; \kappa_{r,2,m,n} = \kappa_{r,m} - \kappa_{r,n}. \tag{71}$$

Random, external eigenfunctions $f_{r,x,l,m,n}$ include sine and cosine waves of all combinations of random, external *x*-wavenumbers and *x*-shifts, as well.

Computation of matrix $\boldsymbol{f}_{r,x,i,m,r,x,j,n}$ returns the following expansions in $f_{r,x,l,m,n}$:

$$\begin{aligned}
&2 f_{r,x,1,m} f_{r,x,1,n} = - f_{r,x,2,m,n} + f_{r,x,4,m,n}, \; 2 f_{r,x,1,m} f_{r,x,2,n} = f_{r,x,1,m,n} + f_{r,x,3,m,n},\\
&2 f_{r,x,1,m} f_{r,x,3,n} = - f_{r,x,6,m,n} + f_{r,x,8,m,n}, \; 2 f_{r,x,1,m} f_{r,x,4,n} = f_{r,x,5,m,n} + f_{r,x,7,m,n},\\
&2 f_{r,x,2,m} f_{r,x,1,n} = + f_{r,x,1,m,n} - f_{r,x,3,m,n}, \; 2 f_{r,x,2,m} f_{r,x,2,n} = f_{r,x,2,m,n} + f_{r,x,4,m,n},\\
&2 f_{r,x,2,m} f_{r,x,3,n} = + f_{r,x,5,m,n} - f_{r,x,7,m,n}, \; 2 f_{r,x,2,m} f_{r,x,4,n} = f_{r,x,6,m,n} + f_{r,x,8,m,n},\\
&2 f_{r,x,3,m} f_{r,x,1,n} = - f_{r,x,10,m,n} + f_{r,x,12,m,n}, 2 f_{r,x,3,m} f_{r,x,2,n} = f_{r,x,9,m,n} + f_{r,x,11,m,n},\\
&2 f_{r,x,3,m} f_{r,x,3,n} = - f_{r,x,14,m,n} + f_{r,x,16,m,n}, 2 f_{r,x,3,m} f_{r,x,4,n} = f_{r,x,13,m,n} + f_{r,x,15,m,n},\\
&2 f_{r,x,4,m} f_{r,x,1,n} = + f_{r,x,9,m,n} - f_{r,x,11,m,n}, \; 2 f_{r,x,4,m} f_{r,x,2,n} = f_{r,x,10,m,n} + f_{r,x,12,m,n},\\
&2 f_{r,x,4,m} f_{r,x,3,n} = + f_{r,x,13,m,n} - f_{r,x,15,m,n}, 2 f_{r,x,4,m} f_{r,x,4,n} = f_{r,x,14,m,n} + f_{r,x,16,m,n}.
\end{aligned} \tag{72}$$

Replacing index r with d results in the deterministic, external eigenfunctions $f_{d,x,l,m,n}$, see (98)-(104) of [5].

2.6. Eigenfunctions of Random, Internal Interaction

If $n = m$, then two wavenumbers $\kappa_{r,1,m,n}$ and $\kappa_{r,2,m,n}$ of $f_{r,x,l,m,n}$ are reduced to a single wavenumber since

$$\kappa_{r,1,m,m} = 2\kappa_{r,m}, \; \kappa_{r,2,m,m} = 0. \tag{73}$$

Eight random, external arguments $\mathrm{A}_{r,x,k,m,n}$ are transformed into four arguments. Therefore, we define four random, internal arguments $\mathrm{B}_{r,x,k,m,m}$ as follows:

$$\begin{aligned}
&\mathrm{B}_{r,x,1,m,m} = 2\kappa_{r,m}\left(x + s_{r,y,\alpha,m}\right), \; \mathrm{B}_{r,x,2,m,m} = \kappa_{r,m}\left(2x + s_{r,y,\alpha,m} + s_{r,y,\beta,m}\right),\\
&\mathrm{B}_{r,x,3,m,m} = 2\kappa_{r,m}\left(x + s_{r,y,\beta,m}\right), \; \mathrm{B}_{r,x,4,m,m} = \kappa_{r,m}\left(s_{r,y,\alpha,m} - s_{r,y,\beta,m}\right).
\end{aligned} \tag{74}$$

Sixteen random, external eigenfunctions $f_{r,x,l,m,n}$ are converted into eight random, internal functions, whereas six random, internal eigenfunctions $g_{r,x,l,m,m}$ and two random shifts $h_{r,y,1,m,m}$ and $h_{r,y,2,m,m}$ are set in the following form:

$$\begin{aligned}
g_{r,x,1,m,m} &= \sin\left(\mathrm{B}_{r,x,1,m,m}\right), \; g_{r,x,2,m,m} = \cos\left(\mathrm{B}_{r,x,1,m,m}\right),\\
g_{r,x,3,m,m} &= \sin\left(\mathrm{B}_{r,x,2,m,m}\right), \; g_{r,x,4,m,m} = \cos\left(\mathrm{B}_{r,x,2,m,m}\right),\\
g_{r,x,5,m,m} &= \sin\left(\mathrm{B}_{r,x,3,m,m}\right), \; g_{r,x,6,m,m} = \cos\left(\mathrm{B}_{r,x,3,m,m}\right),\\
h_{r,y,1,m,m} &= \sin\left(\mathrm{B}_{r,x,4,m,m}\right), \; h_{r,y,2,m,m} = \cos\left(\mathrm{B}_{r,x,4,m,m}\right).
\end{aligned} \tag{75}$$

Matrix $\boldsymbol{f}_{r,x,i,m,r,x,j,m}$ of the random, internal interaction $(n = m)$ in the *x*-direction between $f_{r,x,i,m}$ and $f_{r,x,j,m}$ becomes a symmetrical one, explicitly,

$$\boldsymbol{f}_{r,x,i,m,r,x,j,m} = \begin{bmatrix}
f_{r,x,1,m}^2 & f_{r,x,1,m} f_{r,x,2,m} & f_{r,x,1,m} f_{r,x,3,m} & f_{r,x,1,m} f_{r,x,4,m}\\
f_{r,x,2,m} f_{r,x,1,m} & f_{r,x,2,m}^2 & f_{r,x,2,m} f_{r,x,3,m} & f_{r,x,2,m} f_{r,x,4,m}\\
f_{r,x,3,m} f_{r,x,1,m} & f_{r,x,3,m} f_{r,x,2,m} & f_{r,x,3,m}^2 & f_{r,x,3,m} f_{r,x,4,m}\\
f_{r,x,4,m} f_{r,x,1,m} & f_{r,x,4,m} f_{r,x,2,m} & f_{r,x,4,m} f_{r,x,3,m} & f_{r,x,4,m}^2
\end{bmatrix}. \tag{76}$$

We then compute the following expansions of the elements of matrix $\boldsymbol{f}_{r,x,i,m,r,x,j,m}$ in $g_{r,x,l,m,m}$:

$$\begin{aligned}
&2f^{2}_{r,x,1,m}=-g_{r,x,2,m,m}+1, && 2f_{r,x,1,m}f_{r,x,2,m}=g_{r,x,1,m,m},\\
&2f_{r,x,1,m}f_{r,x,3,m}=-g_{r,x,4,m,m}+h_{r,y,2,m,m}, && 2f_{r,x,1,m}f_{r,x,4,m}=g_{r,x,3,m,m}+h_{r,y,1,m,m},\\
&2f^{2}_{r,x,2,m}=+g_{r,x,2,m,m}+1, && \\
&2f_{r,x,2,m}f_{r,x,3,m}=+g_{r,x,3,m,m}-h_{r,y,1,m,m}, && 2f_{r,x,2,m}f_{r,x,4,m}=g_{r,x,4,m,m}+h_{r,y,2,m,m},\\
&2f^{2}_{r,x,3,m}=-g_{r,x,6,m,m}+1, && 2f_{r,x,3,m}f_{r,x,4,m}=g_{r,x,5,m,m},\\
&2f^{2}_{r,x,4,m}=+g_{r,x,6,m,m}+1. &&
\end{aligned}\tag{77}$$

Replacement of index r with d produces the deterministic, internal eigenfunctions $g_{r,x,l,m,m}$ see (105)-(111) of [5].

3. Oscillons of Deterministic-Random, External Interaction

3.1. The DREE Oscillons

Because of the identity resonance, 16 deterministic-random, external, elementary oscillons (dree oscillons for shortness, see (194) of [7]) are grouped into eight vector dree-x oscillons

$$\boldsymbol{K}_{d,r,e,e,x,q}=\boldsymbol{K}_{d,r,e,e,x,q}\left(\boldsymbol{f}_{d,r,e,e,x,q}\right),\quad q=1,2,\cdots,8,\tag{78}$$

which are formed by the following eight 2-tuples of the deterministic-random, external, elementary interaction in x:

$$\begin{aligned}
&\boldsymbol{f}_{d,r,e,e,x,1}=\left\{f_{d,r,x,1,m,n},\ f_{d,r,x,3,m,n}\right\}, && \boldsymbol{f}_{d,r,e,e,x,2}=\left\{f_{d,r,x,2,m,n},\ f_{d,r,x,4,m,n}\right\},\\
&\boldsymbol{f}_{d,r,e,e,x,3}=\left\{f_{d,r,x,5,m,n},\ f_{d,r,x,7,m,n}\right\}, && \boldsymbol{f}_{d,r,e,e,x,4}=\left\{f_{d,r,x,6,m,n},\ f_{d,r,x,8,m,n}\right\},\\
&\boldsymbol{f}_{d,r,e,e,x,5}=\left\{f_{d,r,x,9,m,n},\ f_{d,r,x,11,m,n}\right\}, && \boldsymbol{f}_{d,r,e,e,x,6}=\left\{f_{d,r,x,10,m,n},\ f_{d,r,x,12,m,n}\right\},\\
&\boldsymbol{f}_{d,r,e,e,x,7}=\left\{f_{d,r,x,13,m,n},\ f_{d,r,x,15,m,n}\right\}, && \boldsymbol{f}_{d,r,e,e,x,8}=\left\{f_{d,r,x,14,m,n},\ f_{d,r,x,16,m,n}\right\}.
\end{aligned}\tag{79}$$

Two-tuple $\boldsymbol{f}_{d,r,e,e,x,1}$ consists of two sine waves $f_{d,r,x,2k-1,m,n}$ and 2-tuple $\boldsymbol{f}_{d,r,e,e,x,2}$ of two cosine waves $f_{d,r,x,2k,m,n}$ for $k=1,2$. Two-tuple $\boldsymbol{f}_{d,r,e,e,x,3}$ comprises two sine waves $f_{d,r,x,2k-1,m,n}$ and 2-tuple $\boldsymbol{f}_{d,r,e,e,x,4}$ two cosine waves $f_{d,r,x,2k,m,n}$ for $k=3,4$. Two-tuple $\boldsymbol{f}_{d,r,e,e,x,5}$ is composed of two sine waves $f_{d,r,x,2k-1,m,n}$ and 2-tuple $\boldsymbol{f}_{d,r,e,e,x,6}$ of two cosine waves $f_{d,r,x,2k,m,n}$ for $k=5,6$. Two-tuple $\boldsymbol{f}_{d,r,e,e,x,7}$ is constructed of two sine waves $f_{d,r,x,2k-1,m,n}$ and 2-tuple $\boldsymbol{f}_{d,r,e,e,x,8}$ of two cosine waves $f_{d,r,x,2k,m,n}$ for $k=7,8$. All 2-tuples $\boldsymbol{f}_{d,r,e,e,x,q}$ are controlled by wavenumbers $\kappa_{d,r,1,m,n},\kappa_{d,r,2,m,n}$ for each m, n.

For any frozen $y=y_0,z=z_0,t=t_0$, application of matrix $\boldsymbol{f}_{d,x,i,m,r,x,j,n}$ yields that the 1st vector dree-x oscillon

$$\boldsymbol{K}_{d,r,e,e,x,1}=\left[K_{o,d,b,m,r,a,n},K_{o,d,a,m,r,b,n}\right]\left(\boldsymbol{f}_{d,r,e,e,x,1}\right)\tag{80}$$

is displayed by a list of two 2-w, deterministic-random, neutral oscillons in x, which are produced by 2-tuple $\boldsymbol{f}_{d,r,e,e,x,1}$ as

$$\begin{aligned}
K_{o,d,b,m,r,a,n}&=+\frac{\rho_c}{2}ez_{d,m}ez_{r,n}Q_{d,y,m}\,Q_{r,y,n}\left(f_{d,r,x,1,m,n}-f_{d,r,x,3,m,n}\right),\\
K_{o,d,a,m,r,b,n}&=+\frac{\rho_c}{2}ez_{d,m}ez_{r,n}\,Q_{d,y,m}\,Q_{r,y,n}\left(f_{d,r,x,1,m,n}+f_{d,r,x,3,m,n}\right).
\end{aligned}\tag{81}$$

The 2nd vector dree-*x* oscillon

$$\boldsymbol{K}_{d,r,e,e,x,2} = \left[K_{o,d,a,m,r,a,n}, K_{o,d,b,m,r,b,n}\right]\left(\boldsymbol{f}_{d,r,e,e,x,2}\right) \tag{82}$$

is presented by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which are generated by 2-tuple $\boldsymbol{f}_{d,r,e,e,x,2}$ since

$$\begin{aligned} K_{o,d,a,m,r,a,n} &= -\frac{\rho_c}{2} ez_{d,m} ez_{r,n} Q_{d,y,m} Q_{r,y,n}\left(f_{d,r,x,2,m,n} - f_{d,r,x,4,m,n}\right), \\ K_{o,d,b,m,r,b,n} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,n} Q_{d,y,m} Q_{r,y,n}\left(f_{d,r,x,2,m,n} + f_{d,r,x,4,m,n}\right). \end{aligned} \tag{83}$$

The 3rd vector dree-*x* oscillon

$$\boldsymbol{K}_{d,r,e,e,x,3} = \left[K_{o,d,b,m,r,c,n}, K_{o,d,a,m,r,d,n}\right]\left(\boldsymbol{f}_{d,r,e,e,x,3}\right) \tag{84}$$

is visualized by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which are formed by 2-tuple $\boldsymbol{f}_{d,r,e,e,x,3}$ because

$$\begin{aligned} K_{o,d,b,m,r,c,n} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,n} Q_{d,y,m} R_{r,y,n}\left(f_{d,r,x,5,m,n} - f_{d,r,x,7,m,n}\right), \\ K_{o,d,a,m,r,d,n} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,n} Q_{d,y,m} R_{r,y,n}\left(f_{d,r,x,5,m,n} + f_{d,r,x,7,m,n}\right). \end{aligned} \tag{85}$$

The 4th vector dree-*x* oscillon

$$\boldsymbol{K}_{d,r,e,e,x,4} = \left[K_{o,d,a,m,r,c,n}, K_{o,d,b,m,r,d,n}\right]\left(\boldsymbol{f}_{d,r,e,e,x,4}\right) \tag{86}$$

is represented by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which are established by 2-tuple $\boldsymbol{f}_{d,r,e,e,x,4}$ in the view of

$$\begin{aligned} K_{o,d,a,m,r,c,n} &= -\frac{\rho_c}{2} ez_{d,m} ez_{r,n} Q_{d,y,m} R_{r,y,n}\left(f_{d,r,x,6,m,n} - f_{d,r,x,8,m,n}\right), \\ K_{o,d,b,m,r,d,n} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,n} Q_{d,y,m} R_{r,y,n}\left(f_{d,r,x,6,m,n} + f_{d,r,x,8,m,n}\right). \end{aligned} \tag{87}$$

The 5th vector dree-*x* oscillon

$$\boldsymbol{K}_{d,r,e,e,x,5} = \left[K_{o,d,d,m,r,a,n}, K_{o,d,c,m,r,b,n}\right]\left(\boldsymbol{f}_{d,r,e,e,x,5}\right) \tag{88}$$

is exposed by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which are created by 2-tuple $\boldsymbol{f}_{d,r,e,e,x,5}$ since

$$\begin{aligned} K_{o,d,d,m,r,a,n} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,n} R_{d,y,m} Q_{r,y,n}\left(f_{d,r,x,9,m,n} - f_{d,r,x,11,m,n}\right), \\ K_{o,d,c,m,r,b,n} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,n} R_{d,y,m} Q_{r,y,n}\left(f_{d,r,x,9,m,n} + f_{d,r,x,11,m,n}\right). \end{aligned} \tag{89}$$

The 6th vector dree-*x* oscillon

$$\boldsymbol{K}_{d,r,e,e,x,6} = \left[K_{o,d,c,m,r,a,n}, K_{o,d,d,m,r,b,n}\right]\left(\boldsymbol{f}_{d,r,e,e,x,6}\right) \tag{90}$$

is exhibited by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which depend on 2-tuple $\boldsymbol{f}_{d,r,e,e,x,6}$ as

$$\begin{aligned} K_{o,d,c,m,r,a,n} &= -\frac{\rho_c}{2} ez_{d,m} ez_{r,n} R_{d,y,m} Q_{r,y,n}\left(f_{d,r,x,10,m,n} - f_{d,r,x,12,m,n}\right), \\ K_{o,d,d,m,r,b,n} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,n} R_{d,y,m} Q_{r,y,n}\left(f_{d,r,x,10,m,n} + f_{d,r,x,12,m,n}\right). \end{aligned} \tag{91}$$

The 7th vector dree-*x* oscillon

$$\boldsymbol{K}_{d,r,e,e,x,7}=\left[K_{o,d,d,m,r,c,n},K_{o,d,c,m,r,d,n}\right]\left(\boldsymbol{f}_{d,r,e,e,x,7}\right) \tag{92}$$

is given by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which are determined by 2-tuple $\boldsymbol{f}_{d,r,e,e,x,7}$ in the view of

$$\begin{aligned}K_{o,d,d,m,r,c,n}&=+\frac{\rho_c}{2}ez_{d,m}ez_{r,n}R_{d,y,m}R_{r,y,n}\left(f_{d,r,x,13,m,n}-f_{d,r,x,15,m,n}\right),\\K_{o,d,c,m,r,d,n}&=+\frac{\rho_c}{2}ez_{d,m}ez_{r,n}R_{d,y,m}R_{r,y,n}\left(f_{d,r,x,13,m,n}+f_{d,r,x,15,m,n}\right).\end{aligned} \tag{93}$$

The 8th vector dree-x oscillon

$$\boldsymbol{K}_{d,r,e,e,x,8}=\left[K_{o,d,c,m,r,c,n},K_{o,d,d,m,r,d,n}\right]\left(\boldsymbol{f}_{d,r,e,e,x,8}\right) \tag{94}$$

is envisioned by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which are governed by 2-tuple $\boldsymbol{f}_{d,r,e,e,x,8}$ because

$$\begin{aligned}K_{o,d,c,m,r,c,n}&=-\frac{\rho_c}{2}ez_{d,m}ez_{r,n}R_{d,y,m}R_{r,y,n}\left(f_{d,r,x,14,m,n}-f_{d,r,x,16,m,n}\right),\\K_{o,d,d,m,r,d,n}&=+\frac{\rho_c}{2}ez_{d,m}ez_{r,n}R_{d,y,m}R_{r,y,n}\left(f_{d,r,x,14,m,n}+f_{d,r,x,16,m,n}\right).\end{aligned} \tag{95}$$

For all vector dree-*x* oscillons, amplitudes of eigenfunctions are controlled by ρ_c,z_0,y_0,t_0 via $ez_{d,m},ez_{r,n},Q_{d,y,m},Q_{r,y,n},R_{d,y,m},R_{r,y,n}$ and *x*-shifts of eigenfunctions depend on y_0,t_0 through $s_{d,y,\alpha,m},s_{d,y,\beta,m},s_{r,y,\alpha,n},s_{r,y,\beta,n}$.

The *x*-periods $L_{d,r,x,1,m,n}$ and $L_{d,r,x,2,m,n}$ of $f_{d,r,x,l,m,n}$ for each $l=1,2,\cdots,16$ are

$$L_{d,r,x,1,m,n}=\frac{2\pi}{\kappa_{d,r,1,m,n}},\ L_{d,r,x,2,m,n}=\frac{2\pi}{\kappa_{d,r,2,m,n}}. \tag{96}$$

The wavelength of the dree-*x* oscillons is given by

$$L_{d,r,x,m,n}=\text{LCM}\left(L_{d,r,x,1,m,n},L_{d,r,x,2,m,n}\right)=k_{d,r,x,1,m,n}L_{d,r,x,1,m,n}=k_{d,r,x,2,m,n}L_{d,r,x,2,m,n}, \tag{97}$$

where $\text{LCM}(a,b)$ is a least common multiple of a and b, $k_{d,r,x,1,m,n}$ and $k_{d,r,x,2,m,n}$ are integers.

Because integrals of cosine waves and sine waves over $L_{d,r,x,1,m,n}$ and $L_{d,r,x,2,m,n}$ vanish, the average of the dree-*x* oscillons over $L_{d,r,x,m,n}$ vanishes, as well. Therefore, the vector dree-*x* oscillons are neutral since on average they do not transfer any kinetic energy along the *x*-axis.

3.2. The RDEE Oscillons

Due to the identity resonance, 16 random-deterministic, external, elementary oscillons (rdee oscillons for concision, see (195) of [7]) are combined into eight vector rdee-*x* oscillons

$$\boldsymbol{K}_{r,d,e,e,x,q}=\boldsymbol{K}_{r,d,e,e,x,q}\left(\boldsymbol{f}_{r,d,e,e,x,q}\right),\ q=1,2,\cdots,8, \tag{98}$$

which are produced by the following eight 2-tuples of the random-deterministic, external, elementary interaction in *x*:

$$\begin{aligned}
&\boldsymbol{f}_{r,d,e,e,x,1}=\left\{f_{r,d,x,1,m,n},\ f_{r,d,x,3,m,n}\right\},\quad \boldsymbol{f}_{r,d,e,e,x,2}=\left\{f_{r,d,x,2,m,n},\ f_{r,d,x,4,m,n}\right\},\\
&\boldsymbol{f}_{r,d,e,e,x,3}=\left\{f_{r,d,x,5,m,n},\ f_{r,d,x,7,m,n}\right\},\quad \boldsymbol{f}_{r,d,e,e,x,4}=\left\{f_{r,d,x,6,m,n},\ f_{r,d,x,8,m,n}\right\},\\
&\boldsymbol{f}_{r,d,e,e,x,5}=\left\{f_{r,d,x,9,m,n},\ f_{r,d,x,11,m,n}\right\},\quad \boldsymbol{f}_{r,d,e,e,x,6}=\left\{f_{r,d,x,10,m,n},\ f_{r,d,x,12,m,n}\right\},\\
&\boldsymbol{f}_{r,d,e,e,x,7}=\left\{f_{r,d,x,13,m,n},\ f_{r,d,x,15,m,n}\right\},\quad \boldsymbol{f}_{r,d,e,e,x,8}=\left\{f_{r,d,x,14,m,n},\ f_{r,d,x,16,m,n}\right\}.
\end{aligned}\tag{99}$$

Two-tuple $\boldsymbol{f}_{r,d,e,e,x,1}$ is constructed of two sine waves $f_{r,d,x,2k-1,m,n}$ and 2-tuple $\boldsymbol{f}_{r,d,e,e,x,2}$ of two cosine waves $f_{r,d,x,2k,m,n}$ for $k=1,2$. Two-tuple $\boldsymbol{f}_{r,d,e,e,x,3}$ is composed of two sine waves $f_{r,d,x,2k-1,m,n}$ and 2-tuple $\boldsymbol{f}_{r,d,e,e,x,4}$ of two cosine waves $f_{r,d,x,2k,m,n}$ for $k=3,4$. Two-tuple $\boldsymbol{f}_{r,d,e,e,x,5}$ comprises two sine waves $f_{r,d,x,2k-1,m,n}$ and 2-tuple $\boldsymbol{f}_{r,d,e,e,x,6}$ two cosine waves $f_{r,d,x,2k,m,n}$ for $k=5,6$. Two-tuple $\boldsymbol{f}_{r,d,e,e,x,7}$ consists of two sine waves $f_{r,d,x,2k-1,m,n}$ and 2-tuple $\boldsymbol{f}_{r,d,e,e,x,8}$ of two cosine waves $f_{r,d,x,2k,m,n}$ for $k=7,8$. All 2-tuples $\boldsymbol{f}_{r,d,e,e,x,q}$ depend on wavenumbers $\kappa_{r,d,1,m,n},\kappa_{r,d,2,m,n}$ for each *m*, *n*.

For any frozen $y=y_0, z=z_0, t=t_0$, we use matrix $\boldsymbol{f}_{r,x,i,m,d,x,j,n}$ to compute that the 1st vector rdee-*x* oscillon

$$\boldsymbol{K}_{r,d,e,e,x,1}=\left[K_{o,r,b,m,d,a,n},K_{o,r,a,m,d,b,n}\right]\left(\boldsymbol{f}_{r,d,e,e,x,1}\right)\tag{100}$$

is exposed by a list of two 2-w, random-deterministic, neutral oscillons in *x*, which depend on 2-tuple $\boldsymbol{f}_{r,d,e,e,x,1}$ in the view of

$$\begin{aligned}
K_{o,r,b,m,d,a,n}&=+\frac{\rho_c}{2}ez_{r,m}ez_{d,n}Q_{r,y,m}\,Q_{d,y,n}\left(f_{r,d,x,1,m,n}-f_{r,d,x,3,m,n}\right),\\
K_{o,r,a,m,d,b,n}&=+\frac{\rho_c}{2}ez_{r,m}ez_{d,n}\,Q_{r,y,m}\,Q_{d,y,n}\left(f_{r,d,x,1,m,n}+f_{r,d,x,3,m,n}\right).
\end{aligned}\tag{101}$$

The 2nd vector rdee-*x* oscillon

$$\boldsymbol{K}_{r,d,e,e,x,2}=\left[K_{o,r,a,m,d,a,n},K_{o,r,b,m,d,b,n}\right]\left(\boldsymbol{f}_{r,d,e,e,x,2}\right)\tag{102}$$

is exhibited by a list of two 2-w, random-deterministic, neutral oscillons in *x*, which are created by 2-tuple $\boldsymbol{f}_{r,d,e,e,x,2}$ because

$$\begin{aligned}
K_{o,r,a,m,d,a,n}&=-\frac{\rho_c}{2}ez_{r,m}ez_{d,n}Q_{r,y,m}\,Q_{d,y,n}\left(f_{r,d,x,2,m,n}-f_{r,d,x,4,m,n}\right),\\
K_{o,r,b,m,d,b,n}&=+\frac{\rho_c}{2}ez_{r,m}ez_{d,n}\,Q_{r,y,m}\,Q_{d,y,n}\left(f_{r,d,x,2,m,n}+f_{r,d,x,4,m,n}\right).
\end{aligned}\tag{103}$$

The 3rd vector rdee-*x* oscillon

$$\boldsymbol{K}_{r,d,e,e,x,3}=\left[K_{o,r,b,m,d,c,n},K_{o,r,a,m,d,d,n}\right]\left(\boldsymbol{f}_{r,d,e,e,x,3}\right)\tag{104}$$

is given by a list of two 2-w, random-deterministic, neutral oscillons in *x*, which are governed by 2-tuple $\boldsymbol{f}_{r,d,e,e,x,3}$ since

$$\begin{aligned}
K_{o,r,b,m,d,c,n}&=+\frac{\rho_c}{2}ez_{r,m}ez_{d,n}Q_{r,y,m}\,R_{d,y,n}\left(f_{r,d,x,5,m,n}-f_{r,d,x,7,m,n}\right),\\
K_{o,r,a,m,d,d,n}&=+\frac{\rho_c}{2}ez_{r,m}ez_{d,n}\,Q_{r,y,m}\,R_{d,y,n}\left(f_{r,d,x,5,m,n}+f_{r,d,x,7,m,n}\right).
\end{aligned}\tag{105}$$

The 4th vector rdee-*x* oscillon

$$\boldsymbol{K}_{r,d,e,e,x,4}=\left[K_{o,r,a,m,d,c,n},K_{o,r,b,m,d,d,n}\right]\left(\boldsymbol{f}_{r,d,e,e,x,4}\right)\tag{106}$$

is envisioned by a list of two 2-w, random-deterministic, neutral oscillons in *x*, which are determined by 2-tuple $\boldsymbol{f}_{r,d,e,e,x,4}$ as

$$\begin{aligned} K_{o,r,a,m,d,c,n} &= -\frac{\rho_c}{2}\, ez_{r,m} ez_{d,n} Q_{r,y,m}\, R_{d,y,n} \left(f_{r,d,x,6,m,n} - f_{r,d,x,8,m,n} \right), \\ K_{o,r,b,m,d,d,n} &= +\frac{\rho_c}{2} ez_{r,m} ez_{d,n}\, Q_{r,y,m}\, R_{d,y,n} \left(f_{r,d,x,6,m,n} + f_{r,d,x,8,m,n} \right). \end{aligned} \tag{107}$$

The 5th vector rdee-*x* oscillon

$$\boldsymbol{K}_{r,d,e,e,x,8} = \left[K_{o,r,d,m,d,a,n}, K_{o,r,c,m,d,b,n} \right] \left(\boldsymbol{f}_{r,d,e,e,x,8} \right) \tag{108}$$

is displayed by a list of two 2-w, random-deterministic, neutral oscillons in *x*, which are generated by 2-tuple $\boldsymbol{f}_{r,d,e,e,x,5}$ because

$$\begin{aligned} K_{o,r,d,m,d,a,n} &= +\frac{\rho_c}{2}\, ez_{r,m} ez_{d,n} R_{r,y,m}\, Q_{d,y,n} \left(f_{r,d,x,9,m,n} - f_{r,d,x,11,m,n} \right), \\ K_{o,r,c,m,d,b,n} &= +\frac{\rho_c}{2} ez_{r,m} ez_{d,n}\, R_{r,y,m}\, Q_{d,y,n} \left(f_{r,d,x,9,m,n} + f_{r,d,x,11,m,n} \right). \end{aligned} \tag{109}$$

The 6th vector rdee-*x* oscillon

$$\boldsymbol{K}_{r,d,e,e,x,6} = \left[K_{o,r,c,m,d,a,n}, K_{o,r,d,m,d,b,n} \right] \left(\boldsymbol{f}_{r,d,e,e,x,6} \right) \tag{110}$$

is presented by a list of two 2-w, random-deterministic, neutral oscillons in *x*, which are produced by 2-tuple $\boldsymbol{f}_{r,d,e,e,x,6}$ in the view of

$$\begin{aligned} K_{o,r,c,m,d,a,n} &= -\frac{\rho_c}{2}\, ez_{r,m} ez_{d,n} R_{r,y,m}\, Q_{d,y,n} \left(f_{r,d,x,10,m,n} - f_{r,d,x,12,m,n} \right), \\ K_{o,r,d,m,d,b,n} &= +\frac{\rho_c}{2} ez_{r,m} ez_{d,n}\, R_{r,y,m}\, Q_{d,y,n} \left(f_{r,d,x,10,m,n} + f_{r,d,x,12,m,n} \right). \end{aligned} \tag{111}$$

The 7th vector rdee-*x* oscillon

$$\boldsymbol{K}_{r,d,e,e,x,7} = \left[K_{o,r,d,m,d,c,n}, K_{o,r,c,m,d,d,n} \right] \left(\boldsymbol{f}_{r,d,e,e,x,7} \right) \tag{112}$$

is visualized by a list of two 2-w, random-deterministic, neutral oscillons in *x*, which are established by 2-tuple $\boldsymbol{f}_{r,d,e,e,x,7}$ as

$$\begin{aligned} K_{o,r,d,m,d,c,n} &= +\frac{\rho_c}{2}\, ez_{r,m} ez_{d,n} R_{r,y,m}\, R_{d,y,n} \left(f_{r,d,x,13,m,n} - f_{r,d,x,15,m,n} \right), \\ K_{o,r,c,m,d,d,n} &= +\frac{\rho_c}{2} ez_{r,m} ez_{d,n}\, R_{r,y,m}\, R_{d,y,n} \left(f_{r,d,x,13,m,n} + f_{r,d,x,15,m,n} \right). \end{aligned} \tag{113}$$

The 8th vector rdee-x oscillon

$$\boldsymbol{K}_{r,d,e,e,x,8} = \left[K_{o,r,c,m,d,c,n}, K_{o,r,d,m,d,d,n} \right] \left(\boldsymbol{f}_{r,d,e,e,x,8} \right) \tag{114}$$

is represented by a list of two 2-w, random-deterministic, neutral oscillons in *x*, which are formed by 2-tuple $\boldsymbol{f}_{r,d,e,e,x,8}$ since

$$\begin{aligned} K_{o,r,c,m,d,c,n} &= -\frac{\rho_c}{2}\, ez_{r,m} ez_{d,n} R_{r,y,m}\, R_{d,y,n} \left(f_{r,d,x,14,m,n} - f_{r,d,x,16,m,n} \right), \\ K_{o,r,d,m,d,d,n} &= +\frac{\rho_c}{2} ez_{r,m} ez_{d,n}\, R_{r,y,m}\, R_{d,y,n} \left(f_{r,d,x,14,m,n} + f_{r,d,x,16,m,n} \right). \end{aligned} \tag{115}$$

For all vector rdee-x oscillons, amplitudes of eigenfunctions are determined by ρ_c, z_0, y_0, t_0 via $ez_{r,m}, ez_{d,n}, Q_{r,y,m}, Q_{d,y,n}, R_{r,y,m}, R_{d,y,n}$ and *x*-shifts of eigenfunctions are governed by y_0, t_0 through $s_{r,y,\alpha,m}, s_{r,y,\beta,m}, s_{d,y,\alpha,n}, s_{d,y,\beta,n}$.

The *x*-periods $L_{r,d,1,x,m,n}$ and $L_{r,d,2,x,m,n}$ of $f_{r,d,x,l,m,n}$ for each $l = 1, 2, \cdots, 16$ are set by the following equations :

$$L_{r,d,x,1,m,n} = \frac{2\pi}{\kappa_{r,d,1,m,n}}, \; L_{r,d,x,2,m,n} = \frac{2\pi}{\kappa_{r,d,2,m,n}}. \tag{116}$$

The wavelength of the rdee-*x* oscillons is determined by

$$\begin{aligned} L_{r,d,x,m,n} = \operatorname{LCM}\left(L_{r,d,x,1,m,n}, L_{r,d,x,2,m,n}\right) &= k_{r,d,x,1,m,n} L_{r,d,x,1,m,n} \\ &= k_{r,d,x,2,m,n} L_{r,d,x,2,m,n}, \end{aligned} \tag{117}$$

where $k_{r,d,x,1,m,n}$ and $k_{r,d,x,2,m,n}$ are integers.

Since integrals of cosine waves and sine waves over $L_{r,d,x,1,m,n}$ and $L_{r,d,x,2,m,n}$ vanish, the average of the rdee-*x* oscillons over $L_{r,d,x,m,n}$ also vanishes. Consequently, the vector rdee-*x* oscillons are neutral since on average they do not transfer any kinetic energy along the *x*-axis.

3.3. The DREW Oscillons

We use the decomposition of deterministic-random, external, wave oscillons (drew oscillons for conciseness, see (197 of [7]) via the dree and rdee oscillons and substitute the inhomogeneous Fourier expansion of the vector dree-*x* and rdee-*x* oscillons to find that 16 drew-*x* oscillons are grouped into four vector drew-*x* oscillons

$$\boldsymbol{K}_{d,r,e,w,x,q} = \boldsymbol{K}_{d,r,e,w,x,q}\left(\boldsymbol{f}_{d,r,e,w,x,q}\right), \; q = 1, 2, 3, 4, \tag{118}$$

which are generated by four 8-tuples of the deterministic-random, external, wave interaction in *x*:

$$\begin{aligned} &\boldsymbol{f}_{d,r,e,w,x,1} = \left\{ f_{d,r,x,2k-1,m,n}, f_{r,d,x,2k-1,m,n} \right\}, \; \boldsymbol{f}_{d,r,e,w,x,2} = \left\{ f_{d,r,x,2k,m,n}, f_{r,d,x,2k,m,n} \right\}, \\ &\boldsymbol{f}_{d,r,e,w,x,3} = \left\{ f_{d,r,x,2k-1,m,n}, f_{r,d,x,2k-1,m,n} \right\}, \; \boldsymbol{f}_{d,r,e,w,x,4} = \left\{ f_{d,r,x,2k,m,n}, f_{r,d,x,2k,m,n} \right\}. \end{aligned} \tag{119}$$

Eight-tuple $\boldsymbol{f}_{d,r,e,w,x,1}$ is constructed of four sine waves $f_{d,r,x,2k-1,m,n}$ and four sine waves $f_{r,d,x,2k-1,m,n}$ and 8-tuple $\boldsymbol{f}_{d,r,e,w,x,2}$ consists of four cosine waves $f_{d,r,x,2k,m,n}$ and four cosine waves $f_{r,d,x,2k,m,n}$ for $k = 1, 2, 7, 8$. Eight-tuple $\boldsymbol{f}_{d,r,e,w,x,3}$ comprises four sine waves $f_{d,r,x,2k-1,m,n}$ and four sine waves $f_{r,d,x,2k-1,m,n}$ and 8-tuple $\boldsymbol{f}_{d,r,e,w,x,4}$ is composed of four cosine waves $f_{d,r,x,2k,m,n}$ and four cosine waves $f_{r,d,x,2k,m,n}$ for $k = 3, 4, 5, 6$. All 8-tuples $\boldsymbol{f}_{d,r,e,w,x,q}$ depend on wavenumbers $\kappa_{d,r,1,m,n}, \kappa_{d,r,2,m,n}$ and $\kappa_{r,d,1,m,n}, \kappa_{r,d,2,m,n}$ for each *m*, *n*.

Due to the identity and wavenumber resonances of the vector dree-*x* and rdee-*x* oscillons for any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector drew-*x* oscillon

$$\boldsymbol{K}_{d,r,e,w,x,1} = \left[K_{w,d,b,m,r,a,n}, K_{w,d,a,m,r,b,n}, K_{w,d,d,m,r,c,n}, K_{w,d,c,m,r,d,n} \right]\left(\boldsymbol{f}_{d,r,e,w,x,1} \right) \tag{120}$$

is visualized by a list of four 4-w, deterministic-random, random-deterministic, neutral oscillons in *x*, which are produced by 8-tuple $\boldsymbol{f}_{d,r,e,w,x,1}$ as

$$\begin{aligned}
K_{w,d,b,m,r,a,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{Q_{d,y,m}\,Q_{r,y,n} \\
&\times\big[\big(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,1,m,n}+\big(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,3,m,n}\big] \\
&- R_{d,y,m}\,R_{r,y,n}\lambda_{d,m}\lambda_{r,n}\big(f_{d,r,x,13,m,n}-f_{d,r,x,15,m,n}\big)\big\} \\
&\qquad + ez_{r,m}ez_{d,n}\big\{Q_{r,y,m}\,Q_{d,y,n} \\
&\times\big[\big(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,1,m,n}-\big(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,3,m,n}\big] \\
&- R_{r,y,m}\,R_{d,y,n}\lambda_{r,m}\lambda_{d,n}\big(f_{r,d,x,13,m,n}+f_{r,d,x,15,m,n}\big)\big\}\Big), \\[1ex]
K_{w,d,a,m,r,b,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{Q_{d,y,m}\,Q_{r,y,n} \\
&\times\big[\big(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,1,m,n}-\big(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,3,m,n}\big] \\
&- R_{d,y,m}\,R_{r,y,n}\lambda_{d,m}\lambda_{r,n}\big(f_{d,r,x,13,m,n}+f_{d,r,x,15,m,n}\big)\big\} \\
&\qquad + ez_{r,m}ez_{d,n}\big\{Q_{r,y,m}\,Q_{d,y,n} \\
&\times\big[\big(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,1,m,n}+\big(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,3,m,n}\big] \\
&- R_{r,y,m}\,R_{d,y,n}\lambda_{r,m}\lambda_{d,n}\big(f_{r,d,x,13,m,n}-f_{r,d,x,15,m,n}\big)\big\}\Big), \\[1ex]
K_{w,d,d,m,r,c,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{R_{d,y,m}\,R_{r,y,n} \\
&\times\big[\big(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,13,m,n}+\big(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,15,m,n}\big] \\
&- Q_{d,y,m}\,Q_{r,y,n}\lambda_{d,m}\lambda_{r,n}\big(f_{d,r,x,1,m,n}-f_{d,r,x,3,m,n}\big)\big\} \\
&\qquad + ez_{r,m}ez_{d,n}\big\{R_{r,y,m}\,R_{d,y,n} \\
&\times\big[\big(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,13,m,n}-\big(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,15,m,n}\big] \\
&- Q_{r,y,m}\,Q_{d,y,n}\lambda_{r,m}\lambda_{d,n}\big(f_{r,d,x,1,m,n}+f_{r,d,x,3,m,n}\big)\big\}\Big), \\[1ex]
K_{w,d,c,m,r,d,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{R_{d,y,m}\,R_{r,y,n} \\
&\times\big[\big(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,13,m,n}-\big(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,15,m,n}\big] \\
&- Q_{d,y,m}\,Q_{r,y,n}\lambda_{d,m}\lambda_{r,n}\big(f_{d,r,x,1,m,n}+f_{d,r,x,3,m,n}\big)\big\} \\
&\qquad + ez_{r,m}ez_{d,n}\big\{R_{r,y,m}\,R_{d,y,n} \\
&\times\big[\big(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,13,m,n}+\big(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,15,m,n}\big] \\
&- Q_{r,y,m}\,Q_{d,y,n}\lambda_{r,m}\lambda_{d,n}\big(f_{r,d,x,1,m,n}-f_{r,d,x,3,m,n}\big)\big\}\Big).
\end{aligned} \tag{121}$$

The 2nd vector drew-*x* oscillon

$$\boldsymbol{K}_{d,r,e,w,x,2} = \big[K_{w,d,a,m,r,a,n}, K_{w,d,b,m,r,b,n}, K_{w,d,c,m,r,c,n}, K_{w,d,d,m,r,d,n}\big]\big(\boldsymbol{f}_{d,r,e,w,x,2}\big) \tag{122}$$

is presented by a list of four 4-w, deterministic-random, random-deterministic, neutral oscillons in *x*, which are generated by 8-tuple $\boldsymbol{f}_{d,r,e,w,x,2}$ because

$$
\begin{aligned}
&K_{w,d,a,m,r,a,n} = +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{Q_{d,y,m}\,Q_{r,y,n}\\
&\times\big[\big(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,2,m,n}+\big(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,4,m,n}\big]\\
&-R_{d,y,m}\,R_{r,y,n}\lambda_{d,m}\lambda_{r,n}\big(f_{d,r,x,14,m,n}-f_{d,r,x,16,m,n}\big)\big\}\\
&\qquad\qquad+ez_{r,m}ez_{d,n}\big\{Q_{r,y,m}\,Q_{d,y,n}\\
&\times\big[\big(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,2,m,n}+\big(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,4,m,n}\big]\\
&-R_{r,y,m}\,R_{d,y,n}\lambda_{r,m}\lambda_{d,n}\big(f_{r,d,x,14,m,n}-f_{r,d,x,16,m,n}\big)\big\}\Big),\\[2ex]
&K_{w,d,b,m,r,b,n} = -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{Q_{d,y,m}\,Q_{r,y,n}\\
&\times\big[\big(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,2,m,n}-\big(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,4,m,n}\big]\\
&-R_{d,y,m}\,R_{r,y,n}\lambda_{d,m}\lambda_{r,n}\big(f_{d,r,x,14,m,n}+f_{d,r,x,16,m,n}\big)\big\}\\
&\qquad\qquad+ez_{r,m}ez_{d,n}\big\{Q_{r,y,m}\,Q_{d,y,n}\\
&\times\big[\big(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,2,m,n}-\big(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,4,m,n}\big]\\
&-R_{r,y,m}\,R_{d,y,n}\lambda_{r,m}\lambda_{d,n}\big(f_{r,d,x,14,m,n}+f_{r,d,x,16,m,n}\big)\big\}\Big),\\[2ex]
&K_{w,d,c,m,r,c,n} = +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{R_{d,y,m}\,R_{r,y,n}\\
&\times\big[\big(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,14,m,n}+\big(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,16,m,n}\big]\\
&-Q_{d,y,m}\,Q_{r,y,n}\lambda_{d,m}\lambda_{r,n}\big(f_{d,r,x,2,m,n}-f_{d,r,x,4,m,n}\big)\big\}\\
&\qquad\qquad+ez_{r,m}ez_{d,n}\big\{R_{r,y,m}\,R_{d,y,n}\\
&\times\big[\big(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,14,m,n}+\big(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,16,m,n}\big]\\
&-Q_{r,y,m}\,Q_{d,y,n}\lambda_{r,m}\lambda_{d,n}\big(f_{r,d,x,2,m,n}-f_{r,d,x,4,m,n}\big)\big\}\Big),\\[2ex]
&K_{w,d,d,m,r,d,n} = -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{R_{d,y,m}\,R_{r,y,n}\\
&\times\big[\big(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,14,m,n}-\big(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\big)f_{d,r,x,16,m,n}\big]\\
&-Q_{d,y,m}\,Q_{r,y,n}\lambda_{d,m}\lambda_{r,n}\big(f_{d,r,x,2,m,n}+f_{d,r,x,4,m,n}\big)\big\}\\
&\qquad\qquad+ez_{r,m}ez_{d,n}\big\{R_{r,y,m}\,R_{d,y,n}\\
&\times\big[\big(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,14,m,n}-\big(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\big)f_{r,d,x,16,m,n}\big]\\
&-Q_{r,y,m}\,Q_{d,y,n}\lambda_{r,m}\lambda_{d,n}\big(f_{r,d,x,2,m,n}+f_{r,d,x,4,m,n}\big)\big\}\Big).
\end{aligned}
\tag{123}
$$

The 3rd vector drew-x oscillon

$$
\boldsymbol{K}_{d,r,e,w,x,3}=\big[K_{w,d,b,m,r,c,n},K_{w,d,a,m,r,d,n},K_{w,d,d,m,r,a,n},K_{w,d,c,m,r,b,n}\big]\big(\boldsymbol{f}_{d,r,e,w,x,3}\big)\tag{124}
$$

is displayed by a list of four 4-w, deterministic-random, random-deterministic, neutral oscillons in x, which are formed by 8-tuple $\boldsymbol{f}_{d,r,e,w,x,3}$ since

$$\begin{aligned}
K_{w,d,b,m,r,c,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m} ez_{r,n} \big\{ Q_{d,y,m} R_{r,y,n} \\
&\times \big[\left(\kappa_{d,m}\kappa_{r,n} - \mu_{d,m}\mu_{r,n} \right) f_{d,r,x,5,m,n} + \left(\kappa_{d,m}\kappa_{r,n} + \mu_{d,m}\mu_{r,n} \right) f_{d,r,x,7,m,n} \big] \\
&+ R_{d,y,m} Q_{r,y,n} \lambda_{d,m}\lambda_{r,n} \left(f_{d,r,x,9,m,n} - f_{d,r,x,11,m,n} \right) \big\} \\
&\qquad + ez_{r,m} ez_{d,n} \big\{ R_{r,y,m} Q_{d,y,n} \\
&\times \big[\left(\kappa_{r,m}\kappa_{d,n} - \mu_{r,m}\mu_{d,n} \right) f_{r,d,x,9,m,n} - \left(\kappa_{r,m}\kappa_{d,n} + \mu_{r,m}\mu_{d,n} \right) f_{r,d,x,11,m,n} \big] \\
&+ Q_{r,y,m} R_{d,y,n} \lambda_{r,m}\lambda_{d,n} \left(f_{r,d,x,5,m,n} + f_{r,d,x,7,m,n} \right) \big\} \Big),
\end{aligned}$$

$$\begin{aligned}
K_{w,d,a,m,r,d,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m} ez_{r,n} \big\{ Q_{d,y,m} R_{r,y,n} \\
&\times \big[\left(\kappa_{d,m}\kappa_{r,n} - \mu_{d,m}\mu_{r,n} \right) f_{d,r,x,5,m,n} - \left(\kappa_{d,m}\kappa_{r,n} + \mu_{d,m}\mu_{r,n} \right) f_{d,r,x,7,m,n} \big] \\
&+ R_{d,y,m} Q_{r,y,n} \lambda_{d,m}\lambda_{r,n} \left(f_{d,r,x,9,m,n} + f_{d,r,x,11,m,n} \right) \big\} \\
&\qquad + ez_{r,m} ez_{d,n} \big\{ R_{r,y,m} Q_{d,y,n} \\
&\times \big[\left(\kappa_{r,m}\kappa_{d,n} - \mu_{r,m}\mu_{d,n} \right) f_{r,d,x,9,m,n} + \left(\kappa_{r,m}\kappa_{d,n} + \mu_{r,m}\mu_{d,n} \right) f_{r,d,x,11,m,n} \big] \\
&+ Q_{r,y,m} R_{d,y,n} \lambda_{r,m}\lambda_{d,n} \left(f_{r,d,x,5,m,n} - f_{r,d,x,7,m,n} \right) \big\} \Big),
\end{aligned}$$

$$\begin{aligned}
K_{w,d,d,m,r,a,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m} ez_{r,n} \big\{ R_{d,y,m} Q_{r,y,n} \\
&\times \big[\left(\kappa_{d,m}\kappa_{r,n} - \mu_{d,m}\mu_{r,n} \right) f_{d,r,x,9,m,n} + \left(\kappa_{d,m}\kappa_{r,n} + \mu_{d,m}\mu_{r,n} \right) f_{d,r,x,11,m,n} \big] \\
&+ Q_{d,y,m} R_{r,y,n} \lambda_{d,m}\lambda_{r,n} \left(f_{d,r,x,5,m,n} - f_{d,r,x,7,m,n} \right) \big\} \\
&\qquad + ez_{r,m} ez_{d,n} \big\{ Q_{r,y,m} R_{d,y,n} \\
&\times \big[\left(\kappa_{r,m}\kappa_{d,n} - \mu_{r,m}\mu_{d,n} \right) f_{r,d,x,5,m,n} - \left(\kappa_{r,m}\kappa_{d,n} + \mu_{r,m}\mu_{d,n} \right) f_{r,d,x,7,m,n} \big] \\
&+ R_{r,y,m} Q_{d,y,n} \lambda_{r,m}\lambda_{d,n} \left(f_{r,d,x,9,m,n} + f_{r,d,x,11,m,n} \right) \big\} \Big),
\end{aligned}$$

$$\begin{aligned}
K_{w,d,c,m,r,b,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m} ez_{r,n} \big\{ R_{d,y,m} Q_{r,y,n} \\
&\times \big[\left(\kappa_{d,m}\kappa_{r,n} - \mu_{d,m}\mu_{r,n} \right) f_{d,r,x,9,m,n} - \left(\kappa_{d,m}\kappa_{r,n} + \mu_{d,m}\mu_{r,n} \right) f_{d,r,x,11,m,n} \big] \\
&+ Q_{d,y,m} R_{r,y,n} \lambda_{d,m}\lambda_{r,n} \left(f_{d,r,x,5,m,n} + f_{d,r,x,7,m,n} \right) \big\} \\
&\qquad + ez_{r,m} ez_{d,n} \big\{ Q_{r,y,m} R_{d,y,n} \\
&\times \big[\left(\kappa_{r,m}\kappa_{d,n} - \mu_{r,m}\mu_{d,n} \right) f_{r,d,x,5,m,n} + \left(\kappa_{r,m}\kappa_{d,n} + \mu_{r,m}\mu_{d,n} \right) f_{r,d,x,7,m,n} \big] \\
&+ R_{r,y,m} Q_{d,y,n} \lambda_{r,m}\lambda_{d,n} \left(f_{r,d,x,9,m,n} - f_{r,d,x,11,m,n} \right) \big\} \Big).
\end{aligned} \tag{125}$$

The 4th vector drew-x oscillon

$$\boldsymbol{K}_{d,r,e,w,x,4} = \left[K_{w,d,a,m,r,c,n}, K_{w,d,b,m,r,d,n}, K_{w,d,c,m,r,a,n}, K_{w,d,d,m,r,b,n} \right]\left(\boldsymbol{f}_{d,r,e,w,x,4} \right) \tag{126}$$

is given by a list of four 4-w, deterministic-random, random-deterministic, neutral oscillons in x, which are established by 8-tuple $\boldsymbol{f}_{d,r,e,w,x,4}$ in the view of

$$
\begin{aligned}
K_{w,d,a,m,r,c,n} &= +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{Q_{d,y,m}\,R_{r,y,n}\\
&\times\big[\left(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\right)f_{d,r,x,6,m,n}+\left(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\right)f_{d,r,x,8,m,n}\big]\\
&+R_{d,y,m}\,Q_{r,y,n}\lambda_{d,m}\lambda_{r,n}\left(f_{d,r,x,10,m,n}-f_{d,r,x,12,m,n}\right)\big\}\\
&\qquad+ez_{r,m}ez_{d,n}\big\{R_{r,y,m}\,Q_{d,y,n}\\
&\times\big[\left(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\right)f_{r,d,x,10,m,n}+\left(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\right)f_{r,d,x,12,m,n}\big]\\
&+Q_{r,y,m}\,R_{d,y,n}\lambda_{r,m}\lambda_{d,n}\left(f_{r,d,x,6,m,n}-f_{r,d,x,8,m,n}\right)\big\}\Big),
\end{aligned}
$$

$$
\begin{aligned}
K_{w,d,b,m,r,d,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{Q_{d,y,m}\,R_{r,y,n}\\
&\times\big[\left(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\right)f_{d,r,x,6,m,n}-\left(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\right)f_{d,r,x,8,m,n}\big]\\
&+R_{d,y,m}\,Q_{r,y,n}\lambda_{d,m}\lambda_{r,n}\left(f_{d,r,x,10,m,n}+f_{d,r,x,12,m,n}\right)\big\}\\
&\qquad+ez_{r,m}ez_{d,n}\big\{R_{r,y,m}\,Q_{d,y,n}\\
&\times\big[\left(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\right)f_{r,d,x,10,m,n}-\left(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\right)f_{r,d,x,12,m,n}\big]\\
&+Q_{r,y,m}\,R_{d,y,n}\lambda_{r,m}\lambda_{d,n}\left(f_{r,d,x,6,m,n}+f_{r,d,x,8,m,n}\right)\big\}\Big),
\end{aligned}
$$

$$
\begin{aligned}
K_{w,d,c,m,r,a,n} &= +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{R_{d,y,m}\,Q_{r,y,n}\\
&\times\big[\left(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\right)f_{d,r,x,10,m,n}+\left(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\right)f_{d,r,x,12,m,n}\big]\\
&+Q_{d,y,m}\,R_{r,y,n}\lambda_{d,m}\lambda_{r,n}\left(f_{d,r,x,6,m,n}-f_{d,r,x,8,m,n}\right)\big\}\\
&\qquad+ez_{r,m}ez_{d,n}\big\{Q_{r,y,m}\,R_{d,y,n}\\
&\times\big[\left(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\right)f_{r,d,x,6,m,n}+\left(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\right)f_{r,d,x,8,m,n}\big]\\
&+R_{r,y,m}\,Q_{d,y,n}\lambda_{r,m}\lambda_{d,n}\left(f_{r,d,x,10,m,n}-f_{r,d,x,12,m,n}\right)\big\}\Big),
\end{aligned}
$$

$$
\begin{aligned}
K_{w,d,d,m,r,b,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,n}\big\{R_{d,y,m}\,Q_{r,y,n}\\
&\times\big[\left(\kappa_{d,m}\kappa_{r,n}-\mu_{d,m}\mu_{r,n}\right)f_{d,r,x,10,m,n}-\left(\kappa_{d,m}\kappa_{r,n}+\mu_{d,m}\mu_{r,n}\right)f_{d,r,x,12,m,n}\big]\\
&+Q_{d,y,m}\,R_{r,y,n}\lambda_{d,m}\lambda_{r,n}\left(f_{d,r,x,6,m,n}+f_{d,r,x,8,m,n}\right)\big\}\\
&\qquad+ez_{r,m}ez_{d,n}\big\{Q_{r,y,m}\,R_{d,y,n}\\
&\times\big[\left(\kappa_{r,m}\kappa_{d,n}-\mu_{r,m}\mu_{d,n}\right)f_{r,d,x,6,m,n}-\left(\kappa_{r,m}\kappa_{d,n}+\mu_{r,m}\mu_{d,n}\right)f_{r,d,x,8,m,n}\big]\\
&+R_{r,y,m}\,Q_{d,y,n}\lambda_{r,m}\lambda_{d,n}\left(f_{r,d,x,10,m,n}+f_{r,d,x,12,m,n}\right)\big\}\Big).
\end{aligned}
\tag{127}
$$

For all vector drew-x oscillons, amplitudes of eigenfunctions are influenced by

$$\rho_c,\kappa_{d,m},\kappa_{d,n},\lambda_{d,m},\lambda_{d,n},\mu_{d,m},\mu_{d,n},\kappa_{r,m},\kappa_{r,n},\lambda_{r,m},\lambda_{r,n},\mu_{r,m},\mu_{r,n},z_0,y_0,t_0 \tag{128}$$

via

$$ez_{d,m},ez_{d,n},ez_{r,m},ez_{r,n},Q_{d,y,m},Q_{d,y,n},R_{d,y,m},R_{d,y,n},Q_{r,y,m},Q_{r,y,n},R_{r,y,m},R_{r,y,n} \tag{129}$$

and x-shifts of eigenfunctions are controlled by y_0,t_0 through

$$s_{d,y,\alpha,m}, s_{d,y,\alpha,n}, s_{d,y,\beta,m}, s_{d,y,\beta,n}, s_{r,y,\alpha,m}, s_{r,y,\alpha,n}, s_{r,y,\beta,m}, s_{r,y,\beta,n}. \tag{130}$$

The wavelength of the drew-x oscillons is computed by

$$L_{t,d,r,x,m,n} = \mathrm{LCM}\left(L_{d,r,x,m,n}, L_{r,d,x,m,n}\right) = k_{d,r,x,m,n} L_{d,r,x,m,n} = k_{r,d,x,m,n} L_{r,d,x,m,n}, \tag{131}$$

where $k_{d,r,x,m,n}$ and $k_{r,d,x,m,n}$ are integers. Similar to the dree-x and rdee-x oscillons, the average of the drew-x oscillons over $L_{t,d,r,x,m,n}$ vanishes. The vector drew-x oscillons do not transfer any kinetic energy along the x-axis, as well.

3.4. The DREG Oscillon

The symmetry and wavenumber resonances of the dree-x and rdee-x oscillons result in reduction of the deterministic-random, external, group oscillon (the dreg-oscillon for briefness, see (198) of [7]) to a 4-w oscillon, which is produced by 16-tuple of the deterministic-random, external, group interaction in x

$$\boldsymbol{f}_{d,r,e,g,x} = \left\{ f_{d,r,x,4k-3,m,n}, f_{d,r,x,4k,m,n}, f_{r,d,x,4k-3,m,n}, f_{r,d,x,4k,m,n} \right\}, \quad k = 1,2,3,4. \tag{132}$$

Sixteen-tuple $\boldsymbol{f}_{d,r,e,g,x}$ includes four sine waves $f_{d,r,x,4k-3,m,n}$ with wavenumber $\kappa_{d,r,1,m,n}$, four cosine waves $f_{d,r,x,4k,m,n}$ with wavenumber $\kappa_{d,r,2,m,n}$, four sine waves $f_{r,d,x,4k-3,m,n}$ with wavenumber $\kappa_{r,d,1,m,n}$, and four cosine waves $f_{r,d,x,4k,m,n}$ with wavenumber $\kappa_{r,d,2,m,n}$ for $k = 1,2,3,4$ and each m, n.

Expressing the dreg oscillon via the dree and rdee oscillons and substituting the inhomogeneous Fourier expansion of the vector drie-x and rdie-x oscillons yields that the dreg-x oscillon for any frozen $y = y_0, z = z_0, t = t_0$ represents the 4-w, deterministic-random, random-deterministic, neutral oscillon in x, which is formed by 16-tuple $\boldsymbol{f}_{d,r,e,g,x}$,

$$K_{g,d,i,m,r,j,n} = K_{g,d,i,m,r,j,n}\left(\boldsymbol{f}_{d,r,e,g,x}\right) \tag{133}$$

because

$$\begin{aligned}
&K_{g,d,i,m,r,j,n} = \\
&\rho_c \Big\{ ez_{d,m} ez_{r,n} \Big[\Lambda_{d,m,r,n} \left(Q_{d,y,m} Q_{r,y,n} f_{d,r,x,1,m,n} + R_{d,y,m} R_{r,y,n} f_{d,r,x,13,m,n} \right) \\
&\qquad - \mathrm{N}_{d,m,r,n} \left(Q_{d,y,m} R_{r,y,n} f_{d,r,x,5,m,n} + R_{d,y,m} Q_{r,y,n} f_{d,r,x,9,m,n} \right) \\
&\qquad + \mathrm{M}_{d,m,r,n} \left(Q_{d,y,m} Q_{r,y,n} f_{d,r,x,4,m,n} + R_{d,y,m} R_{r,y,n} f_{d,r,x,16,m,n} \right) \\
&\qquad + \mathrm{K}_{d,m,r,n} \left(Q_{d,y,m} R_{r,y,n} f_{d,r,x,8,m,n} + R_{d,y,m} Q_{r,y,n} f_{d,r,x,12,m,n} \right) \Big] \\
&\quad + ez_{r,m} ez_{d,n} \Big[\Lambda_{r,m,d,n} \left(Q_{r,y,m} Q_{d,y,n} f_{r,d,x,1,m,n} + R_{r,y,m} R_{d,y,n} f_{r,d,x,13,m,n} \right) \\
&\qquad - \mathrm{N}_{r,m,d,n} \left(Q_{r,y,m} R_{d,y,n} f_{r,d,x,5,m,n} + R_{r,y,m} Q_{d,y,n} f_{r,d,x,9,m,n} \right) \\
&\qquad + \mathrm{M}_{r,m,d,n} \left(Q_{r,y,m} Q_{d,y,n} f_{r,d,x,4,m,n} + R_{r,y,m} R_{d,y,n} f_{r,d,x,16,m,n} \right) \\
&\qquad + \mathrm{K}_{r,m,d,n} \left(Q_{r,y,m} R_{d,y,n} f_{r,d,x,8,m,n} + R_{r,y,m} Q_{d,y,n} f_{r,d,x,12,m,n} \right) \Big] \Big\},
\end{aligned} \tag{134}$$

where nonlinear amplitudes

$$\begin{aligned}
\mathrm{K}_{d,m,r,n} &= +\kappa_{d,m}\kappa_{r,n} - \lambda_{d,m}\lambda_{r,n} + \mu_{d,m}\mu_{r,n}, \\
\Lambda_{d,m,r,n} &= -\kappa_{d,m}\kappa_{r,n} + \lambda_{d,m}\lambda_{r,n} + \mu_{d,m}\mu_{r,n}, \\
\mathrm{M}_{d,m,r,n} &= +\kappa_{d,m}\kappa_{r,n} + \lambda_{d,m}\lambda_{r,n} + \mu_{d,m}\mu_{r,n}, \\
\mathrm{N}_{d,m,r,n} &= +\kappa_{d,m}\kappa_{r,n} + \lambda_{d,m}\lambda_{r,n} - \mu_{d,m}\mu_{r,n}
\end{aligned} \tag{135}$$

and

$$\begin{aligned} \mathrm{K}_{r,m,d,n} &= +\kappa_{r,m}\kappa_{d,n} - \lambda_{r,m}\lambda_{d,n} + \mu_{r,m}\mu_{d,n}, \\ \Lambda_{r,m,d,n} &= -\kappa_{r,m}\kappa_{d,n} + \lambda_{r,m}\lambda_{d,n} + \mu_{r,m}\mu_{d,n}, \\ \mathrm{M}_{r,m,d,n} &= +\kappa_{r,m}\kappa_{d,n} + \lambda_{r,m}\lambda_{d,n} + \mu_{r,m}\mu_{d,n}, \\ \mathrm{N}_{r,m,d,n} &= +\kappa_{r,m}\kappa_{d,n} + \lambda_{r,m}\lambda_{d,n} - \mu_{r,m}\mu_{d,n} \end{aligned} \tag{136}$$

are generated by the deterministic and random wave numbers.

Amplitudes of eigenfunctions depend on

$$\begin{aligned} &\rho_c, \mathrm{K}_{d,m,r,n}, \Lambda_{d,m,r,n}, \mathrm{M}_{d,m,r,n}, \mathrm{N}_{d,m,r,n}, \\ &\quad \mathrm{K}_{r,m,d,n}, \Lambda_{r,m,d,n}, \mathrm{M}_{r,m,d,n}, \mathrm{N}_{r,m,d,n}, z_0, y_0, t_0 \end{aligned} \tag{137}$$

via (129) and *x*-shifts of eigenfunctions are determined by y_0, t_0 through (130). The wavelength of the dreg-*x* oscillon is also equal to $L_{t,d,r,x,m,n}$. The vanishing average of the dreg-*x* oscillon over $L_{t,d,r,x,m,n}$ demonstrates that the dreg-*x* oscillon is also neutral.

3.5. The DREK Oscillon

The deterministic-random, external, kinetic-energy oscillon (the drek oscillon for pithiness, see (92) of [7]) takes the following form:

$$K_{e,d,i,m,r,j,n} = \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} K_{g,d,i,m,r,j,n}. \tag{138}$$

If all wavenumbers of the drek-*x* oscillon are distinct, then the drek-*x* oscillon is displayed as a $2M(M-1)$-w oscillon. For any frozen $y = y_0, z = z_0, t = t_0$, the drek-*x* oscillon is converted into the $2M(M-1)$-w, deterministic-random, random-deterministic, neutral oscillon in *x*, which is formed by $M(M-1)/2$ 16-tuples $\boldsymbol{f}_{d,r,e,g,x}$ with wavenumbers $\kappa_{d,r,1,m,n}, \kappa_{d,r,2,m,n}, \kappa_{r,d,1,m,n}, \kappa_{r,d,2,m,n}$ for all *m*, *n*, *Re*, and wave parameters of the dreg-*x* oscillons.

The wavelength of the drek-*x* oscillon is given by

$$L_{e,d,r,x,e} = \mathrm{LCM}\left(L_{t,d,r,x,m,n}\right) = k_{t,d,r,x,m,n} L_{t,d,r,x,m,n}, \tag{139}$$

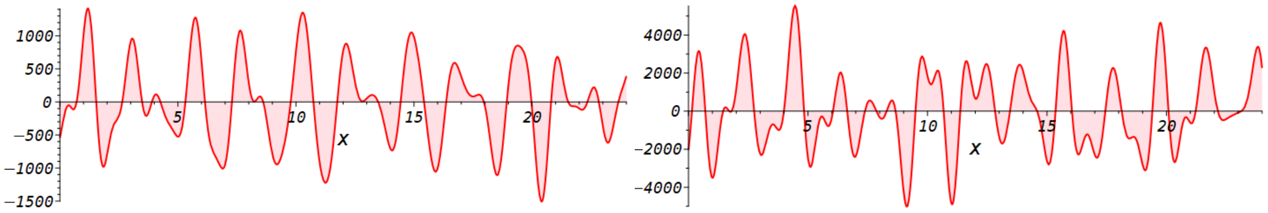
where $k_{t,d,r,x,m,n}$ is an integer. Alike the dreg-*x* oscillon, the vanishing average of the drek-*x* oscillon over $L_{e,d,r,x,e}$ demonstrates that the drek-*x* oscillon on average is also neutral.

The drek-*x* oscillons for $y = y_0, z = z_0, t = t_0$, the Reynolds number $Re = 10^3$, and $Re = 10^5$ are shown in **Figure 1** on wavelength $L_{e,d,x,e} = 24$ of the dek-*x* oscillon. We observe a substantial growth of the range of the 12-w, deterministic-random, random-deterministic, neutral drek-*x* oscillon with *Re*.

In **Figure 1** and sequential figures, the independent deterministic parameters are the same as in [5], namely,

$$\begin{aligned} &\rho_c = 1, \quad M = 3, \quad \eta = 0, \quad x_0 = 0, \quad y_0 = 0, \quad z_0 = 0, \quad t_0 = 5, \\ &\kappa_{d,1} = \frac{\pi}{4}, \ \kappa_{d,2} = \frac{2\pi}{3}, \ \kappa_{d,3} = \frac{3\pi}{2}, \ \lambda_{d,1} = \frac{2\pi}{9}, \ \lambda_{d,2} = \frac{2\pi}{3}, \ \lambda_{d,3} = 2\pi, \\ &U_{d,1} = 4, \quad U_{d,2} = 3, \quad U_{d,3} = 2, \quad V_{d,1} = 3, \quad V_{d,2} = 2, \quad V_{d,3} = 1, \end{aligned}$$

$$\begin{aligned}&X_{d,1,0}=7, X_{d,2,0}=5,\ \ X_{d,3,0}=3,\ \ Y_{d,1,0}=5,\ \ \ Y_{d,2,0}=3,\ \ Y_{d,3,0}=1,\\ &Av_{d,1}=11, Av_{d,2}=7,\ \ Av_{d,3}=3,\ \ Bv_{d,1}=6,\ \ Bv_{d,2}=4,\ \ Bv_{d,3}=2,\\ &Cv_{d,1}=\ 8, Cv_{d,2}=6,\ \ Cv_{d,3}=4,\ \ Dv_{d,1}=7,\ \ Dv_{d,2}=4,\ \ Dv_{d,3}=1.\end{aligned} \tag{140}$$



(a)

(b)

Figure 1. The drek-x oscillons: (a) $-K_{e,d,i,m,r,j,n}(x)$ for $Re=10^3$, (b) $-K_{e,d,i,m,r,j,n}(x)$ for $Re=10^5$.

Using the random oscillatory cn-noise [8], the independent random parameters are computed by

$$\begin{aligned}
U_{r,m}&=S_U(Re)\sum_{l=1}^{5}A_{U,m,l}\operatorname{cn}\left(S_f(Re)\upsilon_{U,m,l}t_0-\mathrm{K}(\varepsilon),\varepsilon\right),\\
V_{r,m}&=S_V(Re)\sum_{l=1}^{5}A_{V,m,l}\operatorname{cn}\left(S_f(Re)\upsilon_{V,m,l}t_0-\mathrm{K}(\varepsilon),\varepsilon\right),\\
X_{r,m,0}&=S_X(Re)\sum_{l=1}^{5}A_{X,m,l}\operatorname{cn}\left(S_f(Re)\upsilon_{X,m,l}t_0-\mathrm{K}(\varepsilon),\varepsilon\right),\\
Y_{r,m,0}&=S_Y(Re)\sum_{l=1}^{5}A_{Y,m,l}\operatorname{cn}\left(S_f(Re)\upsilon_{Y,m,l}t_0-\mathrm{K}(\varepsilon),\varepsilon\right),\\
Av_{r,m}&=S_A(Re)\sum_{l=1}^{5}A_{A,m,l}\operatorname{cn}\left(S_f(Re)\upsilon_{A,m,l}t_0-\mathrm{K}(\varepsilon),\varepsilon\right),\\
Bv_{r,m}&=S_A(Re)\sum_{l=1}^{5}A_{B,m,l}\operatorname{cn}\left(S_f(Re)\upsilon_{B,m,l}t_0-\mathrm{K}(\varepsilon),\varepsilon\right),\\
Cv_{r,m}&=S_A(Re)\sum_{l=1}^{5}A_{C,m,l}\operatorname{cn}\left(S_f(Re)\upsilon_{C,m,l}t_0-\mathrm{K}(\varepsilon),\varepsilon\right),\\
Dv_{r,m}&=S_A(Re)\sum_{l=1}^{5}A_{D,m,l}\operatorname{cn}\left(S_f(Re)\upsilon_{D,m,l}t_0-\mathrm{K}(\varepsilon),\varepsilon\right),\\
\kappa_{r,m}&=S_x(Re)\upsilon_{\kappa,m},\ \ \lambda_{r,m}=S_y(Re)\upsilon_{\lambda,m},
\end{aligned} \tag{141}$$

where $\operatorname{cn}(t,\varepsilon)$ is the elliptic cosine, $\varepsilon=0.9999$ is the elliptic modulus, $\mathrm{K}(\varepsilon)=5.64514827$ is the complete elliptic integral of the first kind,

$$A_{U,m,l}, A_{V,m,l}, A_{X,m,l}, A_{Y,m,l}, A_{A,m,l}, A_{B,m,l}, A_{C,m,l}, A_{D,m,l} \tag{142}$$

are random amplitudes selected from a list of 120 random numbers on [0, 1],

$$\upsilon_{U,m,l}, \upsilon_{V,m,l}, \upsilon_{X,m,l}, \upsilon_{Y,m,l}, \upsilon_{A,m,l}, \upsilon_{B,m,l}, \upsilon_{C,m,l}, \upsilon_{D,m,l}, \upsilon_{\kappa,m}, \upsilon_{\lambda,m} \tag{143}$$

are random frequencies chosen from a list of reciprocals of first 126 prime numbers mixed up randomly,

$$\begin{aligned}
&S_x(Re)=696.7909371\,Re^{0.01952026874},\ \ S_y(Re)=958.7435364\,Re^{0.04994726271},\\
&S_f(Re)=5529.843495\,Re^{0.007252001036},\ \ S_A(Re)=0.6787933393\,Re^{0.1215190243},\\
&S_U(Re)=0.5303300858\,Re^{0.1505149978},\ \ S_V(Re)=0.8838834764\,Re^{0.1505149978},\\
&S_X(Re)=1.237436866\,Re^{0.1505149978},\ \ S_Y(Re)=1.590990258\,Re^{0.1505149978}
\end{aligned} \tag{144}$$

are scales of the random parameters.

For $Re = 10^3$, the independent random parameters have the following values:

$$\begin{array}{lll}
\kappa_{r,1} = \dfrac{797}{383}, & \kappa_{r,2} = \dfrac{797}{599}, & \kappa_{r,3} = \dfrac{797}{163}, \\
\lambda_{r,1} = \dfrac{1353}{467}, & \lambda_{r,2} = \dfrac{1353}{349}, & \lambda_{r,3} = \dfrac{1353}{317}, \\
U_{r,1} = -0.15394, & U_{r,2} = -0.84986, & U_{r,3} = -0.11692 \\
V_{r,1} = -0.041219, & V_{r,2} = -0.030365, & V_{r,3} = +0.21192, \\
X_{r,1,0} = +1.9894, & X_{r,2,0} = -2.5902, & X_{r,3,0} = -1.4556, \\
Y_{r,1,0} = -2.5835, & Y_{r,2,0} = +0.96785, & Y_{r,3,0} = +1.1671, \\
Av_{r,1} = -1.1359, & Av_{r,2} = +0.0089523, & Av_{r,3} = -1.1091, \\
Bv_{r,1} = -0.17456, & Bv_{r,2} = +0.29824, & Bv_{r,3} = +0.65630, \\
Cv_{r,1} = +0.53216, & Cv_{r,2} = +0.49016, & Cv_{r,3} = -0.21265, \\
Dv_{r,1} = +0.69036, & Dv_{r,2} = -0.042070, & Dv_{r,3} = +0.51550.
\end{array} \tag{145}$$

For $Re = 10^5$, the independent random parameters become

$$\begin{array}{lll}
\kappa_{r,1} = \dfrac{872}{383}, & \kappa_{r,2} = \dfrac{872}{599}, & \kappa_{r,3} = \dfrac{872}{163}, \\
\lambda_{r,1} = \dfrac{1703}{467}, & \lambda_{r,2} = \dfrac{1703}{349}, & \lambda_{r,3} = \dfrac{1703}{317}, \\
U_{r,1} = -1.3184, & U_{r,2} = -0.045891, & U_{r,3} = +2.8227, \\
V_{r,1} = +0.0036312, & V_{r,2} = -5.5983, & V_{r,3} = +0.38011, \\
X_{r,1,0} = +0.48879, & X_{r,2,0} = -0.18596, & X_{r,3,0} = -5.3347, \\
Y_{r,1,0} = -6.9272, & Y_{r,2,0} = +0.12119, & Y_{r,3,0} = +5.7784, \\
Av_{r,1} = +2.0121, & Av_{r,2} = -1.7599, & Av_{r,3} = +1.8337, \\
Bv_{r,1} = +0.68324, & Bv_{r,2} = +0.46434, & Bv_{r,3} = +0.74922, \\
Cv_{r,1} = +0.16767, & Cv_{r,2} = +3.8844, & Cv_{r,3} = +0.70078, \\
Dv_{r,1} = -0.010623, & Dv_{r,2} = +2.0548, & Dv_{r,3} = +2.2215.
\end{array} \tag{146}$$

Maple programs for computation of spatial quantization in the x-eigenfunctions will be published elsewhere.

4. Oscillons of Deterministic-Random, Internal Interaction

4.1. The DRIE Oscillons

In the view of the identity resonance, 16 deterministic-random, internal, elementary oscillons (the drie oscillons for briefness, see (188) of [7]) are arranged into eight vector drie-x oscillons:

$$\boldsymbol{K}_{d,r,i,e,x,q} = \boldsymbol{K}_{d,r,i,e,x,q}\left(\boldsymbol{f}_{d,r,i,e,x,q}\right), \ q = 1, 2, \cdots, 8, \tag{147}$$

which are formed by eight 2-tuples of the deterministic-random, internal, elementary interaction in x:

$$\begin{aligned}
&\boldsymbol{f}_{d,r,i,e,x,1}=\left\{g_{d,r,x,1,m,m},\ g_{d,r,x,3,m,m}\right\},\quad \boldsymbol{f}_{d,r,i,e,x,2}=\left\{g_{d,r,x,2,m,m},\ g_{d,r,x,4,m,m}\right\},\\
&\boldsymbol{f}_{d,r,i,e,x,3}=\left\{g_{d,r,x,5,m,m},\ g_{d,r,x,7,m,m}\right\},\quad \boldsymbol{f}_{d,r,i,e,x,4}=\left\{g_{d,r,x,6,m,m},\ g_{d,r,x,8,m,m}\right\},\\
&\boldsymbol{f}_{d,r,i,e,x,5}=\left\{g_{d,r,x,9,m,m},\ g_{d,r,x,11,m,m}\right\},\quad \boldsymbol{f}_{d,r,i,e,x,6}=\left\{g_{d,r,x,10,m,m},g_{d,r,x,12,m,m}\right\},\\
&\boldsymbol{f}_{d,r,i,e,x,7}=\left\{g_{d,r,x,13,m,m},g_{d,r,x,15,m,m}\right\},\quad \boldsymbol{f}_{d,r,i,e,x,8}=\left\{g_{d,r,x,14,m,m},g_{d,r,x,16,m,m}\right\}.
\end{aligned} \tag{148}$$

Two-tuple $\boldsymbol{f}_{d,r,i,e,x,1}$ consists of two sine waves $g_{d,r,x,2k-1,m,m}$ and 2-tuple $\boldsymbol{f}_{d,r,i,e,x,2}$ of two cosine waves $g_{d,r,x,2k,m,m}$ for $k=1,2$. Two-tuple $\boldsymbol{f}_{d,r,i,e,x,3}$ comprises two sine waves $g_{d,r,x,2k-1,m,m}$ and 2-tuple $\boldsymbol{f}_{d,r,i,e,x,4}$ two cosine waves $g_{d,r,x,2k,m,m}$ for $k=3,4$. Two-tuple $\boldsymbol{f}_{d,r,i,e,x,5}$ is composed of two sine waves $g_{d,r,x,2k-1,m,m}$ and 2-tuple $\boldsymbol{f}_{d,r,i,e,x,6}$ of two cosine waves $g_{d,r,x,2k,m,m}$ for $k=5,6$. Two-tuple $\boldsymbol{f}_{d,r,i,e,x,7}$ is constructed of two sine waves $g_{d,r,x,2k-1,m,m}$ and 2-tuple $\boldsymbol{f}_{d,r,i,e,x,8}$ of two cosine waves $g_{d,r,x,2k,m,m}$ for $k=7,8$. All 2-tuples $\boldsymbol{f}_{d,r,i,e,x,q}$ are parametrized by wavenumbers $\kappa_{d,r,1,m,m},\kappa_{d,r,2,m,m}$ for each *m*.

For any frozen $y=y_0,z=z_0,t=t_0$, usage of matrix $\boldsymbol{f}_{d,x,i,m,r,x,j,m}$ yields that the 1st vector drie-*x* oscillon

$$\boldsymbol{K}_{d,r,i,e,x,1}=\left[K_{o,d,b,m,r,a,m},K_{o,d,a,m,r,b,m}\right]\left(\boldsymbol{f}_{d,r,i,e,x,1}\right) \tag{149}$$

is exposed by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which depend on 2-tuple $\boldsymbol{f}_{d,r,i,e,x,1}$ since

$$\begin{aligned}
K_{o,d,b,m,r,a,m}&=+\frac{\rho_c}{2}ez_{d,m}ez_{r,m}Q_{d,y,m}\,Q_{r,y,m}\left(g_{d,r,x,1,m,m}-g_{d,r,x,3,m,m}\right),\\
K_{o,d,a,m,r,b,m}&=+\frac{\rho_c}{2}ez_{d,m}ez_{r,m}\,Q_{d,y,m}\,Q_{r,y,m}\left(g_{d,r,x,1,m,m}+g_{d,r,x,3,m,m}\right).
\end{aligned} \tag{150}$$

The 2nd vector drie-*x* oscillon

$$\boldsymbol{K}_{d,r,i,e,x,2}=\left[K_{o,d,a,m,r,a,m},K_{o,d,b,m,r,b,m}\right]\left(\boldsymbol{f}_{d,r,i,e,x,2}\right) \tag{151}$$

is exhibited by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which are created by 2-tuple $\boldsymbol{f}_{d,r,i,e,x,2}$ as

$$\begin{aligned}
K_{o,d,a,m,r,a,m}&=-\frac{\rho_c}{2}\,ez_{d,m}ez_{r,m}Q_{d,y,m}\,Q_{r,y,m}\left(g_{d,r,x,2,m,m}-g_{d,r,x,4,m,m}\right),\\
K_{o,d,b,m,r,b,m}&=+\frac{\rho_c}{2}ez_{d,m}ez_{r,m}\,Q_{d,y,m}\,Q_{r,y,m}\left(g_{d,r,x,2,m,m}+g_{d,r,x,4,m,m}\right).
\end{aligned} \tag{152}$$

The 3rd vector drie-*x* oscillon

$$\boldsymbol{K}_{d,r,i,e,x,3}=\left[K_{o,d,b,m,r,c,m},K_{o,d,a,m,r,d,m}\right]\left(\boldsymbol{f}_{d,r,i,e,x,3}\right) \tag{153}$$

is given by a list of two 2-w, deterministic-random, neutral oscillons in *x*, which are governed by 2-tuple $\boldsymbol{f}_{d,r,i,e,x,3}$ in the view of

$$\begin{aligned}
K_{o,d,b,m,r,c,m}&=+\frac{\rho_c}{2}\,ez_{d,m}ez_{r,m}Q_{d,y,m}\,R_{r,y,m}\left(g_{d,r,x,5,m,m}-g_{d,r,x,7,m,m}\right),\\
K_{o,d,a,m,r,d,m}&=+\frac{\rho_c}{2}ez_{d,m}ez_{r,m}\,Q_{d,y,m}\,R_{r,y,m}\left(g_{d,r,x,5,m,m}+g_{d,r,x,7,m,m}\right).
\end{aligned} \tag{154}$$

The 4th vector drie-*x* oscillon

$$\boldsymbol{K}_{d,r,i,e,x,4}=\left[K_{o,d,a,m,r,c,m},K_{o,d,b,m,r,d,m}\right]\left(\boldsymbol{f}_{d,r,i,e,x,4}\right) \tag{155}$$

is envisioned by a list of two 2-w, deterministic-random, neutral oscillons in x, which are determined by 2-tuple $\boldsymbol{f}_{d,r,i,e,x,4}$ because

$$\begin{aligned} K_{o,d,a,m,r,c,m} &= -\frac{\rho_c}{2} ez_{d,m} ez_{r,m} Q_{d,y,m} R_{r,y,m} \left(g_{d,r,x,6,m,m} - g_{d,r,x,8,m,m} \right), \\ K_{o,d,b,m,r,d,m} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,m} Q_{d,y,m} R_{r,y,m} \left(g_{d,r,x,6,m,m} + g_{d,r,x,8,m,m} \right). \end{aligned} \tag{156}$$

The 5th vector drie-x oscillon

$$\boldsymbol{K}_{d,r,i,e,x,5} = \left[K_{o,d,d,m,r,a,m}, K_{o,d,c,m,r,b,m} \right] \left(\boldsymbol{f}_{d,r,i,e,x,5} \right) \tag{157}$$

is displayed by a list of two 2-w, deterministic-random, neutral oscillons in x, which are generated by 2-tuple $\boldsymbol{f}_{d,r,i,e,x,5}$ as

$$\begin{aligned} K_{o,d,d,m,r,a,m} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,m} R_{d,y,m} Q_{r,y,m} \left(g_{d,r,x,9,m,m} - g_{d,r,x,11,m,m} \right), \\ K_{o,d,c,m,r,b,m} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,m} R_{d,y,m} Q_{r,y,m} \left(g_{d,r,x,9,m,m} + g_{d,r,x,11,m,m} \right). \end{aligned} \tag{158}$$

The 6th vector drie-x oscillon

$$\boldsymbol{K}_{d,r,i,e,x,6} = \left[K_{o,d,c,m,r,a,m}, K_{o,d,d,m,r,b,m} \right] \left(\boldsymbol{f}_{d,r,i,e,x,6} \right) \tag{159}$$

is presented by a list of two 2-w, deterministic-random, neutral oscillons in x, which are produced by 2-tuple $\boldsymbol{f}_{d,r,i,e,x,6}$ since

$$\begin{aligned} K_{o,d,c,m,r,a,m} &= -\frac{\rho_c}{2} ez_{d,m} ez_{r,m} R_{d,y,m} Q_{r,y,m} \left(g_{d,r,x,10,m,m} - g_{d,r,x,12,m,m} \right), \\ K_{o,d,d,m,r,b,m} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,m} R_{d,y,m} Q_{r,y,m} \left(g_{d,r,x,10,m,m} + g_{d,r,x,12,m,m} \right). \end{aligned} \tag{160}$$

The 7th vector drie-x oscillon

$$\boldsymbol{K}_{d,r,i,e,x,7} = \left[K_{o,d,d,m,r,c,m}, K_{o,d,c,m,r,d,m} \right] \left(\boldsymbol{f}_{d,r,i,e,x,7} \right) \tag{161}$$

is visualized by a list of two 2-w, deterministic-random, neutral oscillons in x, which are established by 2-tuple $\boldsymbol{f}_{d,r,i,e,x,7}$ because

$$\begin{aligned} K_{o,d,d,m,r,c,m} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,m} R_{d,y,m} R_{r,y,m} \left(g_{d,r,x,13,m,m} - g_{d,r,x,15,m,m} \right), \\ K_{o,d,c,m,r,d,m} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,m} R_{d,y,m} R_{r,y,m} \left(g_{d,r,x,13,m,m} + g_{d,r,x,15,m,m} \right). \end{aligned} \tag{162}$$

The 8th vector drie-x oscillon

$$\boldsymbol{K}_{d,r,i,e,x,8} = \left[K_{o,d,c,m,r,c,m}, K_{o,d,d,m,r,d,m} \right] \left(\boldsymbol{f}_{d,r,i,e,x,8} \right) \tag{163}$$

is represented by a list of two 2-w, deterministic-random, neutral oscillons in x, which are formed by 2-tuple $\boldsymbol{f}_{d,r,i,e,x,8}$ in the view of

$$\begin{aligned} K_{o,d,c,m,r,c,m} &= -\frac{\rho_c}{2} ez_{d,m} ez_{r,m} R_{d,y,m} R_{r,y,m} \left(g_{d,r,x,14,m,m} - g_{d,r,x,16,m,m} \right), \\ K_{o,d,d,m,r,d,m} &= +\frac{\rho_c}{2} ez_{d,m} ez_{r,m} R_{d,y,m} R_{r,y,m} \left(g_{d,r,x,14,m,m} + g_{d,r,x,16,m,m} \right). \end{aligned} \tag{164}$$

For all vector drie-x oscillons, amplitudes of eigenfunctions are governed by ρ_c, z_0, y_0, t_0 via $ez_{d,m}, ez_{r,m}, Q_{d,y,m}, Q_{r,y,m}, R_{d,y,m}, R_{r,y,m}$ and x-shifts of eigenfunctions are influenced by y_0, t_0 through $s_{d,y,\alpha,m}, s_{d,y,\beta,m}, s_{r,y,\alpha,m}, s_{r,y,\beta,m}$.

The x-periods $L_{d,r,x,1,m,m}$ and $L_{d,r,x,2,m,m}$ of $g_{d,r,x,l,m,m}$ for each $l = 1, 2, \cdots, 16$ become

$$L_{d,r,x,1,m,m} = \frac{2\pi}{\kappa_{d,r,1,m,m}}, \quad L_{d,r,x,2,m,m} = \frac{2\pi}{\kappa_{d,r,2,m,m}}. \tag{165}$$

The wavelength of the drie-x oscillons is determined by

$$\begin{aligned} L_{d,r,x,m,m} = \mathrm{LCM}\left(L_{d,r,x,1,m,m}, L_{d,r,x,2,m,m}\right) &= k_{d,r,x,1,m,m} L_{d,r,x,1,m,m} \\ &= k_{d,r,x,2,m,m} L_{d,r,x,2,m,m}, \end{aligned} \tag{166}$$

where $k_{d,r,x,1,m,m}$ and $k_{d,r,x,2,m,m}$ are integers.

Because integrals of cosine and sine waves over $L_{d,r,x,1,m,m}$ and $L_{d,r,x,2,m,m}$ vanish, the average of the drie-x oscillons over $L_{d,r,x,m,m}$ also vanishes. The vector drie-x oscillons are neutral since on average they do not transfer any kinetic energy along the x-axis.

4.2. The DRIW Oscillons

We apply the decomposition of the deterministic-random, internal, wave oscillons (the driw oscillons for straightforwardness, see (190) of [7]) via the drie oscillons and substitute the inhomogeneous Fourier expansion of the vector drie-x oscillons to obtain that 16 driw-x oscillons are assembled into four vector driw-x oscillons

$$\boldsymbol{K}_{d,r,i,w,x,q} = \boldsymbol{K}_{d,r,i,w,x,q}\left(\boldsymbol{f}_{d,r,i,w,x,q}\right), \quad q = 1, 2, 3, 4, \tag{167}$$

which are produced by four 4-tuples of the deterministic-random, internal, wave interaction in x:

$$\begin{aligned} &\boldsymbol{f}_{d,r,i,w,x,1} = \left\{g_{d,r,x,2k-1,m,m}\right\}, \quad \boldsymbol{f}_{d,r,i,w,x,2} = \left\{g_{d,r,x,2k,m,m}\right\}, \\ &\boldsymbol{f}_{d,r,i,w,x,3} = \left\{g_{d,r,x,2k-1,m,m}\right\}, \quad \boldsymbol{f}_{d,r,i,w,x,4} = \left\{g_{d,r,x,2k,m,m}\right\}. \end{aligned} \tag{168}$$

Four-tuple $\boldsymbol{f}_{d,r,i,w,x,1}$ consists of four sine waves $g_{d,r,x,2k-1,m,m}$ and 4-tuple $\boldsymbol{f}_{d,r,i,w,x,2}$ is constructed of four cosine waves $g_{d,r,x,2k,m,m}$ for $k = 1, 2, 7, 8$. Four-tuple $\boldsymbol{f}_{d,r,i,w,x,3}$ is composed of four sine waves $g_{d,r,x,2k-1,m,m}$ and 4-tuple $\boldsymbol{f}_{d,r,i,w,x,4}$ comprises four cosine waves $g_{d,r,x,2k,m,m}$ $k = 3, 4, 5, 6$. All 4-tuples $\boldsymbol{f}_{d,r,i,w,x,q}$ are controlled by wavenumbers $\kappa_{d,r,1,m,m}, \kappa_{d,r,2,m,m}$ for each m.

Because of the identity and wavenumber resonances of the vector drie-x oscillons for any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector driw-$x$ oscillon

$$\boldsymbol{K}_{d,r,i,w,x,1} = \left[K_{w,d,b,m,r,a,m}, K_{w,d,a,m,r,b,m}, K_{w,d,d,m,r,c,m}, K_{w,d,c,m,r,d,m}\right]\left(\boldsymbol{f}_{d,r,i,w,x,1}\right) \tag{169}$$

is displayed by a list of four 2-w, deterministic-random, neutral oscillons in x, which depend on 4-tuple $\boldsymbol{f}_{d,r,i,w,x,1}$ in agreement with

$$\begin{aligned} K_{w,d,b,m,r,a,m} = &-\frac{\rho_c}{2}\Big(ez_{d,m} ez_{r,m} \Big\{Q_{d,y,m} Q_{r,y,m} \\ &\times \Big[\left(\kappa_{d,m}\kappa_{r,m} - \mu_{d,m}\mu_{r,m}\right) g_{d,r,x,1,m,m} + \left(\kappa_{d,m}\kappa_{r,m} + \mu_{d,m}\mu_{r,m}\right) g_{d,r,x,3,m,m}\Big] \\ &- R_{d,y,m} R_{r,y,m} \lambda_{d,m}\lambda_{r,m}\left(g_{d,r,x,13,m,m} - g_{d,r,x,15,m,m}\right)\Big\}\Big), \end{aligned}$$

$$
\begin{aligned}
K_{w,d,a,m,r,b,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{Q_{d,y,m}\,Q_{r,y,m} \\
&\times\Big[\left(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,1,m,m}-\left(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,3,m,m}\Big] \\
&- R_{d,y,m}\,R_{r,y,m}\lambda_{d,m}\lambda_{r,m}\left(g_{d,r,x,13,m,m}+g_{d,r,x,15,m,m}\right)\Big\}\Big), \\
K_{w,d,d,m,r,c,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{R_{d,y,m}\,R_{r,y,m} \\
&\times\Big[\left(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,13,m,m}+\left(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,15,m,m}\Big] \\
&- Q_{d,y,m}\,Q_{r,y,m}\lambda_{d,m}\lambda_{r,m}\left(g_{d,r,x,1,m,m}-g_{d,r,x,3,m,m}\right)\Big\}\Big), \\
K_{w,d,c,m,r,d,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{R_{d,y,m}\,R_{r,y,m} \\
&\times\Big[\left(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,13,m,m}-\left(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,15,m,m}\Big] \\
&- Q_{d,y,m}\,Q_{r,y,m}\lambda_{d,m}\lambda_{r,m}\left(g_{d,r,x,1,m,m}+g_{d,r,x,3,m,m}\right)\Big\}\Big).
\end{aligned}
\tag{170}
$$

The 2nd vector driw-x oscillon

$$
\boldsymbol{K}_{d,r,i,w,x,2}=\left[K_{w,d,a,m,r,a,m},K_{w,d,b,m,r,b,m},K_{w,d,c,m,r,c,m},K_{w,d,d,m,r,d,m}\right]\left(\boldsymbol{f}_{d,r,i,w,x,2}\right)
\tag{171}
$$

is represented by a list of four 2-w, deterministic-random, neutral oscillons in x, which are formed by 4-tuple $\boldsymbol{f}_{d,r,i,w,x,2}$ as

$$
\begin{aligned}
K_{w,d,a,m,r,a,m} &= +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{Q_{d,y,m}\,Q_{r,y,m} \\
&\times\Big[\left(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,2,m,m}+\left(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,4,m,m}\Big] \\
&- R_{d,y,m}\,R_{r,y,m}\lambda_{d,m}\lambda_{r,m}\left(g_{d,r,x,14,m,m}-g_{d,r,x,16,m,m}\right)\Big\}\Big), \\
K_{w,d,b,m,r,b,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{Q_{d,y,m}\,Q_{r,y,m} \\
&\times\Big[\left(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,2,m,m}-\left(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,4,m,m}\Big] \\
&- R_{d,y,m}\,R_{r,y,m}\lambda_{d,m}\lambda_{r,m}\left(g_{d,r,x,14,m,m}+g_{d,r,x,16,m,m}\right)\Big\}\Big), \\
K_{w,d,c,m,r,c,m} &= +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{R_{d,y,m}\,R_{r,y,m} \\
&\times\Big[\left(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,14,m,m}+\left(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,16,m,m}\Big] \\
&- Q_{d,y,m}\,Q_{r,y,m}\lambda_{d,m}\lambda_{r,m}\left(g_{d,r,x,2,m,m}-g_{d,r,x,4,m,m}\right)\Big\}\Big), \\
K_{w,d,d,m,r,d,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{R_{d,y,m}\,R_{r,y,m} \\
&\times\Big[\left(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,14,m,m}-\left(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\right)g_{d,r,x,16,m,m}\Big] \\
&- Q_{d,y,m}\,Q_{r,y,m}\lambda_{d,m}\lambda_{r,m}\left(g_{d,r,x,2,m,m}+g_{d,r,x,4,m,m}\right)\Big\}\Big).
\end{aligned}
\tag{172}
$$

The 3rd vector driw-x oscillon

$$
\boldsymbol{K}_{d,r,i,w,x,3}=\left[K_{w,d,b,m,r,c,m},K_{w,d,a,m,r,d,m},K_{w,d,d,m,r,a,m},K_{w,d,c,m,r,b,m}\right]\left(\boldsymbol{f}_{d,r,i,w,x,3}\right)
\tag{173}
$$

is visualized by a list of four 2-w, deterministic-random, neutral oscillons in x, which are generated by 4-tuple $\boldsymbol{f}_{d,r,i,w,x,3}$ in the view of

$$\begin{aligned}K_{w,d,b,m,r,c,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{Q_{d,y,m}\,R_{r,y,m}\\&\times\Big[\big(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,5,m,m}+\big(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,7,m,m}\Big]\\&+R_{d,y,m}\,Q_{r,y,m}\lambda_{d,m}\lambda_{r,m}\big(g_{d,r,x,9,m,m}-g_{d,r,x,11,m,m}\big)\Big\}\Big),\end{aligned}$$

$$\begin{aligned}K_{w,d,a,m,r,d,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{Q_{d,y,m}\,R_{r,y,m}\\&\times\Big[\big(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,5,m,m}-\big(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,7,m,m}\Big]\\&+R_{d,y,m}\,Q_{r,y,m}\lambda_{d,m}\lambda_{r,m}\big(g_{d,r,x,9,m,m}+g_{d,r,x,11,m,m}\big)\Big\}\Big),\end{aligned}$$

$$\begin{aligned}K_{w,d,d,m,r,a,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{R_{d,y,m}\,Q_{r,y,m}\\&\times\Big[\big(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,9,m,m}+\big(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,11,m,m}\Big]\\&+Q_{d,y,m}\,R_{r,y,m}\lambda_{d,m}\lambda_{r,m}\big(g_{d,r,x,5,m,m}-g_{d,r,x,7,m,m}\big)\Big\}\Big),\end{aligned}$$

$$\begin{aligned}K_{w,d,c,m,r,b,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{R_{d,y,m}\,Q_{r,y,m}\\&\times\Big[\big(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,9,m,m}-\big(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,11,m,m}\Big]\\&+Q_{d,y,m}\,R_{r,y,m}\lambda_{d,m}\lambda_{r,m}\big(g_{d,r,x,5,m,m}+g_{d,r,x,7,m,m}\big)\Big\}\Big).\end{aligned}\tag{174}$$

The 4th vector driw-x oscillon

$$\boldsymbol{K}_{d,r,i,w,x,4}=\Big[K_{w,d,a,m,r,c,m},K_{w,d,b,m,r,d,m},K_{w,d,c,m,r,a,m},K_{w,d,d,m,r,b,m}\Big]\Big(\boldsymbol{f}_{d,r,i,w,x,4}\Big)\tag{175}$$

is given by a list of four 2-w, deterministic-random, neutral oscillons in x, which are determined by 4-tuple $\boldsymbol{f}_{d,r,i,w,x,4}$ because

$$\begin{aligned}K_{w,d,a,m,r,c,m} &= +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{Q_{d,y,m}\,R_{r,y,m}\\&\times\Big[\big(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,6,m,m}+\big(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,8,m,m}\Big]\\&+R_{d,y,m}\,Q_{r,y,m}\lambda_{d,m}\lambda_{r,m}\big(g_{d,r,x,10,m,m}-g_{d,r,x,12,m,m}\big)\Big\}\Big),\end{aligned}$$

$$\begin{aligned}K_{w,d,b,m,r,d,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{Q_{d,y,m}\,R_{r,y,m}\\&\times\Big[\big(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,6,m,m}-\big(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,8,m,m}\Big]\\&+R_{d,y,m}\,Q_{r,y,m}\lambda_{d,m}\lambda_{r,m}\big(g_{d,r,x,10,m,m}+g_{d,r,x,12,m,m}\big)\Big\}\Big),\end{aligned}$$

$$\begin{aligned}K_{w,d,c,m,r,a,m} &= +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{R_{d,y,m}\,Q_{r,y,m}\\&\times\Big[\big(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,10,m,m}+\big(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,12,m,m}\Big]\\&+Q_{d,y,m}\,R_{r,y,m}\lambda_{d,m}\lambda_{r,m}\big(g_{d,r,x,6,m,m}-g_{d,r,x,8,m,m}\big)\Big\}\Big),\end{aligned}$$

$$\begin{aligned}K_{w,d,d,m,r,b,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{r,m}\Big\{R_{d,y,m}\,Q_{r,y,m}\\&\times\Big[\big(\kappa_{d,m}\kappa_{r,m}-\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,10,m,m}-\big(\kappa_{d,m}\kappa_{r,m}+\mu_{d,m}\mu_{r,m}\big)g_{d,r,x,12,m,m}\Big]\\&+Q_{d,y,m}\,R_{r,y,m}\lambda_{d,m}\lambda_{r,m}\big(g_{d,r,x,6,m,m}+g_{d,r,x,8,m,m}\big)\Big\}\Big).\end{aligned}\tag{176}$$

For all vector driw-*x* oscillons, amplitudes of eigenfunctions are controlled by $\rho_c,\kappa_{d,m},\lambda_{d,m},\mu_{d,m},\kappa_{r,m},\lambda_{r,m},\mu_{r,m},z_0,y_0,t_0$ via $ez_{d,m},ez_{r,m},Q_{d,y,m},R_{d,y,m},Q_{r,y,m},R_{r,y,m}$ and *x*-shifts of eigenfunctions depend on y_0,t_0 through $s_{d,y,\alpha,m},s_{d,y,\beta,m},s_{r,y,\alpha,m},s_{r,y,\beta,m}$. The wavelength of the driw-*x* oscillons is also $L_{d,r,x,m,m}$. Similar to the drie-*x* oscillons, the average of the driw-*x* oscillons over $L_{d,r,x,m,m}$ vanishes, *i.e.* the vector driw-*x*-oscillons do not transfer any kinetic energy along the *x*-axis.

4.3. The DRIG Oscillon

The symmetry and wavenumber resonances of the drie-*x* oscillons produce reduction of the deterministic-random, internal, group oscillon (the drig oscillon for easiness, see (191) of [7]) to a 2-w oscillon, which is generated by 8-tuple of the deterministic-random, internal, group interaction in *x*

$$\boldsymbol{f}_{d,r,i,g,x}=\left\{g_{d,r,x,4k-3,m,m},g_{d,r,x,4k,m,m}\right\},\ k=1,2,3,4.\tag{177}$$

Eight-tuple $\boldsymbol{f}_{d,r,i,g,x}$ consists of four sine waves $g_{d,r,x,4k-3,m,m}$ with wavenumber $\kappa_{d,r,1,m,m}$, and four cosine waves $g_{d,r,x,4k,m,m}$ with wavenumber $\kappa_{d,r,2,m,m}$ for $k=1,2,3,4$, and each *m*.

We then express the drig oscillon via the drie oscillons and substitute the inhomogeneous Fourier expansion of the vector drie-*x* oscillons to obtain for any frozen $y=y_0,z=z_0,t=t_0$ that the drig-*x* oscillon is converted into the 2-w, deterministic-random, neutral oscillon in *x*, which is determined by 8-tuple $\boldsymbol{f}_{d,r,i,g,x}$,

$$K_{g,d,i,m,r,j,m}=K_{g,d,i,m,r,j,m}\left(\boldsymbol{f}_{d,r,i,g,x}\right)\tag{178}$$

since

$$\begin{aligned}K_{g,d,i,m,r,j,m} &= \rho_c ez_{d,m}ez_{r,m}\\&\times\Big[\Lambda_{d,m,r,m}\big(Q_{d,y,m}\,Q_{r,y,m}g_{d,r,x,1,m,m}+R_{d,y,m}\,R_{r,y,m}g_{d,r,x,13,m,m}\big)\\&-\mathrm{N}_{d,m,r,m}\big(Q_{d,y,m}\,R_{r,y,m}g_{d,r,x,5,m,m}+R_{d,y,m}\,Q_{r,y,m}g_{d,r,x,9,m,m}\big)\\&+\mathrm{M}_{d,m,r,m}\big(Q_{d,y,m}\,Q_{r,y,m}g_{d,r,x,4,m,m}+R_{d,y,m}\,R_{r,y,m}g_{d,r,x,16,m,m}\big)\\&+\mathrm{K}_{d,m,r,m}\big(Q_{d,y,m}\,R_{r,y,m}g_{d,r,x,8,m,m}+R_{d,y,m}\,Q_{r,y,m}g_{d,r,x,12,m,m}\big)\Big].\end{aligned}\tag{179}$$

Amplitudes of eigenfunctions are determined by

$$\rho_c,\mathrm{K}_{d,m,r,m},\Lambda_{d,m,r,m},\mathrm{M}_{d,m,r,m},\mathrm{N}_{d,m,r,m},z_0,y_0,t_0\tag{180}$$

via $ez_{d,m},ez_{r,m},Q_{d,y,m},R_{d,y,m},Q_{r,y,m},R_{r,y,m}$ and *x*-shifts of eigenfunctions are governed by y_0,t_0 through $s_{d,y,\alpha,m},s_{d,y,\beta,m},s_{r,y,\alpha,m},s_{r,y,\beta,m}$. The wavelength of the drig-*x* oscillon is $L_{d,r,x,m,m}$, as well. The vanishing average of the drig-*x* oscillon over $L_{d,r,x,m,m}$ shows that the drig-*x* oscillon on average is neutral since it does not transfer the kinetic energy along the *x*-axis.

4.4. The DRIK Oscillon

The deterministic-random, internal, kinetic-energy oscillon (the drik oscillon for brevity, see (86) of [7]) may be written as follows:

$$K_{e,d,i,m,r,j,m} = \sum_{m=1}^{M} K_{g,d,i,m,r,j,m}. \tag{181}$$

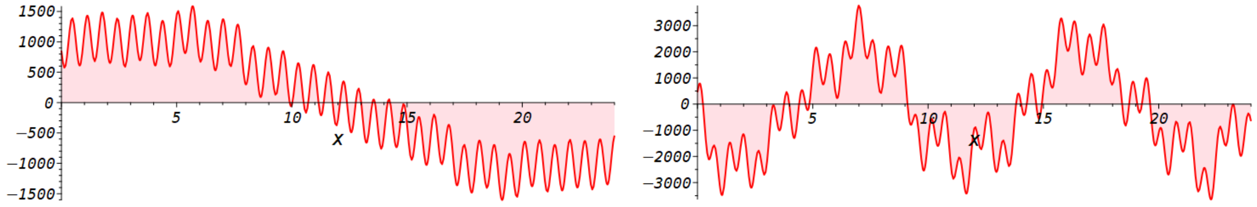
Since all wavenumbers of the drik-*x* oscillons are distinct, the drik-*x* oscillon is visualized as a 2*M*-w oscillon. For any frozen $y = y_0, z = z_0, t = t_0$, the drik-*x* oscillon is presented by the 2*M*-w, deterministic-random, neutral oscillon in *x*, which is formed by *M* 8-tuples $\boldsymbol{f}_{d,r,i,g,x}$ with wavenumbers $\kappa_{d,r,1,m,n}, \kappa_{d,r,2,m,n}$ for all *m*, *Re*, and wave parameters of the drig-*x* oscillons.

The wavelength of the drik-x oscillon is computed by

$$L_{e,d,r,x,i} = \mathrm{LCM}\left(L_{d,r,x,m,m}\right) = k_{d,r,x,m,m} L_{d,r,x,m,m}, \tag{182}$$

where $k_{d,r,x,m,m}$ is an integer. Similar to the drig-*x* oscillon, the vanishing average of the drik-*x* oscillon over $L_{e,d,r,x,i}$ shows that the drik-*x* oscillon on average does not transfer any kinetic energy along the *x*-axis.

The drik-x oscillons for $y = y_0, z = z_0, t = t_0, Re = 10^3, Re = 10^5$, wave parameters (140), (145), (146) are displayed on $L_{e,d,x,e}$ in **Figure 2**. The Reynolds number strongly affects both the range and the shape of the 6-w, deterministic-random, neutral drik-*x* oscillon.



(a) (b)

Figure 2. The drik-*x* oscillons: (a) $-K_{e,d,i,m,r,j,m}(x)$ for $Re = 10^3$, (b) $-K_{e,d,i,m,r,j,m}(x)$ for $Re = 10^5$.

5. Oscillons of Turbulent, External Interaction

5.1. The TEE Oscillons

The turbulent, external, elementary oscillons (the tee oscillons for simplicity) are computed as the superposition of the dee oscillons ((149) of [7]) and the ree oscillons ((181) of [7]) as follows:

$$K_{o,t,i,m,t,j,n} = K_{o,d,i,m,d,j,n} + K_{o,r,i,m,r,j,n}, \quad i = a,b,c, \quad j = i+1,\cdots,d. \tag{183}$$

Similar to the vector dee-*x* and ree-*x* oscillons, six tee oscillons are grouped in three vector tee-*x* oscillons

$$\boldsymbol{K}_{t,e,e,x,q} = \boldsymbol{K}_{t,e,e,x,q}\left(\boldsymbol{f}_{t,e,e,x,q}\right), \quad q = 1,2,3, \tag{184}$$

which are formed by 4-tuple and two 8-tuples of the turbulent, external, elementary interaction in *x*:

$$
\begin{aligned}
\boldsymbol{f}_{t,e,e,x,1} &= \left\{ \boldsymbol{f}_{d,e,e,x,1}, \boldsymbol{f}_{r,e,e,x,1} \right\} = \left\{ f_{d,x,2k-1,m,n}, f_{r,x,2k-1,m,n} \right\}, \\
\boldsymbol{f}_{t,e,e,x,2} &= \left\{ \boldsymbol{f}_{d,e,e,x,2}, \boldsymbol{f}_{r,e,e,x,2} \right\} = \left\{ f_{d,x,2k,m,n}, \quad f_{r,x,2k,m,n} \right\}, \\
\boldsymbol{f}_{t,e,e,x,3} &= \left\{ \boldsymbol{f}_{d,e,e,x,3}, \boldsymbol{f}_{r,e,e,x,3} \right\} = \left\{ f_{d,x,2k-1,m,n}, f_{r,x,2k-1,m,n} \right\}.
\end{aligned} \tag{185}
$$

Four-tuple $\boldsymbol{f}_{t,e,e,x,1}$ consists of two deterministic sine waves $f_{d,x,2k-1,m,n}$ with wavenumber $\kappa_{d,1,m,n}$ and two random sine waves $f_{r,x,2k-1,m,n}$ with wavenumber $\kappa_{r,1,m,n}$ for $k=1,7$ and each *m*, *n*. Eight-tuple $\boldsymbol{f}_{t,e,e,x,2}$ comprises four deterministic cosine waves $f_{d,x,2k,m,n}$ with wavenumbers $\kappa_{d,1,m,n}, \kappa_{d,2,m,n}$ and four random cosine waves $f_{r,x,2k,m,n}$ with wavenumbers $\kappa_{r,1,m,n}, \kappa_{r,2,m,n}$ for $k=3,4,5,6$, and each *m*, *n*. Eight-tuple $\boldsymbol{f}_{t,e,e,x,3}$ is composed of four deterministic sine waves $f_{d,x,2k-1,m,n}$ with wavenumbers $\kappa_{d,1,m,n}, \kappa_{d,2,m,n}$ and four random sine waves $f_{r,x,2k-1,m,n}$ with wavenumbers $\kappa_{r,1,m,n}, \kappa_{r,2,m,n}$ for $k=3,4,5,6,$ and each *m*, *n*.

For any frozen $y=y_0, z=z_0, t=t_0$, the 1st vector tee-*x* oscillon

$$
\boldsymbol{K}_{t,e,e,x,1} = \left[K_{o,t,a,m,t,b,n}, K_{o,t,c,m,t,d,n} \right] \left(\boldsymbol{f}_{t,e,e,x,1} \right) \tag{186}
$$

is reduced to a list of two 2-w, turbulent, neutral oscillons in *x*, which are produced by 4-tuple $\boldsymbol{f}_{t,e,e,x,1}$ as

$$
\begin{aligned}
K_{o,t,a,m,t,b,n} &= \rho_c \left(ez_{d,m} ez_{d,n} Q_{d,y,m} Q_{d,y,n} f_{d,x,1,m,n} \right. \\
&\quad \left. + ez_{r,m} ez_{r,n} Q_{r,y,m} Q_{r,y,n} f_{r,x,1,m,n} \right), \\
K_{o,t,c,m,t,d,n} &= \rho_c \left(ez_{d,m} ez_{d,n} R_{d,y,m} R_{d,y,n} f_{d,x,13,m,n} \right. \\
&\quad \left. + ez_{r,m} ez_{r,n} R_{r,y,m} R_{r,y,n} f_{r,x,13,m,n} \right).
\end{aligned} \tag{187}
$$

The 2nd vector tee-*x* oscillon

$$
\boldsymbol{K}_{t,e,e,x,2} = \left[K_{o,t,a,m,t,c,n}, K_{o,t,b,m,t,d,n} \right] \left(\boldsymbol{f}_{t,e,e,x,2} \right) \tag{188}
$$

is presented by a list of two 4-w, turbulent, neutral oscillons in *x*, which are generated by 8-tuple $\boldsymbol{f}_{t,e,e,x,2}$ since

$$
\begin{aligned}
K_{o,t,a,m,t,c,n} &= -\frac{\rho_c}{2} \left\{ ez_{d,m} ez_{d,n} \left[Q_{d,y,m} R_{d,y,n} \left(f_{d,x,6,m,n} - f_{d,x,8,m,n} \right) \right. \right. \\
&\quad \left. + R_{d,y,m} Q_{d,y,n} \left(f_{d,x,10,m,n} - f_{d,x,12,m,n} \right) \right] \\
&\quad + ez_{r,m} ez_{r,n} \left[Q_{r,y,m} R_{r,y,n} \left(f_{r,x,6,m,n} - f_{r,x,8,m,n} \right) \right. \\
&\quad \left. \left. + R_{r,y,m} Q_{r,y,n} \left(f_{r,x,10,m,n} - f_{r,x,12,m,n} \right) \right] \right\}, \\
K_{o,t,b,m,t,d,n} &= +\frac{\rho_c}{2} \left\{ ez_{d,m} ez_{d,n} \left[Q_{d,y,m} R_{d,y,n} \left(f_{d,x,6,m,n} + f_{d,x,8,m,n} \right) \right. \right. \\
&\quad \left. + R_{d,y,m} Q_{d,y,n} \left(f_{d,x,10,m,n} + f_{d,x,12,m,n} \right) \right] \\
&\quad + ez_{r,m} ez_{r,n} \left[Q_{r,y,m} R_{r,y,n} \left(f_{r,x,6,m,n} + f_{r,x,8,m,n} \right) \right. \\
&\quad \left. \left. + R_{r,y,m} Q_{r,y,n} \left(f_{r,x,10,m,n} + f_{r,x,12,m,n} \right) \right] \right\}.
\end{aligned} \tag{189}
$$

The 3rd vector tee-*x* oscillon

$$\boldsymbol{K}_{t,e,e,x,3}=\left[K_{o,t,b,m,t,c,n},K_{o,t,a,m,t,d,n}\right]\left(\boldsymbol{f}_{t,e,e,x,3}\right)\tag{190}$$

is visualized by a list of two 4-w, turbulent, neutral oscillons in *x*, which are formed by 8-tuple $\boldsymbol{f}_{t,e,e,x,3}$ because

$$\begin{aligned}
K_{o,t,b,m,t,c,n}&=+\frac{\rho_c}{2}\Big\{ez_{d,m}ez_{d,n}\Big[Q_{d,y,m}R_{d,y,n}\left(f_{d,x,5,m,n}-f_{d,x,7,m,n}\right)\\
&\qquad+R_{d,y,m}Q_{d,y,n}\left(f_{d,x,9,m,n}+f_{d,x,11,m,n}\right)\Big]\\
&\quad+ez_{r,m}ez_{r,n}\Big[Q_{r,y,m}R_{r,y,n}\left(f_{r,x,5,m,n}-f_{r,x,7,m,n}\right)\\
&\qquad+R_{r,y,m}Q_{r,y,n}\left(f_{r,x,9,m,n}+f_{r,x,11,m,n}\right)\Big]\Big\},\\
K_{o,t,a,m,t,d,n}&=+\frac{\rho_c}{2}\Big\{ez_{d,m}ez_{d,n}\Big[Q_{d,y,m}R_{d,y,n}\left(f_{d,x,5,m,n}+f_{d,x,7,m,n}\right)\\
&\qquad+R_{d,y,m}Q_{d,y,n}\left(f_{d,x,9,m,n}-f_{d,x,11,m,n}\right)\Big]\\
&\quad+ez_{r,m}ez_{r,n}\Big[Q_{r,y,m}R_{r,y,n}\left(f_{r,x,5,m,n}+f_{r,x,7,m,n}\right)\\
&\qquad+R_{r,y,m}Q_{r,y,n}\left(f_{r,x,19,m,n}-f_{r,x,11,m,n}\right)\Big]\Big\}.
\end{aligned}\tag{191}$$

Wave parameters of the vector tee-*x* oscillons are given by correspondent parameters of the vector dee-*x* and ree-*x* oscillons. The wavelength of the first vector tee-*x* oscillon

$$L_{t,x,1,m,n}=\mathrm{LCM}\left(L_{d,x,1,m,n},L_{r,x,1,m,n}\right)=k_{t,d,x,1,m,n}L_{d,x,1,m,n}=k_{t,r,x,1,m,n}L_{r,x,1,m,n},\tag{192}$$

where $k_{t,d,x,1,m,n}$ and $k_{t,r,x,1,m,n}$ are integers,

$$L_{d,x,1,m,n}=\frac{2\pi}{\kappa_{d,1,m,n}},\ L_{r,x,1,m,n}=\frac{2\pi}{\kappa_{r,1,m,n}}.\tag{193}$$

The wavelength of the second and third vector tee-*x* oscillons is given by wavelength of the turbulent external interaction in the *x*-direction

$$L_{t,x,m,n}=\mathrm{LCM}\left(L_{d,x,m,n},L_{r,x,m,n}\right)=k_{t,d,x,m,n}L_{d,x,m,n}=k_{t,r,x,m,n}L_{r,x,m,n},\tag{194}$$

where $k_{t,d,x,m,n}$ and $k_{t,r,x,m,n}$ are integers,

$$\begin{aligned}
L_{d,x,m,n}&=\mathrm{LCM}\left(L_{d,x,1,m,n},L_{d,x,2,m,n}\right)=k_{d,x,1,m,n}L_{d,x,1,m,n}=k_{d,x,2,m,n}L_{d,x,2,m,n},\\
L_{r,x,m,n}&=\mathrm{LCM}\left(L_{r,x,1,m,n},L_{r,x,2,m,n}\right)=k_{r,x,1,m,n}L_{r,x,1,m,n}=k_{r,x,2,m,n}L_{r,x,2,m,n},
\end{aligned}\tag{195}$$

$$L_{d,x,2,m,n}=\frac{2\pi}{\kappa_{d,2,m,n}},\ L_{r,x,2,m,n}=\frac{2\pi}{\kappa_{r,2,m,n}},\tag{196}$$

where $k_{d,x,1,m,n},k_{d,x,2,m,n},k_{r,x,1,m,n},k_{r,x,2,m,n}$ are integers. The average of the tee-*x* oscillons over the correspondent wavelength vanishes, *i.e.* the vector tee-*x* oscillons are neutral because on average they do not transfer any kinetic energy along the *x*-axis.

5.2. The TEW Oscillons

The turbulent, external, wave oscillons (the tew oscillons for straightforwardness) are constructed as the superposition of the dew oscillons ((151) of [7]) and the rew oscillons ((183) of [7]) in the following form:

$$K_{w,t,i,m,t,j,n} = K_{w,d,i,m,d,j,n} + K_{w,r,i,m,r,j,n}, \quad i=a,b,c, \quad j=i+1,\cdots,d. \tag{197}$$

Like the vector dew-x and rew-x oscillons, six tew oscillons are combined into three vector tew-x oscillons

$$\boldsymbol{K}_{t,e,w,x,q} = \boldsymbol{K}_{t,e,w,x,q}\left(\boldsymbol{f}_{t,e,e,x,q}\right), \quad q=1,2,3, \tag{198}$$

which are produced by 4-tuple $\boldsymbol{f}_{t,e,e,x,1}$ and 8-tuples $\boldsymbol{f}_{t,e,e,x,2}$ and $\boldsymbol{f}_{t,e,e,x,3}$.

For any frozen $y=y_0, z=z_0, t=t_0$, the 1st vector tew-x oscillon

$$\boldsymbol{K}_{t,e,w,x,1} = \left[K_{w,t,a,m,t,b,n}, K_{w,t,c,m,t,d,n}\right]\left(\boldsymbol{f}_{t,e,e,x,1}\right) \tag{199}$$

is represented by a list of two 2-w, turbulent, neutral oscillons in x, which are formed by 4-tuple $\boldsymbol{f}_{t,e,e,x,1}$ in view of

$$\begin{aligned}
K_{w,t,a,m,t,b,n} = -\rho_c \Big\{ & ez_{d,m} ez_{d,n} \Big[Q_{d,y,m} Q_{d,y,n} \left(\kappa_{d,m}\kappa_{d,n} - \mu_{d,m}\mu_{d,n}\right) f_{d,x,1,m,n} \\
& \qquad - R_{d,y,m} R_{d,y,n} \lambda_{d,m} \lambda_{d,n} f_{d,x,13,m,n} \Big] \\
& + ez_{r,m} ez_{r,n} \Big[Q_{r,y,m} Q_{r,y,n} \left(\kappa_{r,m}\kappa_{r,n} - \mu_{r,m}\mu_{r,n}\right) f_{r,x,1,m,n} \\
& \qquad - R_{r,y,m} R_{r,y,n} \lambda_{r,m} \lambda_{r,n} f_{r,x,13,m,n} \Big] \Big\}, \\
K_{w,t,c,m,t,d,n} = -\rho_c \Big\{ & ez_{d,m} ez_{d,n} \Big[R_{d,y,m} R_{d,y,n} \left(\kappa_{d,m}\kappa_{d,n} - \mu_{d,m}\mu_{d,n}\right) f_{d,x,13,m,n} \\
& \qquad - Q_{d,y,m} Q_{d,y,n} \lambda_{d,m} \lambda_{d,n} f_{d,x,1,m,n} \Big] \\
& + ez_{r,m} ez_{r,n} \Big[R_{r,y,m} R_{r,y,n} \left(\kappa_{r,m}\kappa_{r,n} - \mu_{r,m}\mu_{r,n}\right) f_{r,x,13,m,n} \\
& \qquad - Q_{r,y,m} Q_{r,y,n} \lambda_{r,m} \lambda_{r,n} f_{r,x,1,m,n} \Big] \Big\}.
\end{aligned} \tag{200}$$

The 2nd vector tew-x oscillon

$$\boldsymbol{K}_{t,e,w,x,2} = \left[K_{w,t,a,m,t,c,n}, K_{w,t,b,m,t,d,n}\right]\left(\boldsymbol{f}_{t,e,e,x,2}\right) \tag{201}$$

is exposed by a list of two 4-w, turbulent, neutral oscillons in x, which are established by 8-tuple $\boldsymbol{f}_{t,e,e,x,2}$ since

$$\begin{aligned}
& K_{w,t,a,m,t,c,n} = \\
& +\frac{\rho_c}{2} \Big\{ ez_{d,m} ez_{d,n} \Big[\mathrm{K}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,8,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,12,m,n} \right) \\
& \qquad\qquad + \mathrm{N}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,6,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,10,m,n} \right) \Big] \\
& \qquad + ez_{r,m} ez_{r,n} \Big[\mathrm{K}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,8,m,n} + R_{r,y,m} Q_{r,y,n} f_{r,x,12,m,n} \right) \\
& \qquad\qquad + \mathrm{N}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,6,m,n} + R_{r,y,m} Q_{r,y,n} f_{r,x,10,m,n} \right) \Big] \Big\}, \\
& K_{w,t,b,m,t,d,n} = \\
& +\frac{\rho_c}{2} \Big\{ ez_{d,m} ez_{d,n} \Big[\mathrm{K}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,8,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,12,m,n} \right) \\
& \qquad\qquad - \mathrm{N}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,6,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,10,m,n} \right) \Big] \\
& \qquad + ez_{r,m} ez_{r,n} \Big[\mathrm{K}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,8,m,n} + R_{r,y,m} Q_{r,y,n} f_{r,x,12,m,n} \right) \\
& \qquad\qquad - \mathrm{N}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,6,m,n} + R_{r,y,m} Q_{r,y,n} f_{r,x,10,m,n} \right) \Big] \Big\}.
\end{aligned} \tag{202}$$

The 3rd vector tew-x oscillon

$$\boldsymbol{K}_{t,e,w,x,3} = \left[K_{w,t,b,m,t,c,n} K_{w,t,a,m,t,d,n}, \right] \left(\boldsymbol{f}_{t,e,e,x,3} \right) \tag{203}$$

is exhibited by a list of two 4-w, turbulent, neutral oscillons in *x*, which depend on 8-tuple $\boldsymbol{f}_{t,e,e,x,3}$ as

$$\begin{aligned}
&K_{w,t,b,m,t,c,n} = \\
&-\frac{\rho_c}{2} \Big\{ ez_{d,m} ez_{d,n} \Big[\mathrm{M}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,7,m,n} - R_{d,y,m} Q_{d,y,n} f_{d,x,11,m,n} \right) \\
&\qquad + \mathrm{N}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,5,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,9,m,n} \right) \Big] \\
&\quad + ez_{r,m} ez_{r,n} \Big[\mathrm{M}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,7,m,n} - R_{r,y,m} Q_{r,y,n} f_{r,x,11,m,n} \right) \\
&\qquad + \mathrm{N}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,5,m,n} + R_{r,y,m} Q_{r,y,n} f_{r,x,9,m,n} \right) \Big] \Big\}, \\
&K_{w,t,a,m,t,d,n} = \\
&+\frac{\rho_c}{2} \Big\{ ez_{d,m} ez_{d,n} \Big[\mathrm{M}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,7,m,n} - R_{d,y,m} Q_{d,y,n} f_{d,x,11,m,n} \right) \\
&\qquad - \mathrm{N}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,5,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,9,m,n} \right) \Big] \\
&\quad + ez_{r,m} ez_{r,n} \Big[\mathrm{M}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,7,m,n} - R_{r,y,m} Q_{r,y,n} f_{r,x,11,m,n} \right) \\
&\qquad - \mathrm{N}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,5,m,n} + R_{r,y,m} Q_{r,y,n} f_{r,x,9,m,n} \right) \Big] \Big\}.
\end{aligned} \tag{204}$$

where

$$\begin{aligned}
\mathrm{K}_{d,m,d,n} &= +\kappa_{d,m}\kappa_{d,n} - \lambda_{d,m}\lambda_{d,n} + \mu_{d,m}\mu_{d,n}, \\
\Lambda_{d,m,d,n} &= -\kappa_{d,m}\kappa_{d,n} + \lambda_{d,m}\lambda_{d,n} + \mu_{d,m}\mu_{d,n}, \\
\mathrm{M}_{d,m,d,n} &= +\kappa_{d,m}\kappa_{d,n} + \lambda_{d,m}\lambda_{d,n} + \mu_{d,m}\mu_{d,n}, \\
\mathrm{N}_{d,m,d,n} &= +\kappa_{d,m}\kappa_{d,n} + \lambda_{d,m}\lambda_{d,n} - \mu_{d,m}\mu_{d,n},
\end{aligned} \tag{205}$$

and

$$\begin{aligned}
\mathrm{K}_{r,m,r,n} &= +\kappa_{r,m}\kappa_{r,n} - \lambda_{r,m}\lambda_{r,n} + \mu_{r,m}\mu_{r,n}, \\
\Lambda_{r,m,r,n} &= -\kappa_{r,m}\kappa_{r,n} + \lambda_{r,m}\lambda_{r,n} + \mu_{r,m}\mu_{r,n}, \\
\mathrm{M}_{r,m,r,n} &= +\kappa_{r,m}\kappa_{r,n} + \lambda_{r,m}\lambda_{r,n} + \mu_{r,m}\mu_{r,n}, \\
\mathrm{N}_{r,m,r,n} &= +\kappa_{r,m}\kappa_{r,n} + \lambda_{r,m}\lambda_{r,n} - \mu_{r,m}\mu_{r,n}.
\end{aligned} \tag{206}$$

Wave parameters of the vector tew-*x* oscillons are provided by relevant parameters of the vector dew-*x* and rew-*x* oscillons. The wavelength of the first vector tew-*x* oscillon is $L_{t,x,1,m,n}$ and the wavelength of the second and third vector tew-*x* oscillons is $L_{t,x,m,n}$. Likewise the vector tee-*x* oscillons, the average of the tew-*x* oscillons over the correspondent wavelength vanishes. Therefore, the vector tee-*x* oscillons are neutral because on average they do not transfer any kinetic energy along the *x*-axis.

5.3. The TEG Oscillon

The turbulent, external, group oscillon (the teg oscillon for easiness) is composed as the superposition of the deg oscillon ((152) of [7]) and the reg oscillon ((184) of [7]), *i.e.*

$$K_{g,t,i,m,t,j,n} = K_{g,d,i,m,d,j,n} + K_{g,r,i,m,r,j,n}. \tag{207}$$

Analogous to the deg-x and reg-x oscillons, the teg-x oscillon

$$K_{g,t,i,m,t,j,n} = K_{g,t,i,m,t,j,n}\left(\boldsymbol{f}_{t,e,g,x}\right), \tag{208}$$

where 12-tuple of the turbulent, external, group interaction in the x-direction

$$\boldsymbol{f}_{t,e,g,x} = \left\{\boldsymbol{f}_{d,e,g,x}, \boldsymbol{f}_{r,e,g,x}\right\} = \left\{f_{d,x,4k-3,m,n}, f_{d,x,4l,m,n}, f_{r,x,4k-3,m,n}, f_{r,x,4l,m,n}\right\} \tag{209}$$

includes four deterministic sine waves $f_{d,x,4k-3,m,n}$ with wavenumber $\kappa_{d,1,m,n}$ for $k=1,2,3,4$ and each m, n, two deterministic cosine waves $f_{d,x,4l,m,n}$ with wavenumber $\kappa_{d,2,m,n}$ for $l=2,3$ and each m, n, four random sine waves $f_{r,x,4k-3,m,n}$ with wavenumber $\kappa_{r,1,m,n}$ for $k=1,2,3,4$ and each m, n, and two random cosine waves $f_{r,x,4l,m,n}$ with wavenumber $\kappa_{r,2,m,n}$ for $l=2,3$ and each m, n.

For any frozen $y=y_0, z=z_0, t=t_0$, the teg-x oscillon is displayed by a 4-w, turbulent, neutral oscillon in x, which is generated by 12-tuple $\boldsymbol{f}_{t,e,g,x}$ since

$$\begin{aligned}
&K_{g,t,i,m,t,j,n} = \\
&\rho_c\Big\{ez_{d,m}ez_{d,n}\Big[\Lambda_{d,m,d,n}\left(Q_{d,y,m}Q_{d,y,n}f_{d,x,1,m,n} + R_{d,y,m}R_{d,y,n}f_{d,x,13,m,n}\right)\\
&\qquad -\mathrm{N}_{d,m,d,n}\left(Q_{d,y,m}R_{d,y,n}f_{d,x,5,m,n} + R_{d,y,m}Q_{d,y,n}f_{d,x,9,m,n}\right)\\
&\qquad +\mathrm{K}_{d,m,d,n}\left(Q_{d,y,m}R_{d,y,n}f_{d,x,8,m,n} + R_{d,y,m}Q_{d,y,n}f_{d,x,12,m,n}\right)\Big]\\
&\quad + ez_{r,m}ez_{r,n}\Big[\Lambda_{r,m,r,n}\left(Q_{r,y,m}Q_{r,y,n}f_{r,x,1,m,n} + R_{r,y,m}R_{r,y,n}f_{r,x,13,m,n}\right)\\
&\qquad -\mathrm{N}_{r,m,r,n}\left(Q_{r,y,m}R_{r,y,n}f_{r,x,5,m,n} + R_{r,y,m}Q_{r,y,n}f_{r,x,9,m,n}\right)\\
&\qquad +\mathrm{K}_{r,m,r,n}\left(Q_{r,y,m}R_{r,y,n}f_{r,x,8,m,n} + R_{r,y,m}Q_{r,y,n}f_{r,x,12,m,n}\right)\Big]\Big\}.
\end{aligned} \tag{210}$$

Wave parameters of the teg-x oscillons are specified by appropriate parameters of the deg-x and reg-x oscillons. The wavelength of the teg-x oscillon is given by $L_{t,x,m,n}$. The average of the teg-x oscillon over $L_{t,x,m,n}$ vanishes. Therefore, the teg-x oscillon on average is neutral since it does not transfer any kinetic energy along the x-axis, as well.

5.4. The TEK Oscillon

The turbulent, external, kinetic-energy oscillon (the tek oscillon for fastness), which is set as the superposition of the dek and rek oscillons, may be represented as

$$K_{e,t,i,m,t,j,n} = K_{e,d,i,m,d,j,n} + K_{e,r,i,m,r,j,n}. \tag{211}$$

With the help of (77) and (120) of [7], the inhomogeneous Fourier expansion the tek-x oscillon becomes

$$K_{e,t,i,m,t,j,n} = \sum_{m=1}^{M-1}\sum_{n=m+1}^{M} K_{g,t,i,m,t,j,n}. \tag{212}$$

If all wavenumbers of the tek-x oscillon are distinct, then the tek-x oscillon is visualized as a $2M(M-1)$-w oscillon. For any frozen $y=y_0, z=z_0, t=t_0$, the tek-x oscillon is represented by the $2M(M-1)$-w, turbulent, neutral oscillon in x, which is generated by $M(M-1)/2$ 12-tuples $\boldsymbol{f}_{t,e,g,x}$ with wavenumbers $\kappa_{d,1,m,n}, \kappa_{d,2,m,n}, \kappa_{r,1,m,n}, \kappa_{r,2,m,n}$ for all m, n, Re, and wave parameters of the teg-x

oscillons.

The wavelength of the tek-x oscillon is specified by

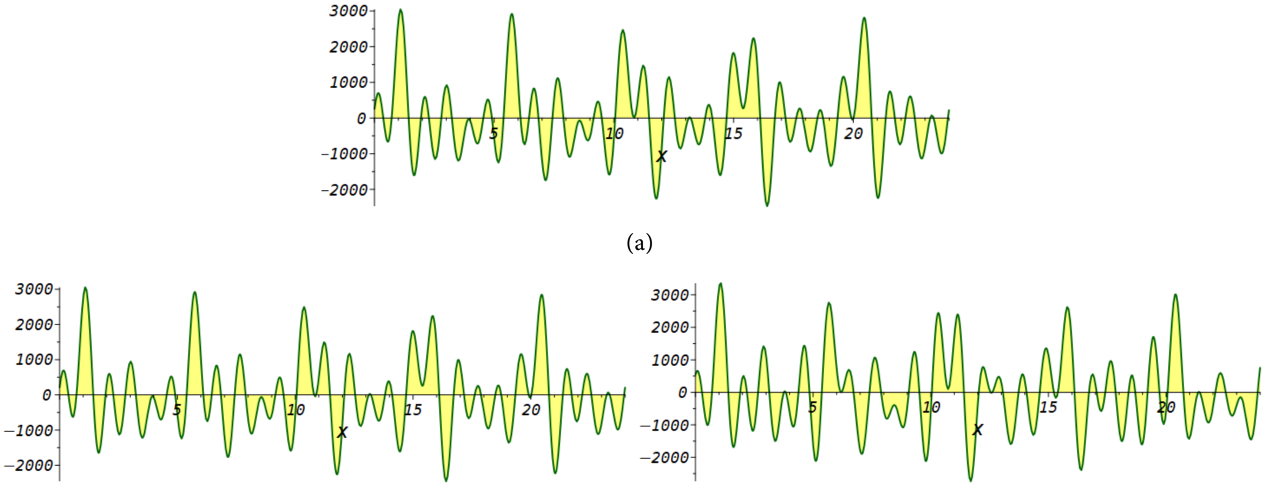
$$L_{e,t,x,e} = \mathrm{LCM}\left(L_{e,d,x,e}, L_{e,r,x,e}\right) = k_{e,t,d,x,e} L_{e,d,x,e} = k_{e,t,r,x,e} L_{e,r,x,e}, \tag{213}$$

where $k_{e,t,d,x,e}$ and $k_{e,t,r,x,e}$ are integers,

$$\begin{aligned} L_{e,d,x,e} &= \mathrm{LCM}\left(L_{d,x,m,n}\right) = k_{d,x,m,n} L_{d,x,m,n}, \\ L_{e,r,x,e} &= \mathrm{LCM}\left(L_{r,x,m,n}\right) = k_{r,x,m,n} L_{r,x,m,n} \end{aligned} \tag{214}$$

are the wavelengths of the dek-x and rek-x oscillons, $k_{d,x,m,n}$ and $k_{r,x,m,n}$ are integers. The vanishing average of the tek-x oscillon over $L_{e,t,x,e}$ shows that the tek-x oscillon on average does not transfer any kinetic energy along the x-axis.

The dek-x and tek-x oscillons for $y = y_0, z = z_0, t = t_0, Re = 10^3, Re = 10^5$, and wave parameters (140), (145), (146) are visualized on $L_{e,d,x,e}$ in **Figure 3**. Since the range of the 6-w, random, neutral rek-x oscillon at $Re = 10^3$ is $[-58, 52]$, the 12-w, turbulent, neutral tek-x oscillon in **Figure 3(b)** alters insignificantly compared with the 6-w, deterministic, neutral dek-x oscillon in **Figure 3(a)**. The range of the 6-w, random, neutral rek-x oscillon at $Re = 10^5$ becomes $[-1180, 1250]$, therefore shape of the 12-w, turbulent, neutral tek-x oscillon in **Figure 3(c)** modifies significantly.



(a)

(b)

(c)

Figure 3. The dek-x and tek-x oscillons: (a) $-K_{e,d,i,m,d,j,n}(x)$, (b) $-K_{e,t,i,m,t,j,n}(x)$ for $Re = 10^3$, (c) $-K_{e,t,i,m,t,j,n}(x)$ for $Re = 10^5$.

6. Oscillons of Turbulent, Diagonal Interaction

6.1. The TDE Oscillons

The turbulent, diagonal, elementary oscillons (the tde oscillons for swiftness) are set as the superposition of the dde oscillons ((144) of [7]) and the rde oscillons ((176) of [7]), namely,

$$K_{o,t,i,m,t,i,n} = K_{o,d,i,m,d,i,n} + K_{o,r,i,m,r,i,n}, \quad i = a, b, c, d. \tag{215}$$

Like the vector dde-x and rde-x oscillons, four tde-x oscillons are grouped into two vector tde-x oscillons

$$\boldsymbol{K}_{t,d,e,x,q} = \boldsymbol{K}_{t,d,e,x,q}\left(\boldsymbol{f}_{t,d,e,x,q}\right),\ q = 1,2, \tag{216}$$

which are constructed using two 4-tuples of the turbulent, diagonal, elementary interaction in x, specifically,

$$\begin{aligned} \boldsymbol{f}_{t,d,e,x,1} &= \left\{\boldsymbol{f}_{d,d,e,x,1}, \boldsymbol{f}_{r,d,e,x,1}\right\} = \left\{f_{d,x,2,m,n},\ f_{d,x,4,m,n},\ f_{r,x,2,m,n},\ f_{r,x,4,m,n}\right\}, \\ \boldsymbol{f}_{t,d,e,x,2} &= \left\{\boldsymbol{f}_{d,d,e,x,2}, \boldsymbol{f}_{r,d,e,x,2}\right\} = \left\{f_{d,x,14,m,n}, f_{d,x,16,m,n}, f_{r,x,14,m,n}, f_{r,x,16,m,n}\right\}. \end{aligned} \tag{217}$$

Four-tuple $\boldsymbol{f}_{t,d,e,x,1}$ embraces two deterministic cosine waves $f_{d,x,2k,m,n}$ with wavenumbers $\kappa_{d,1,m,n}, \kappa_{d,2,m,n}$, respectively, and two random cosine waves $f_{r,x,2k,m,n}$ with wavenumbers $\kappa_{r,1,m,n}, \kappa_{r,2,m,n}$, likely, for $k = 1,2$, and each m, n. Four-tuple $\boldsymbol{f}_{t,d,e,x,2}$ includes two deterministic cosine waves $f_{d,x,2k,m,n}$ with wavenumbers $\kappa_{d,1,m,n}, \kappa_{d,2,m,n}$, similarly, and two random cosine waves $f_{r,x,2k,m,n}$ with wavenumbers $\kappa_{r,1,m,n}, \kappa_{r,2,m,n}$, accordingly, for $k = 7,8$, and each m, n.

For any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector tde-$x$ oscillon

$$\boldsymbol{K}_{t,d,e,x,1} = \left[K_{o,t,a,m,t,a,n}, K_{o,t,b,m,t,b,n}\right]\left(\boldsymbol{f}_{t,d,e,x,1}\right) \tag{218}$$

is represented by a list of two 4-w, turbulent, neutral oscillons in x, which depend on 4-tuple $\boldsymbol{f}_{t,d,e,x,1}$ seeing that

$$\begin{aligned} K_{o,t,a,m,t,a,n} &= -\frac{\rho_c}{2}\Big[ez_{d,m} ez_{d,n}\, Q_{d,y,m}\, Q_{d,y,n}\left(f_{d,x,2,m,n} - f_{d,x,4,m,n}\right) \\ &\qquad + ez_{r,m}\, ez_{r,n}\, Q_{r,y,m}\, Q_{r,y,n}\left(f_{r,x,2,m,n} - f_{r,x,4,m,n}\right)\Big], \\ K_{o,t,b,m,t,b,n} &= +\frac{\rho_c}{2}\Big[ez_{d,m} ez_{d,n}\, Q_{d,y,m}\, Q_{d,y,n}\left(f_{d,x,2,m,n} + f_{d,x,4,m,n}\right) \\ &\qquad + ez_{r,m}\, ez_{r,n}\, Q_{r,y,m}\, Q_{r,y,n}\left(f_{r,x,2,m,n} + f_{r,x,4,m,n}\right)\Big]. \end{aligned} \tag{219}$$

The 2nd vector tde-x oscillon

$$\boldsymbol{K}_{t,d,e,x,2} = \left[K_{o,t,c,m,t,c,n}, K_{o,t,d,m,t,d,n}\right]\left(\boldsymbol{f}_{t,d,e,x,2}\right) \tag{220}$$

is displayed by a list of two 4-w, turbulent, neutral oscillons in x, which are controlled by 4-tuple $\boldsymbol{f}_{t,d,e,x,2}$ considering that

$$\begin{aligned} K_{o,t,c,m,t,c,n} &= -\frac{\rho_c}{2}\Big[ez_{d,m} ez_{d,n}\, R_{d,y,m}\, R_{d,y,n}\left(f_{d,x,14,m,n} - f_{d,x,16,m,n}\right) \\ &\qquad + ez_{r,m}\, ez_{r,n}\, R_{r,y,m}\, R_{r,y,n}\left(f_{r,x,14,m,n} - f_{r,x,16,m,n}\right)\Big], \\ K_{o,t,d,m,t,d,n} &= +\frac{\rho_c}{2}\Big[ez_{d,m} ez_{d,n}\, R_{d,y,m}\, R_{d,y,n}\left(f_{d,x,14,m,n} + f_{d,x,16,m,n}\right) \\ &\qquad + ez_{r,m}\, ez_{r,n}\, R_{r,y,m}\, R_{r,y,n}\left(f_{r,x,14,m,n} + f_{r,x,16,m,n}\right)\Big]. \end{aligned} \tag{221}$$

Wave parameters of the vector tde-x oscillons are indicated by similar parameters of the vector dde-x and rde-x oscillons. The wavelength of the tde-x oscillons is $L_{t,x,m,n}$. The average of the tde-x oscillons over $L_{t,x,m,n}$ vanishes. Therefore, the vector tde-x oscillons are neutral since on average they do not transfer any kinetic energy along the x-axis.

6.2. The TDW Oscillons

The turbulent, diagonal, wave oscillons (the tdw oscillons for quickness) are composed with the help of the superposition of the ddw oscillons ((146) of [7]) and the rdw oscillons ((178) of [7]) oscillons, explicitly,

$$K_{w,t,i,m,t,i,n} = K_{w,d,i,m,d,i,n} + K_{w,r,i,m,r,i,n}, \quad i = a,b,c,d. \tag{222}$$

Parallel to the vector ddw-x and rdw-x oscillons, four tdw-x oscillons are assembled into two vector tdw-x oscillons

$$\boldsymbol{K}_{t,d,w,x,q} = \boldsymbol{K}_{t,d,w,x,q}\left(\boldsymbol{f}_{t,d,w,x,}\right), \quad q = 1,2, \tag{223}$$

which are formed by 8-tuple of the turbulent, diagonal, wave interaction in x:

$$\boldsymbol{f}_{t,d,w,x} = \left\{\boldsymbol{f}_{d,d,w,x}, \boldsymbol{f}_{r,d,w,x}\right\} = \left\{f_{d,x,2k,m,n}, f_{r,x,2k,m,n}\right\}. \tag{224}$$

Eight-tuple $\boldsymbol{f}_{t,d,w,x}$ consists of four deterministic cosine waves $f_{d,x,2k,m,n}$ with wavenumbers $\kappa_{d,1,m,n}, \kappa_{d,2,m,n}$ and four random cosine waves $f_{r,x,2k,m,n}$ with wavenumbers $\kappa_{r,1,m,n}, \kappa_{r,2,m,n}$ for $l = 1,2,7,8$, and each m, n.

For any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector tdw-$x$ oscillon

$$\boldsymbol{K}_{t,d,w,x,1} = \left[K_{w,t,a,m,t,a,n}, K_{w,t,b,m,t,b,n}\right]\left(\boldsymbol{f}_{t,d,w,x}\right) \tag{225}$$

is exhibited by a list of two 4-w, turbulent, neutral oscillons in x, which are established by 8-tuple $\boldsymbol{f}_{t,d,w,x}$ for the reason that

$$\begin{aligned}
K_{w,t,a,m,t,a,m} &= +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{d,n}\Big\{Q_{d,y,m}\,Q_{d,y,n} \\
&\times\Big[\left(\kappa_{d,m}\kappa_{d,n} - \mu_{d,m}\mu_{d,n}\right) f_{d,x,2,m,n} + \left(\kappa_{d,m}\kappa_{d,n} + \mu_{d,m}\mu_{d,n}\right) f_{d,x,4,m,n}\Big] \\
&- R_{d,y,m}\,R_{d,y,n}\lambda_{d,m}\lambda_{d,n}\left(f_{d,x,14,m,n} - f_{d,x,16,m,n}\right)\Big\} \\
&\qquad + ez_{r,m}ez_{r,n}\Big\{Q_{r,y,m}\,Q_{r,y,n} \\
&\times\Big[\left(\kappa_{r,m}\kappa_{r,n} - \mu_{r,m}\mu_{r,n}\right) f_{r,x,2,m,n} + \left(\kappa_{r,m}\kappa_{r,n} + \mu_{r,m}\mu_{r,n}\right) f_{r,x,4,m,n}\Big] \\
&- R_{r,y,m}\,R_{r,y,n}\lambda_{r,m}\lambda_{r,n}\left(f_{r,x,14,m,n} - f_{r,x,16,m,n}\right)\Big\}\Big),
\end{aligned}$$

$$\begin{aligned}
K_{w,t,b,m,t,b,m} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{d,n}\Big\{Q_{d,y,m}\,Q_{d,y,n} \\
&\times\Big[\left(\kappa_{d,m}\kappa_{d,n} - \mu_{d,m}\mu_{d,n}\right) f_{d,x,2,m,n} - \left(\kappa_{d,m}\kappa_{d,n} + \mu_{d,m}\mu_{d,n}\right) f_{d,x,4,m,n}\Big] \\
&- R_{d,y,m}\,R_{d,y,n}\lambda_{d,m}\lambda_{d,n}\left(f_{d,x,14,m,n} + f_{d,x,16,m,n}\right)\Big\} \\
&\qquad + ez_{r,m}ez_{r,n}\Big\{Q_{r,y,m}\,Q_{r,y,n} \\
&\times\Big[\left(\kappa_{r,m}\kappa_{r,n} - \mu_{r,m}\mu_{r,n}\right) f_{r,x,2,m,n} - \left(\kappa_{r,m}\kappa_{r,n} + \mu_{r,m}\mu_{r,n}\right) f_{r,x,4,m,n}\Big] \\
&- R_{r,y,m}\,R_{r,y,n}\lambda_{r,m}\lambda_{r,n}\left(f_{r,x,14,m,n} + f_{r,x,16,m,n}\right)\Big\}\Big).
\end{aligned} \tag{226}$$

The 2nd vector tdw-x oscillon

$$\boldsymbol{K}_{t,d,w,x,2} = \left[K_{w,t,c,m,t,c,n}, K_{w,t,d,m,t,d,n}\right]\left(\boldsymbol{f}_{t,d,w,x}\right) \tag{227}$$

is exposed by a list of two 4-w, turbulent, neutral oscillons in x, which are also set by 8-tuple $\boldsymbol{f}_{t,d,w,x}$ because

$$\begin{aligned}
K_{w,t,c,m,t,c,n} &= +\frac{\rho_c}{2}\Big(ez_{d,m}ez_{d,n}\Big\{R_{d,y,m}\,R_{d,y,n}\\
&\times\Big[\big(\kappa_{d,m}\kappa_{d,n}-\mu_{d,m}\mu_{d,n}\big)\,f_{d,x,14,m,n}+\big(\kappa_{d,m}\kappa_{d,n}+\mu_{d,m}\mu_{d,n}\big)\,f_{d,x,16,m,n}\Big]\\
&-Q_{d,y,m}\,Q_{d,y,n}\lambda_{d,m}\lambda_{d,n}\big(f_{d,x,2,m,n}-f_{d,x,4,m,n}\big)\Big\}\\
&\qquad+ez_{r,m}ez_{r,n}\Big\{R_{r,y,m}\,R_{r,y,n}\\
&\times\Big[\big(\kappa_{r,m}\kappa_{r,n}-\mu_{r,m}\mu_{r,n}\big)\,f_{r,x,14,m,n}+\big(\kappa_{r,m}\kappa_{r,n}+\mu_{r,m}\mu_{r,n}\big)\,f_{r,x,16,m,n}\Big]\\
&-Q_{r,y,m}\,Q_{r,y,n}\lambda_{r,m}\lambda_{r,n}\big(f_{r,x,2,m,n}-f_{r,x,4,m,n}\big)\Big\}\Big),\\[1ex]
K_{w,t,d,m,t,d,n} &= -\frac{\rho_c}{2}\Big(ez_{d,m}ez_{d,n}\Big\{R_{d,y,m}\,R_{d,y,n}\\
&\times\Big[\big(\kappa_{d,m}\kappa_{d,n}-\mu_{d,m}\mu_{d,n}\big)\,f_{d,x,14,m,n}-\big(\kappa_{d,m}\kappa_{d,n}+\mu_{d,m}\mu_{d,n}\big)\,f_{d,x,16,m,n}\Big]\\
&-Q_{d,y,m}\,Q_{d,y,n}\lambda_{d,m}\lambda_{d,n}\big(f_{d,x,2,m,n}+f_{d,x,4,m,n}\big)\Big\}\\
&\qquad+ez_{r,m}ez_{r,n}\Big\{R_{r,y,m}\,R_{r,y,n}\\
&\times\Big[\big(\kappa_{r,m}\kappa_{r,n}-\mu_{r,m}\mu_{r,n}\big)\,f_{r,x,14,m,n}-\big(\kappa_{r,m}\kappa_{r,n}+\mu_{r,m}\mu_{r,n}\big)\,f_{r,x,16,m,n}\Big]\\
&-Q_{r,y,m}\,Q_{r,y,n}\lambda_{r,m}\lambda_{r,n}\big(f_{r,x,2,m,n}+f_{r,x,4,m,n}\big)\Big\}\Big).
\end{aligned}\tag{228}$$

The vector tdw-x oscillons vary by the structure of amplitudes of eigenfunctions. Wave parameters of the vector tdw-x oscillons are given by relevant parameters of the vector ddw-x and rdw-x oscillons. The wavelength of the tdw-x oscillons is also provided by $L_{t,x,m,n}$. The average over $L_{t,x,m,n}$ of the tdw-x oscillons also vanishes. Therefore, the vector tdw-x oscillons are neutral since on average they do not transfer any kinetic energy along the x-axis.

6.3. The TDG Oscillon

The turbulent, diagonal, group oscillon (the tdg oscillon for curtness) is set as the superposition of the ddg oscillons ((147) of [7]) and the rdg oscillons ((179) of [7]), *viz.*

$$K_{g,t,i,m,t,i,n}=K_{g,d,i,m,d,i,n}+K_{g,r,i,m,r,i,n}.\tag{229}$$

Alike the vector ddg-x and rdg-x oscillons, the tdg-x oscillon

$$K_{g,t,i,m,t,i,n}=K_{g,t,i,m,t,i,n}\big(\mathbf{f}_{t,d,g,x}\big),\tag{230}$$

where 4-tuple of the turbulent, diagonal, group, interaction in the x-direction

$$\mathbf{f}_{t,d,g,x}=\big\{\mathbf{f}_{d,d,g,x},\mathbf{f}_{r,d,g,x}\big\}=\big\{f_{d,x,2k,m,n},f_{r,x,2k,m,n}\big\}\tag{231}$$

is composed of two deterministic cosine waves $f_{d,x,2k,m,n}$ with wavenumber $\kappa_{d,2,m,n}$ and two random cosine waves $f_{r,x,2k,m,n}$ with wavenumber $\kappa_{r,2,m,n}$ for $k=2,8$, and each m, n.

For any frozen $y=y_0,z=z_0,t=t_0$, the tdg-x oscillon is represented by a 2-w, turbulent, neutral oscillon in x, which is produced by 4-tuple $\mathbf{f}_{t,d,g,x}$ because

$$\begin{aligned}
&K_{g,t,i,m,t,i,n}=\\
&\rho_c\Big[ez_{d,m}ez_{d,n}\,\mathrm{M}_{d,m,d,n}\big(Q_{d,y,m}\,Q_{d,y,n}\,f_{d,x,4,m,n}+R_{d,y,m}\,R_{d,y,n}\,f_{d,x,16,m,n}\big)\\
&\quad+ez_{r,m}\,ez_{r,n}\,\mathrm{M}_{r,m,r,n}\big(Q_{r,y,m}\,Q_{r,y,n}\,f_{r,x,4,m,n}+R_{r,y,m}\,R_{r,y,n}\,f_{r,x,16,m,n}\big)\Big].
\end{aligned}\tag{232}$$

Wave parameters of the tdg-x oscillon are indicated by relevant parameters of the ddg-x and rdg-x oscillons. The wavelength of the tdg-x oscillon

$$L_{t,x,2,m,n} = \mathrm{LCM}\left(L_{d,x,2,m,n}, L_{r,x,2,m,n}\right) = k_{t,d,x,2,m,n} L_{d,x,2,m,n} = k_{t,r,x,2,m,n} L_{r,x,2,m,n}, \quad (233)$$

where $k_{t,d,x,2,m,n}$ and $k_{t,r,x,2,m,n}$ are integers. The vanishing average of the tdg-x oscillon over $L_{t,x,2,m,n}$ shows that the tdg-x oscillon on average also is neutral since it does not transfer any kinetic energy along the x-axis.

6.4. The TDK Oscillon

The turbulent, diagonal, kinetic-energy oscillon (the tdk oscillon for terseness), which is established by the superposition of the ddk and rdk oscillons, becomes

$$K_{e,t,i,m,t,i,n} = K_{e,d,i,m,d,i,n} + K_{e,r,i,m,r,i,n}. \quad (234)$$

Using (64) and (107) of [7], we compute the inhomogeneous Fourier form of the tdk-x oscillon as follows:

$$K_{e,t,i,m,t,i,n} = \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} K_{g,t,i,m,t,i,n}. \quad (235)$$

If all wavenumbers of the tdg-x oscillons are distinct, then the tdk-x oscillon is displayed as a $M(M-1)$-w oscillon. For any frozen $y = y_0, z = z_0, t = t_0$, the tdk-$x$ oscillon is converted into the $M(M-1)$-w, turbulent, neutral oscillon in x, which is formed by $M(M-1)/2$ 4-tuples $f_{t,d,g,x}$ with wavenumbers $\kappa_{d,2,m,n}, \kappa_{r,2,m,n}$ for each m, n, Re, and wave parameters of the tdg-x oscillons.

The wavelength of the tdk-x oscillon

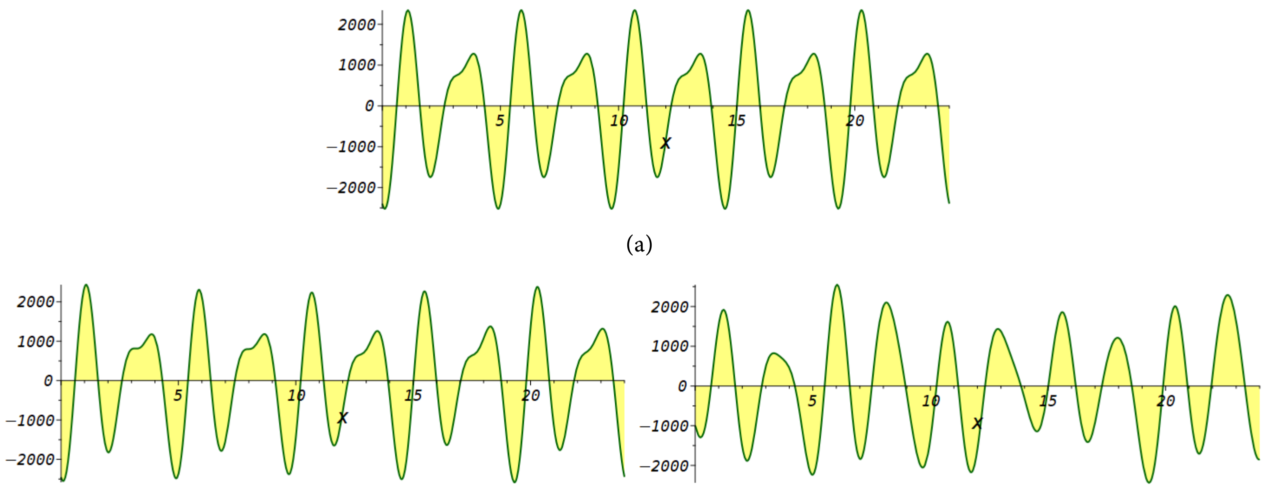
$$L_{e,t,x,d} = \mathrm{LCM}\left(L_{e,d,x,d}, L_{e,r,x,d}\right) = k_{e,t,d,x,d} L_{e,d,x,d} = k_{e,t,r,x,d} L_{e,r,x,d}, \quad (236)$$

where $k_{e,t,d,x,d}$ and $k_{e,t,r,x,d}$ are integers,

$$\begin{aligned} L_{e,d,x,d} &= \mathrm{LCM}\left(L_{d,x,2,m,n}\right) = k_{e,d,x,m,n} L_{d,x,2,m,n}, \\ L_{e,r,x,d} &= \mathrm{LCM}\left(L_{r,x,2,m,n}\right) = k_{e,r,x,m,n} L_{r,x,2,m,n} \end{aligned} \quad (237)$$

are wavelengths of the ddk-x and rdk-x oscillons, $k_{e,d,x,m,n}$ and $k_{e,r,x,m,n}$ are integers. The vanishing average of the tdk-x oscillon over $L_{e,t,x,d}$ demonstrates that the tdk-x oscillon on average does not transfer any kinetic energy along the x-axis, as well.

The ddk-x and tdk-x oscillons for $y = y_0, z = z_0, t = t_0, Re = 10^3, Re = 10^5$, and wave parameters (140), (145), (146) are shown on $L_{e,d,x,e}$ in **Figure 4**. Because the range of the 3-w, random, neutral rdk-x oscillon at $Re = 10^3$ is $[-105, 143]$, the 6-w, turbulent, neutral tdk-x oscillon in **Figure 4(b)** is modified insignificantly compared with the 3-w, deterministic, neutral ddk-x oscillon in **Figure 4(a)**. As the range of the 3-w, random, neutral rdk-x oscillon at $Re = 10^5$ increases to $[-850, 1500]$, shape of the 6-w, turbulent, neutral tdk-x oscillon in **Figure 4(c)** is changed significantly.



(a)

(b) (c)

Figure 4. The ddk-x and tdk-x oscillons: (a) $-K_{e,d,i,m,d,i,n}(x)$, (b) $-K_{e,t,i,m,t,i,n}(x)$ for $Re = 10^3$, (c) $-K_{e,t,i,m,t,i,n}(x)$ for $Re = 10^5$.

7. Oscillons of Turbulent, Internal Interaction

7.1. The TIE Oscillons

The turbulent, internal, elementary oscillons (the tie oscillons for pithiness) are defined as the superposition of the die oscillons ((139) of [7]) and the rie oscillons ((171) of [7]) in the following form:

$$K_{o,t,i,m,t,j,m} = K_{o,d,i,m,d,j,m} + K_{o,r,i,m,r,j,m}, \quad i = a,b,c, \quad j = i+1,\cdots,d. \tag{238}$$

Similar to the vector die-x and rie-x oscillons, six tie-x oscillons are grouped into three vector tie-x oscillons

$$\boldsymbol{K}_{t,i,e,x,q} = \boldsymbol{K}_{t,i,e,x,q}\left(\boldsymbol{f}_{t,i,e,x,q}\right), \quad q = 1,2,3, \tag{239}$$

which are formed by three 4-tuples of the turbulent, internal, elementary interaction in the x-direction:

$$\begin{aligned} \boldsymbol{f}_{t,i,e,x,1} &= \left\{\boldsymbol{f}_{d,i,e,x,1}, \boldsymbol{f}_{r,i,e,x,1}\right\} = \left\{g_{d,x,1,m,m}, g_{d,x,5,m,m}, g_{r,x,1,m,m}, g_{r,x,5,m,m}\right\}, \\ \boldsymbol{f}_{t,i,e,x,2} &= \left\{\boldsymbol{f}_{d,i,e,x,2}, \boldsymbol{f}_{r,i,e,x,2}\right\} = \left\{g_{d,x,4,m,m}, g_{r,x,4,m,m}, h_{d,y,2,m,m}, h_{r,y,2,m,m}\right\}, \\ \boldsymbol{f}_{t,i,e,x,3} &= \left\{\boldsymbol{f}_{d,i,e,x,3}, \boldsymbol{f}_{r,i,e,x,3}\right\} = \left\{g_{d,x,3,m,m}, g_{r,x,3,m,m}, h_{d,y,1,m,m}, h_{r,y,1,m,m}\right\}. \end{aligned} \tag{240}$$

Four-tuple $\boldsymbol{f}_{t,i,e,x,1}$ consists of two deterministic sine waves $g_{d,x,2k-1,m,m}$ with wavenumber $\kappa_{d,1,m,m}$ and two random sine waves $g_{r,x,2k-1,m,m}$ with wavenumber $\kappa_{r,1,m,m}$ for $k = 1,3$, and each m. Four-tuple $\boldsymbol{f}_{t,i,e,x,2}$ comprises deterministic cosine wave $g_{d,x,4,m,m}$ with wavenumber $\kappa_{d,1,m,m}$, random cosine wave $g_{r,x,4,m,m}$ with wavenumber $\kappa_{r,1,m,m}$, deterministic uniform pulson $h_{d,y,2,m,m}$, and random uniform pulson $h_{r,y,2,m,m}$ for each m. Four-tuple $\boldsymbol{f}_{t,i,e,x,3}$ is composed of deterministic sine wave $g_{d,x,3,m,m}$ with wavenumber $\kappa_{d,1,m,m}$, random sine wave $g_{r,x,3,m,m}$ with wavenumber $\kappa_{r,1,m,m}$, deterministic uniform pulson $h_{d,y,1,m,m}$, and random uniform pulson $h_{r,y,1,m,m}$ for each m.

For any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector tie-$x$ oscillon

$$\boldsymbol{K}_{t,i,e,x,1} = \left[K_{o,t,a,m,t,b,m}, K_{o,t,c,m,t,d,m}\right]\left(\boldsymbol{f}_{t,i,e,x,1}\right) \tag{241}$$

is reduced to a list of two 2-w, turbulent, neutral oscillons in x, which are produced by 4-tuple $\boldsymbol{f}_{t,i,e,x,1}$ as

$$\begin{aligned} K_{o,t,a,m,t,b,m} &= +\frac{\rho_c}{2}\left(ez_{d,m}^2 Q_{d,y,m}^2 g_{d,x,1,m,m} + ez_{r,m}^2 Q_{r,y,m}^2 g_{r,x,1,m,m}\right), \\ K_{o,t,c,m,t,d,m} &= +\frac{\rho_c}{2}\left(ez_{d,m}^2 R_{d,y,m}^2 g_{d,x,5,m,m} + ez_{r,m}^2 R_{r,y,m}^2 g_{r,x,5,m,m}\right). \end{aligned} \tag{242}$$

The 2nd vector tie-x oscillon

$$\boldsymbol{K}_{t,i,e,x,2} = \left[K_{o,t,a,m,t,c,m}, K_{o,t,b,m,t,d,m}\right]\left(\boldsymbol{f}_{t,i,e,x,2}\right) \tag{243}$$

is presented by a list of two 2-w, turbulent, non-neutral oscillons in x, which are generated by 4-tuple $\boldsymbol{f}_{t,i,e,x,2}$ since

$$\begin{aligned} K_{o,t,a,m,t,c,m} &= -\frac{\rho_c}{2}\Big[ez_{d,m}^2 Q_{d,y,m} R_{d,y,m}\left(g_{d,x,4,m,m} - h_{d,y,2,m,m}\right) \\ &\qquad + ez_{r,m}^2 Q_{r,y,m} R_{r,y,m}\left(g_{r,x,4,m,m} - h_{r,y,2,m,m}\right)\Big], \\ K_{o,t,b,m,t,d,m} &= +\frac{\rho_c}{2}\Big[ez_{d,m}^2 Q_{d,y,m} R_{d,y,m}\left(g_{d,x,4,m,m} + h_{d,y,2,m,m}\right) \\ &\qquad + ez_{r,m}^2 Q_{r,y,m} R_{r,y,m}\left(g_{r,x,4,m,m} + h_{r,y,2,m,m}\right)\Big]. \end{aligned} \tag{244}$$

The 3rd vector tie-x oscillon

$$\boldsymbol{K}_{t,i,e,x,3} = \left[K_{o,t,a,m,t,d,m}, K_{o,t,b,m,t,c,m}\right]\left(\boldsymbol{f}_{t,i,e,x,3}\right) \tag{245}$$

is visualized by a list of two 2-w, turbulent, non-neutral oscillons in x, which are formed by 4-tuple $\boldsymbol{f}_{t,i,e,x,3}$ because

$$\begin{aligned} K_{o,t,b,m,t,c,m} &= +\frac{\rho_c}{2}\Big[ez_{d,m}^2 Q_{d,y,m} R_{d,y,m}\left(g_{d,x,3,m,m} - h_{d,y,1,m,m}\right) \\ &\qquad + ez_{r,m}^2 Q_{r,y,m} R_{r,y,m}\left(g_{r,x,3,m,m} - h_{r,y,1,m,m}\right)\Big], \\ K_{o,t,a,m,t,d,m} &= +\frac{\rho_c}{2}\Big[ez_{d,m}^2 Q_{d,y,m} R_{d,y,m}\left(g_{d,x,3,m,m} + h_{d,y,1,m,m}\right) \\ &\qquad + ez_{r,m}^2 Q_{r,y,m} R_{r,y,m}\left(g_{r,x,3,m,m} + h_{r,y,1,m,m}\right)\Big]. \end{aligned} \tag{246}$$

Wave parameters of the vector tie-x oscillons are given by correspondent parameters of the vector die-x and rie-x oscillons. The wavelength of the tie-x oscillons

$$L_{t,x,m,m} = \mathrm{LCM}\left(L_{d,x,m,m}, L_{r,x,m,m}\right) = k_{t,d,x,m,m} L_{d,x,m,m} = k_{t,r,x,m,m} L_{r,x,m,m}, \tag{247}$$

where $k_{t,d,x,m,m}$ and $k_{t,r,x,m,m}$ are integers,

$$L_{d,x,m,m} = \frac{\pi}{\kappa_{d,m}}, \ L_{r,x,m,m} = \frac{\pi}{\kappa_{r,m}} \tag{248}$$

are wavelengths of the die-x and rie-x oscillons.

The average of members of the first vector tie-x oscillon over $L_{t,x,m,m}$ vanishes. The average of members of the second and third vector tie-x oscillons over $L_{t,x,m,m}$ takes the following form:

$$\begin{aligned}
&\frac{1}{L_{t,x,m,m}}\int_{0}^{L_{t,x,m,m}} K_{o,t,a,m,t,c,m}\mathrm{d}x = +\frac{1}{L_{t,x,m,m}}\int_{0}^{L_{t,x,m,m}} K_{o,t,b,m,t,d,m}\mathrm{d}x \\
&= +\frac{\rho_c}{2}\left(ez_{d,m}^{2}Q_{d,y,m}R_{d,y,m}h_{d,y,2,m,m} + ez_{r,m}^{2}Q_{r,y,m}R_{r,y,m}h_{r,y,2,m,m}\right), \\
&\frac{1}{L_{t,x,m,m}}\int_{0}^{L_{t,x,m,m}} K_{o,t,b,m,t,c,m}\mathrm{d}x = -\frac{1}{L_{t,x,m,m}}\int_{0}^{L_{t,x,m,m}} K_{o,t,a,m,t,d,m}\mathrm{d}x \\
&= -\frac{\rho_c}{2}\left(ez_{d,m}^{2}Q_{d,y,m}R_{d,y,m}h_{d,y,1,m,m} + ez_{r,m}^{2}Q_{r,y,m}R_{r,y,m}h_{r,y,1,m,m}\right).
\end{aligned} \tag{249}$$

Thus, the members of the first vector tie-x oscillon on average do not transfer the kinetic energy along the x-axis, the members of the second vector tie-x oscillon transfer equal positive amounts of the kinetic energy, and the members of the third vector tie-x oscillon transfer equal amounts of the kinetic energy of opposite signs. See (157)-(160) of [5], for the pulsatory nature of the kinetic energy of propagation of vector fields and the oscillatory nature of the kinetic energy of interaction of vector fields.

7.2. The TIW Oscillons

The turbulent, internal, wave oscillons (the tiw oscillons for pithiness) are constructed as the superposition of the diw oscillons ((141) of [7]) and the riw oscillons ((173) of [7]) as follows:

$$\begin{aligned}
K_{w,t,a,m,t,b,m} &= +K_{w,d,a,m,d,b,m} + K_{w,r,a,m,r,b,m} \\
= +K_{w,t,c,m,t,d,m} &= +K_{w,d,c,m,d,d,m} + K_{w,r,c,m,r,d,m}, \\
K_{w,t,a,m,t,c,m} &= +K_{w,d,a,m,d,c,m} + K_{w,r,a,m,r,c,m} \\
= +K_{w,t,b,m,t,d,m} &= +K_{w,d,b,m,d,d,m} + K_{w,r,b,m,r,d,m}, \\
K_{w,t,b,m,t,c,m} &= +K_{w,d,b,m,d,c,m} + K_{w,r,b,m,r,c,m} \\
= -K_{w,t,a,m,t,d,m} &= -K_{w,d,a,m,d,d,m} - K_{w,r,a,m,r,d,m}.
\end{aligned} \tag{250}$$

Alike the vector diw-x and riw-x oscillons, six tiw-x oscillons are combined in three vector tiw-x oscillons

$$\boldsymbol{K}_{t,i,w,x,q} = \boldsymbol{K}_{t,i,w,x,q}\left(\boldsymbol{f}_{t,i,w,x,q}\right),\ q = 1,2,3, \tag{251}$$

which are produced by 4-tuple and two 2-tuples of the turbulent, internal, wave interaction in x:

$$\begin{aligned}
\boldsymbol{f}_{t,i,w,x,1} &= \left\{\boldsymbol{f}_{d,i,w,x,1}, \boldsymbol{f}_{r,i,w,x,1}\right\} = \left\{g_{d,x,1,m,m}, g_{d,x,5,m,m}, g_{r,x,1,m,m}, g_{r,x,5,m,m}\right\}, \\
\boldsymbol{f}_{t,i,w,x,2} &= \left\{\boldsymbol{f}_{d,i,w,x,2}, \boldsymbol{f}_{r,i,w,x,2}\right\} = \left\{h_{d,y,2,m,m}, h_{r,y,2,m,m}\right\}, \\
\boldsymbol{f}_{t,i,w,x,3} &= \left\{\boldsymbol{f}_{d,i,w,x,3}, \boldsymbol{f}_{r,i,w,x,3}\right\} = \left\{h_{d,y,1,m,m}, h_{r,y,1,m,m}\right\}.
\end{aligned} \tag{252}$$

Four-tuple $\boldsymbol{f}_{t,i,w,x,1}$ is composed of two deterministic sine waves $g_{d,x,2k-1,m,m}$ with wavenumber $\kappa_{d,1,m,m}$ and two random sine waves $g_{r,x,2k-1,m,m}$ with wavenumber $\kappa_{r,1,m,m}$ for $k = 1,3$, and each m. Two-tuple $\boldsymbol{f}_{t,i,w,x,2}$ is constructed of deterministic uniform pulson $h_{d,y,2,m,m}$, and random uniform pulson $h_{r,y,2,m,m}$ for each m. Two-tuple $\boldsymbol{f}_{t,i,w,x,3}$ includes deterministic uniform pulson $h_{d,y,1,m,m}$, and random uniform pulson $h_{r,y,1,m,m}$ for each m.

For any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector tiw-$x$ oscillon

$$\boldsymbol{K}_{t,i,w,x,1} = \left[K_{w,t,a,m,t,b,m}, K_{w,t,c,m,t,d,m}\right]\left(\boldsymbol{f}_{t,i,w,x,1}\right) \tag{253}$$

is represented by a list of two 2-w, turbulent, neutral oscillons in x, which are formed by 4-tuple $\boldsymbol{f}_{t,i,w,x,1}$ in view of

$$\begin{aligned} K_{w,t,a,m,t,b,m} &= +K_{w,t,c,m,t,d,m} \\ &= +\frac{\rho_c}{2}\Big[ez_{d,m}^2\lambda_{d,m}^2\left(Q_{d,y,m}^2 g_{d,x,1,m,m} + R_{d,y,m}^2 g_{d,x,5,m,m}\right) \\ &\quad + ez_{r,m}^2\lambda_{r,m}^2\left(Q_{r,y,m}^2 g_{r,x,1,m,m} + R_{r,y,m}^2 g_{r,x,5,m,m}\right)\Big]. \end{aligned} \tag{254}$$

The 2nd vector tiw-x oscillon

$$\boldsymbol{K}_{t,i,w,x,2} = \left[K_{w,t,a,m,t,c,m}, K_{w,t,b,m,t,d,m}\right]\left(\boldsymbol{f}_{t,i,w,x,2}\right) \tag{255}$$

is exposed by a list of two 0-w, turbulent pulsons in x, which are established by 2-tuple $\boldsymbol{f}_{t,i,w,x,2}$ since

$$\begin{aligned} K_{w,t,a,m,t,c,m} = +K_{w,t,b,m,t,d,m} = +\rho_c\big(&ez_{d,m}^2\kappa_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,2,m,m} \\ &+ ez_{r,m}^2\kappa_{r,m}^2 Q_{r,y,m} R_{r,y,m} h_{r,y,2,m,m}\big). \end{aligned} \tag{256}$$

The 3rd vector tiw-x oscillon

$$\boldsymbol{K}_{t,i,e,x,3} = \left[K_{o,t,a,m,t,d,m}, K_{o,t,b,m,t,c,m}\right]\left(\boldsymbol{f}_{t,i,e,x,3}\right) \tag{257}$$

is exhibited by a list of two 0-w, turbulent pulsons in x, which depend on 2-tuple $\boldsymbol{f}_{t,i,w,x,3}$ as

$$\begin{aligned} K_{w,t,b,m,t,c,m} = -K_{w,t,a,m,t,d,m} = -\rho_c\big(&ez_{d,m}^2\mu_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,1,m,m} \\ &+ ez_{r,m}^2\mu_{r,m}^2 Q_{r,y,m} R_{r,y,m} h_{r,y,1,m,m}\big). \end{aligned} \tag{258}$$

Wave parameters of the vector tiw-x oscillons are provided by relevant parameters of the vector diw-x and riw-x oscillons. The wavelength of the tiw-x oscillons is given by $L_{t,x,m,m}$, as well. The average of members of the first vector tiw-x oscillon over $L_{t,x,m,m}$ also vanishes. The average of members of the second and third vector tiw-x oscillons over $L_{t,x,m,m}$ becomes

$$\begin{aligned} &\frac{1}{L_{t,x,m,m}}\int_0^{L_{t,x,m,m}} K_{w,t,a,m,t,c,m}\mathrm{d}x = +\frac{1}{L_{t,x,m,m}}\int_0^{L_{t,x,m,m}} K_{w,t,b,m,t,d,m}\mathrm{d}x \\ &= +\rho_c\left(ez_{d,m}^2\kappa_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,2,m,m} + ez_{r,m}^2\kappa_{r,m}^2 Q_{r,y,m} R_{r,y,m} h_{r,y,2,m,m}\right), \\ &\frac{1}{L_{t,x,m,m}}\int_0^{L_{t,x,m,m}} K_{w,t,b,m,t,c,m}\mathrm{d}x = -\frac{1}{L_{t,x,m,m}}\int_0^{L_{t,x,m,m}} K_{w,t,a,m,t,d,m}\mathrm{d}x \\ &= -\rho_c\left(ez_{d,m}^2\mu_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,1,m,m} + ez_{r,m}^2\mu_{r,m}^2 Q_{r,y,m} R_{r,y,m} h_{r,y,1,m,m}\right). \end{aligned} \tag{259}$$

So, the members of the first vector tiw-x oscillon on average does not transfer any kinetic energy along the x-axis, the members of the second vector tiw-x oscillon on average transfer along the x-axis equal amounts of the kinetic energy, and the members of the third vector tie-x oscillon on average transfer along the x-axis equal amounts of the kinetic energy of opposite signs.

7.3. The TIG Oscillon

The turbulent, internal, group oscillon (the tig oscillon for concision) is constructed as the superposition of the dig oscillon ((142) of [7]) and the rig oscillon ((174) of [7]), *i.e.*

$$K_{g,t,i,m,t,j,m} = K_{g,d,i,m,d,j,m} + K_{g,r,i,m,r,j,m}. \tag{260}$$

Summation of the dig-x and rig-x oscillons yields the inhomogeneous Fourier expansion of the tig-x oscillon in the following form:

$$K_{g,t,i,m,t,j,m} = K_{g,t,i,m,t,j,m}\left(\boldsymbol{f}_{t,i,g,x}\right), \tag{261}$$

where 6-tuple of the turbulent, internal, group interaction in the x-direction

$$\boldsymbol{f}_{t,i,g,x} = \left\{\boldsymbol{f}_{d,i,g,x}, \boldsymbol{f}_{r,i,g,x}\right\} = \left\{g_{d,x,2k-1,m,m}, g_{r,x,2k-1,m,m}, h_{d,y,2,m,m}, h_{r,y,2,m,m}\right\} \tag{262}$$

is composed of two deterministic sine waves $g_{d,x,2k-1,m,m}$ with wavenumber $\kappa_{d,1,m,m}$, two random sine waves $g_{r,x,2k-1,m,m}$ with wavenumber $\kappa_{r,1,m,m}$, deterministic uniform pulson $h_{d,y,2,m,m}$, and random uniform pulson $h_{r,y,2,m,m}$ for $k = 1,3$, and each m.

For any frozen $y = y_0, z = z_0, t = t_0$, the tig-$x$ oscillon is displayed by a 2-w, turbulent, non-neutral oscillon in x, which is generated by 6-tuple $\boldsymbol{f}_{t,i,g,x}$ since

$$\begin{aligned} &K_{g,t,i,m,t,j,m} = \\ &\rho_c \left\{ez_{d,m}^2\left[\lambda_{d,m}^2\left(Q_{d,y,m}^2 g_{d,x,1,m,m} + R_{d,y,m}^2 g_{d,x,5,m,m}\right) + 2\kappa_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,2,m,m}\right]\right. \\ &\quad \left.+ ez_{r,m}^2\left[\lambda_{r,m}^2\left(Q_{r,y,m}^2 g_{r,x,1,m,m} + R_{r,y,m}^2 g_{r,x,5,m,m}\right) + 2\kappa_{r,m}^2 Q_{r,y,m} R_{r,y,m} h_{r,y,2,m,m}\right]\right\}. \end{aligned} \tag{263}$$

Wave parameters of the tig-x oscillon are specified by appropriate parameters of the dig-x and rig-x oscillons. The wavelength of the tig-x oscillon is also given by $L_{t,x,m,m}$. The average of the tig-x oscillon over $L_{t,x,m,m}$

$$\begin{aligned} &\frac{1}{L_{t,x,m,m}} \int_0^{L_{t,x,m,m}} K_{g,t,i,m,t,j,m} \mathrm{d}x = \\ &2\rho_c\left(ez_{d,m}^2 \kappa_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,2,m,m} + ez_{r,m}^2 \kappa_{r,m}^2 Q_{r,y,m} R_{r,y,m} h_{r,y,2,m,m}\right) \end{aligned} \tag{264}$$

proves that the tig-x oscillon on average is non-neutral as it transfers the kinetic energy along the x-axis.

7.4. The TIK Oscillon

The turbulent, internal, kinetic-energy oscillon (the tik oscillon for shortness), which is defined as the superposition of the dik-x and rik-x oscillons, takes the following form:

$$K_{e,t,i,m,t,j,m} = K_{e,d,i,m,d,j,m} + K_{e,r,i,m,r,j,m}. \tag{265}$$

With the help of (72) and (115) of [7], the inhomogeneous Fourier expansion of the tik-x oscillon becomes

$$K_{e,t,i,m,t,j,m} = \sum_{m=1}^{M} K_{g,t,i,m,t,j,m}. \tag{266}$$

Since all wavenumbers of the tik-x oscillon are distinct, the tik-x oscillon is visualized as a $2M$-w oscillon. For any frozen $y = y_0, z = z_0, t = t_0$, the tik-$x$ oscillon is represented by the $2M$-w, turbulent, non-neutral oscillon in x that is produced by M 6-tuples $f_{t,i,g,x}$ with wavenumbers $\kappa_{d,1,m,m}, \kappa_{r,1,m,m}$ for each m, Re, and wave parameters of the tig-x oscillons.

The wavelength of the tik-x oscillon

$$L_{e,t,x,i} = \mathrm{LCM}\left(L_{e,d,x,i}, L_{e,r,x,i}\right) = k_{e,t,d,x,i}L_{e,d,x,i} = k_{e,t,r,x,i}L_{e,r,x,i}, \tag{267}$$

where $k_{e,t,d,x,i}$ and $k_{e,t,r,x,i}$ are integers,

$$\begin{aligned} L_{e,d,x,i} &= \mathrm{LCM}\left(L_{d,x,m,m}\right) = k_{d,x,m,m}L_{d,x,m,m}, \\ L_{e,r,x,i} &= \mathrm{LCM}\left(L_{r,x,m,m}\right) = k_{r,x,m,m}L_{r,x,m,m} \end{aligned} \tag{268}$$

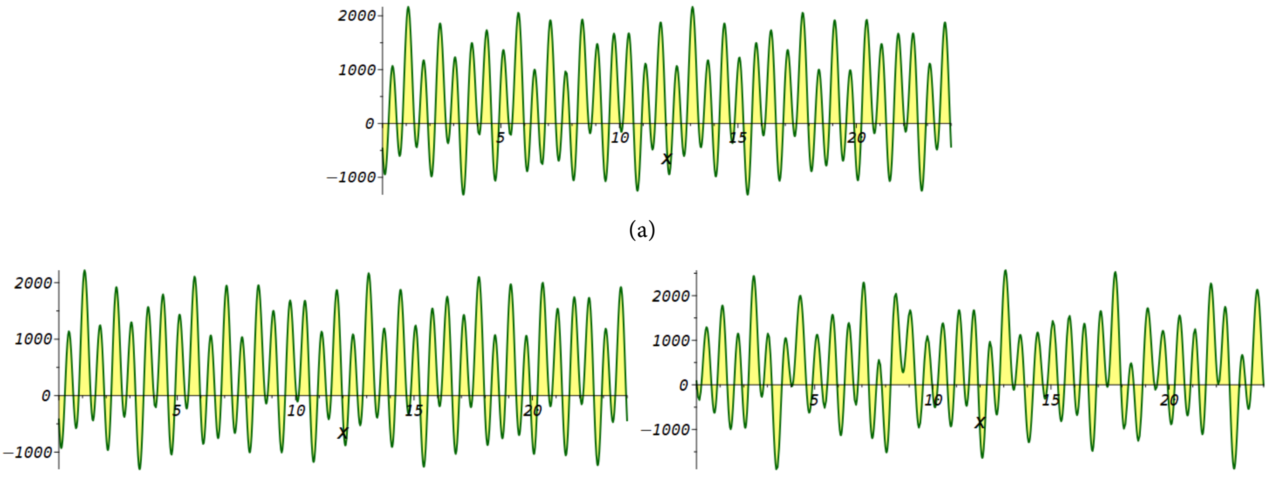
are the wavelengths of the dik-x and rik-x oscillons, respectively, $k_{d,x,m,m}$ and $k_{r,x,m,m}$ are integers.

The average of the tik-x oscillon over $L_{e,t,x,i}$,

$$\begin{aligned} &\frac{1}{L_{e,t,x,i}}\int_0^{L_{e,t,x,i}} K_{e,t,i,m,t,j,m}\mathrm{d}x = \\ &2\rho_c\sum_{m=1}^{M}\left(ez_{d,m}^2\kappa_{d,m}^2 Q_{d,y,m}R_{d,y,m}h_{d,y,2,m,m} + ez_{r,m}^2\kappa_{r,m}^2 Q_{r,y,m}R_{r,y,m}h_{r,y,2,m,m}\right), \end{aligned} \tag{269}$$

shows that the tik-x oscillon on average is also non-neutral since it transfers the kinetic energy along the x-axis.

The dik-x and tik-x oscillons for $y = y_0, z = z_0, t = t_0, Re = 10^3, Re = 10^5$, and wave parameters (140), (145), (146) are presented on $L_{e,d,x,e}$ in Figure 5. The 6-w, turbulent, non-neutral tik-x oscillon in Figure 5(b) changes insignificantly compared with the 3-w, deterministic, non-neutral dik-x oscillon in Figure 5(a) because the range of the 3-w, random, non-neutral rik-x oscillon at $Re = 10^3$ is $[-8, 83]$. Shape of the 6-w, turbulent, non-neutral tik-x oscillon in Figure 5(c) is modified significantly as the range of the 3-w, random, non-neutral rik-x oscillon at $Re = 10^5$ becomes $[-720, 630]$.



(a)

(b)

(c)

Figure 5. The dik-x and tik-x oscillons: (a) $-K_{e,t,i,m,t,j,m}(x)$, (b) $-K_{e,t,i,m,t,j,m}(x)$ for $Re = 10^3$, (c) $-K_{e,t,i,m,t,j,m}(x)$ for $Re = 10^5$.

8. Turbulent and Cumulative Pulsons

8.1. The TE Pulsons

The turbulent, elementary pulsons (the te pulsons for briefness) are set as the superposition of the de pulsons ((134) of [7]) and the re pulsons ((166) of [7]) as follows:

$$K_{p,t,i,m,t,i,m} = K_{p,d,i,m,d,i,m} + K_{p,r,i,m,r,i,m}, \quad i = a,b,c,d. \tag{270}$$

Like the vector de-x and re-x pulsons, four te-x pulsons are grouped into two vector te-x pulsons

$$\boldsymbol{K}_{t,e,x,q} = \boldsymbol{K}_{t,e,x,q}\left(\boldsymbol{f}_{t,e,x,q}\right), \quad q = 1,2, \tag{271}$$

which are constructed using two 3-tuples of the turbulent, elementary, pulsons propagating in the x-direction:

$$\begin{aligned} \boldsymbol{f}_{t,e,x,1} &= \left\{\boldsymbol{f}_{d,e,x,1}, \boldsymbol{f}_{r,e,x,1}\right\} = \left\{1, g_{d,x,2,m,m}, g_{r,x,2,m,m}\right\}, \\ \boldsymbol{f}_{t,e,x,2} &= \left\{\boldsymbol{f}_{d,e,x,2}, \boldsymbol{f}_{r,e,x,2}\right\} = \left\{1, g_{d,x,6,m,m}, g_{r,x,6,m,m}\right\}. \end{aligned} \tag{272}$$

Three-tuple $\boldsymbol{f}_{t,e,x,1}$ embraces the unit pulson, deterministic cosine wave $g_{d,x,2,m,m}$ with wavenumber $\kappa_{d,1,m,m}$ and random cosine wave $g_{r,x,2,m,m}$ with wavenumber $\kappa_{r,1,m,m}$ for each m. Three-tuple $\boldsymbol{f}_{t,e,x,2}$ includes the unit pulson, deterministic cosine wave $g_{d,x,6,m,m}$ with wavenumber $\kappa_{d,1,m,m}$ and random cosine wave $g_{r,x,6,m,m}$ with wavenumber $\kappa_{r,1,m,m}$ for each m.

For any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector te-$x$ pulson

$$\boldsymbol{K}_{t,e,x,1} = \left[K_{p,t,a,m,t,a,m}, K_{p,t,b,m,t,b,m}\right]\left(\boldsymbol{f}_{t,e,x,1}\right) \tag{273}$$

is represented by a list of two 2-w, turbulent, supercritical pulsons in x, which depend on 3-tuple $\boldsymbol{f}_{t,e,x,1}$ seeing that

$$\begin{aligned} K_{p,t,a,m,t,a,m} &= -\frac{\rho_c}{4}\left[ez_{d,m}^2 Q_{d,y,m}^2\left(g_{d,x,2,m,m} - 1\right) + ez_{r,m}^2 Q_{r,y,m}^2\left(g_{r,x,2,m,m} - 1\right)\right], \\ K_{p,t,b,m,t,b,m} &= +\frac{\rho_c}{4}\left[ez_{d,m}^2 Q_{d,y,m}^2\left(g_{d,x,2,m,m} + 1\right) + ez_{r,m}^2 Q_{r,y,m}^2\left(g_{r,x,2,m,m} + 1\right)\right]. \end{aligned} \tag{274}$$

The 2nd vector te-x pulson

$$\boldsymbol{K}_{t,e,x,2} = \left[K_{p,t,c,m,t,c,m}, K_{p,t,d,m,t,d,m}\right]\left(\boldsymbol{f}_{t,e,x,2}\right) \tag{275}$$

is displayed by a list of two 2-w, turbulent, supercritical pulsons in x, which are controlled by 3-tuple $\boldsymbol{f}_{t,e,x,2}$ considering that

$$\begin{aligned} K_{p,t,c,m,t,c,m} &= -\frac{\rho_c}{4}\left[ez_{d,m}^2 R_{d,y,m}^2\left(g_{d,x,6,m,m} - 1\right) + ez_{r,m}^2 R_{r,y,m}^2\left(g_{r,x,6,m,m} - 1\right)\right], \\ K_{p,t,d,m,t,d,m} &= +\frac{\rho_c}{4}\left[ez_{d,m}^2 R_{d,y,m}^2\left(g_{d,x,6,m,m} + 1\right) + ez_{r,m}^2 R_{r,y,m}^2\left(g_{r,x,6,m,m} + 1\right)\right]. \end{aligned} \tag{276}$$

Wave parameters of the vector te-x pulsons are indicated by similar parameters of the vector de-x and re-x pulsons. The wavelength of the te-x pulsons is given by $L_{t,x,m,m}$. The average of the te-x pulsons over $L_{t,x,m,m}$ becomes

$$\begin{aligned}
&\frac{1}{L_{t,x,m,m}}\int_{0}^{L_{t,x,m,m}} K_{p,t,a,m,t,a,m}\mathrm{d}x = \frac{1}{L_{t,x,m,m}}\int_{0}^{L_{t,x,m,m}} K_{p,t,b,m,t,b,m}\mathrm{d}x \\
&\qquad = \frac{\rho_c}{4}\left(ez_{d,m}^2 Q_{d,y,m}^2 + ez_{r,m}^2 Q_{r,y,m}^2\right), \\
&\frac{1}{L_{t,x,m,m}}\int_{0}^{L_{t,x,m,m}} K_{p,t,c,m,t,c,m}\mathrm{d}x = \frac{1}{L_{t,x,m,m}}\int_{0}^{L_{t,x,m,m}} K_{p,t,d,m,t,d,m}\mathrm{d}x \\
&\qquad = \frac{\rho_c}{4}\left(ez_{d,m}^2 R_{d,y,m}^2 + ez_{r,m}^2 R_{r,y,m}^2\right).
\end{aligned} \tag{277}$$

Thus, the vector te-x pulsons on average transfer positive amounts of the kinetic energy along the x-axis.

8.2. The TW Pulsons

The turbulent, wave pulsons (the tw pulsons for conciseness) are composed with the help of the superposition of the dw pulsons ((136) of [7]) and the rw pulsons ((168) of [7]) in the following form:

$$K_{w,t,i,m,t,i,m} = K_{w,d,i,m,d,i,m} + K_{w,r,i,m,r,i,m}, \quad i = a,b,c,d, \tag{278}$$

Parallel to the vector dw-x and rw-x pulsons, four tw-x pulsons are assembled into two vector tw-x pulsons

$$\boldsymbol{K}_{t,w,x,q} = \boldsymbol{K}_{t,w,x,q}\left(\boldsymbol{f}_{t,w,x}\right), \quad q = 1,2, \tag{279}$$

which are formed by 5-tuple of the turbulent, wave pulsons propagating in the x-direction:

$$\boldsymbol{f}_{t,w,x} = \left\{\boldsymbol{f}_{d,w,x}, \boldsymbol{f}_{r,w,x}\right\} = \left\{1, g_{d,x,2k,m,m}, g_{r,x,2k,m,m}\right\}. \tag{280}$$

Five-tuple $\boldsymbol{f}_{t,w,x}$ consists of the unit pulson, deterministic cosine wave $g_{d,x,2k,m,m}$ with wavenumber $\kappa_{d,1,m,m}$ and random cosine waves $g_{r,x,2k,m,m}$ with wavenumber $\kappa_{r,1,m,m}$ for $k = 1,3$, and each m.

For any frozen $y = y_0, z = z_0, t = t_0$, the 1st vector tw-$x$ pulson

$$\boldsymbol{K}_{t,w,x,1} = \left[K_{w,t,a,m,t,a,m}, K_{w,t,b,m,t,b,m}\right]\left(\boldsymbol{f}_{t,w,x}\right) \tag{281}$$

is exhibited by a list of two 2-w, turbulent, supercritical pulsons in x, which are established by 5-tuple $\boldsymbol{f}_{t,w,x}$ for the reason that

$$\begin{aligned}
K_{w,t,a,m,t,a,m} = \frac{\rho_c}{4}\Big\{ & ez_{d,m}^2\Big[\left(2\kappa_{d,m}^2 + \lambda_{d,m}^2\right)Q_{d,y,m}^2 + \lambda_{d,m}^2 R_{d,y,m}^2 \\
& - \lambda_{d,m}^2\left(Q_{d,y,m}^2 g_{d,x,2,m,m} + R_{d,y,m}^2 g_{d,x,6,m,m}\right)\Big] \\
& + ez_{r,m}^2\Big[\left(2\kappa_{r,m}^2 + \lambda_{r,m}^2\right)Q_{r,y,m}^2 + \lambda_{r,m}^2 R_{r,y,m}^2 \\
& - \lambda_{r,m}^2\left(Q_{r,y,m}^2 g_{r,x,2,m,m} + R_{r,y,m}^2 g_{r,x,6,m,m}\right)\Big]\Big\}, \\
K_{w,t,b,m,t,b,m} = \frac{\rho_c}{4}\Big\{ & ez_{d,m}^2\Big[\left(2\kappa_{d,m}^2 + \lambda_{d,m}^2\right)Q_{d,y,m}^2 + \lambda_{d,m}^2 R_{d,y,m}^2 \\
& + \lambda_{d,m}^2\left(Q_{d,y,m}^2 g_{d,x,2,m,m} + R_{d,y,m}^2 g_{d,x,6,m,m}\right)\Big] \\
& + ez_{r,m}^2\Big[\left(2\kappa_{r,m}^2 + \lambda_{r,m}^2\right)Q_{r,y,m}^2 + \lambda_{r,m}^2 R_{r,y,m}^2 \\
& + \lambda_{r,m}^2\left(Q_{r,y,m}^2 g_{r,x,2,m,m} + R_{r,y,m}^2 g_{r,x,6,m,m}\right)\Big]\Big\}.
\end{aligned} \tag{282}$$

The 2nd vector tw-x pulson

$$\boldsymbol{K}_{t,w,x,2} = \left[K_{w,t,c,m,t,c,m}, K_{w,t,d,m,t,d,m}\right]\left(\boldsymbol{f}_{t,w,x}\right) \tag{283}$$

is exposed by a list of two 2-w, turbulent, supercritical pulsons in x, which are also set by 5-tuple $\boldsymbol{f}_{t,w,x}$ because

$$\begin{aligned} K_{w,t,c,m,t,c,m} = \frac{\rho_c}{4}\Big\{ & ez_{d,m}^2\Big[\lambda_{d,m}^2 Q_{d,y,m}^2 + \left(2\kappa_{d,m}^2 + \lambda_{d,m}^2\right)R_{d,y,m}^2 \\ & - \lambda_{d,m}^2\left(Q_{d,y,m}^2 g_{d,x,2,m,m} + R_{d,y,m}^2 g_{d,x,6,m,m}\right)\Big] \\ & + ez_{r,m}^2\Big[\lambda_{r,m}^2 Q_{r,y,m}^2 + \left(2\kappa_{r,m}^2 + \lambda_{r,m}^2\right)R_{r,y,m}^2 \\ & - \lambda_{r,m}^2\left(Q_{r,y,m}^2 g_{r,x,2,m,m} + R_{r,y,m}^2 g_{r,x,6,m,m}\right)\Big]\Big\}, \end{aligned}$$

$$\begin{aligned} K_{w,t,d,m,t,d,m} = \frac{\rho_c}{4}\Big\{ & ez_{d,m}^2\Big[\lambda_{d,m}^2 Q_{d,y,m}^2 + \left(2\kappa_{d,m}^2 + \lambda_{d,m}^2\right)R_{d,y,m}^2 \\ & + \lambda_{d,m}^2\left(Q_{d,y,m}^2 g_{d,x,2,m,m} + R_{d,y,m}^2 g_{d,x,6,m,m}\right)\Big] \\ & + ez_{r,m}^2\Big[\lambda_{r,m}^2 Q_{r,y,m}^2 + \left(2\kappa_{r,m}^2 + \lambda_{r,m}^2\right)R_{r,y,m}^2 \\ & + \lambda_{r,m}^2\left(Q_{r,y,m}^2 g_{r,x,2,m,m} + R_{r,y,m}^2 g_{r,x,6,m,m}\right)\Big]\Big\}. \end{aligned} \tag{284}$$

The vector tw-x pulsons vary by the structure of amplitudes of eigenfunctions. Wave parameters of the vector tw-x pulsons are given by relevant parameters of the vector dw-x and rw-x pulsons. The wavelength of the tw-x pulsons is also provided by $L_{t,x,m,m}$.

The average of the tw-x pulsons over $L_{t,x,m,m}$ becomes

$$\begin{aligned} & \frac{1}{L_{t,x,m,m}}\int_0^{L_{t,x,m,m}} K_{w,t,a,m,t,a,m}\mathrm{d}x = \frac{1}{L_{t,x,m,m}}\int_0^{L_{t,x,m,m}} K_{w,t,b,m,t,b,m}\mathrm{d}x \\ & = \frac{\rho_c}{4}\Big\{ez_{d,m}^2\Big[\lambda_{d,m}^2\left(Av_{d,m}^2 + Bv_{d,m}^2 + Cv_{d,m}^2 + Dv_{d,m}^2\right) + 2\kappa_{d,m}^2 Q_{d,y,m}^2\Big] \\ & \quad + ez_{r,m}^2\Big[\lambda_{r,m}^2\left(Av_{r,m}^2 + Bv_{r,m}^2 + Cv_{r,m}^2 + Dv_{r,m}^2\right) + 2\kappa_{r,m}^2 Q_{r,y,m}^2\Big]\Big\}, \\ & \frac{1}{L_{t,x,m,m}}\int_0^{L_{t,x,m,m}} K_{w,t,c,m,t,c,m}\mathrm{d}x = \frac{1}{L_{t,x,m,m}}\int_0^{L_{t,x,m,m}} K_{w,t,d,m,t,d,m}\mathrm{d}x \\ & = \frac{\rho_c}{4}\Big\{ez_{d,m}^2\Big[\lambda_{d,m}^2\left(Av_{d,m}^2 + Bv_{d,m}^2 + Cv_{d,m}^2 + Dv_{d,m}^2\right) + 2\kappa_{d,m}^2 R_{d,y,m}^2\Big] \\ & \quad + ez_{r,m}^2\Big[\lambda_{r,m}^2\left(Av_{r,m}^2 + Bv_{r,m}^2 + Cv_{r,m}^2 + Dv_{r,m}^2\right) + 2\kappa_{r,m}^2 R_{r,y,m}^2\Big]\Big\}. \end{aligned} \tag{285}$$

So, the members of the vector tw-x pulsons on average transfer equal amounts of the kinetic energy along the x-axis.

8.3. The TG Pulson

The turbulent, group pulson (the tg pulson for brevity) is set as the superposition of the dg pulson ((137) of [7]) and the rg pulson ((169) of [7]), *viz.*

$$K_{g,t,i,m,t,i,m} = K_{g,d,i,m,d,i,m} + K_{g,r,i,m,r,i,m}. \tag{286}$$

Summation of the dg-x and rg-x pulsons yields the tg-x pulson in the following form:

$$K_{g,t,i,m,t,i,m} = K_{g,t,i,m,t,i,m}\left(\boldsymbol{f}_{t,g,x}\right), \tag{287}$$

where 1-tuple of the turbulent, group pulsons propagating in the *x*-direction

$$\boldsymbol{f}_{t,g,x} = \left\{\boldsymbol{f}_{d,g,x}, \boldsymbol{f}_{r,g,x}\right\} = \{1\} \tag{288}$$

consists of the unit pulson because of the identity resonance.

For any frozen $z = z_0$ and all y, t, the tg-*x* pulson is represented by the 0-w (uniform), turbulent, supercritical pulson in *x*, which is produced by 1-tuple $\boldsymbol{f}_{t,g,x}$ with wave parameters indicated by correspondent parameters of the dg-*x* and rg-*x* pulsons since

$$\begin{aligned} K_{g,t,i,m,t,i,m} = \rho_c\Big[& ez^2_{d,m}\mu^2_{d,m}\left(Av^2_{d,m} + Bv^2_{d,m} + Cv^2_{d,m} + Dv^2_{d,m}\right) \\ &+ ez^2_{r,m}\mu^2_{r,m}\left(Av^2_{r,m} + Bv^2_{r,m} + Cv^2_{r,m} + Dv^2_{r,m}\right)\Big]. \end{aligned} \tag{289}$$

Since the tg-*x* pulson is uniform in space,

$$\begin{aligned} \frac{1}{L_{t,x,m,m}}\int_0^{L_{t,x,m,m}} K_{g,t,i,m,t,i,m}\,\mathrm{d}x = \rho_c\Big[& ez^2_{d,m}\mu^2_{d,m}\left(Av^2_{d,m} + Bv^2_{d,m} + Cv^2_{d,m} + Dv^2_{d,m}\right) \\ &+ ez^2_{r,m}\mu^2_{r,m}\left(Av^2_{r,m} + Bv^2_{r,m} + Cv^2_{r,m} + Dv^2_{r,m}\right)\Big]. \end{aligned} \tag{290}$$

Therefore, the tg-*x* pulson transfers a positive amount of the kinetic energy along the *x*-axis.

8.4. The TK Pulson

The turbulent, kinetic-energy pulson (the tk pulson for easiness) is written as the superposition of the dk pulson ((58) of [7]) and the rk pulson ((101) of [7]) by

$$K_{e,t,i,m,t,i,m} = K_{e,d,i,m,d,i,m} + K_{e,r,i,m,r,i,m} = \sum_{m=1}^{M} K_{g,t,i,m,t,i,m}. \tag{291}$$

For any frozen $z = z_0$ and all y, t, the tk-*x* pulson also becomes a 0-w, turbulent, supercritical pulson in *x*, which is formed by 1-tuple $\boldsymbol{f}_{t,g,x}$ with the same wave parameters as the tg-*x* pulson for each *m*, *Re*.

Because the tk pulson is also uniform in space,

$$\begin{aligned} \frac{1}{L_{e,t,x,i}}\int_0^{L_{e,t,x,i}} K_{e,t,a,m,t,a,m}\,\mathrm{d}x = \rho_c\sum_{m=1}^{M}\Big[& ez^2_{d,m}\mu^2_{d,m}\left(Av^2_{d,m} + Bv^2_{d,m} + Cv^2_{d,m} + Dv^2_{d,m}\right) \\ &+ ez^2_{r,m}\mu^2_{r,m}\left(Av^2_{r,m} + Bv^2_{r,m} + Cv^2_{r,m} + Dv^2_{r,m}\right)\Big]. \end{aligned} \tag{292}$$

Thus, the tk-*x* pulson on average transfers a positive amount of the kinetic energy along the *x*-axis.

8.5. The DCK Pulson

The deterministic, cumulative, kinetic-energy pulson in the *x*-direction (the dck-*x* pulson for simplicity), which is the superposition of the dk-*x* pulson and the dik-*x*, ddk-*x*, and dek-*x* oscillons, namely,

$$\begin{aligned} K_{e,d,d} &= K_{e,d,i,m,d,i,m} + K_{e,d,i,m,d,j,m} + K_{e,d,i,m,d,i,n} + K_{e,d,i,m,d,j,n} \\ &= K_{e,d,d}\left(\boldsymbol{f}_{d,c,i,x}, \boldsymbol{f}_{d,c,e,x}\right) \end{aligned} \tag{293}$$

depends on 4-tuple of the deterministic, cumulative, internal interaction along the x-axis

$$\boldsymbol{f}_{d,c,i,x} = \left\{ \boldsymbol{f}_{d,g,x}, \boldsymbol{f}_{d,i,g,x} \right\} = \left\{ 1, g_{d,x,4k-3,m,m}, h_{d,y,2,m,m} \right\} \tag{294}$$

and 8-tuple of the deterministic, cumulative, external interaction in the x-direction

$$\boldsymbol{f}_{d,c,e,x} = \left\{ \boldsymbol{f}_{d,d,g,x}, \boldsymbol{f}_{d,e,g,x} \right\} = \left\{ f_{d,x,4k-3,m,n}, f_{d,x,4k,m,n} \right\}. \tag{295}$$

Four-tuple $\boldsymbol{f}_{d,c,i,x}$ consists of the unit pulson, two sine waves $g_{d,x,4k-3,m,m}$ with wavenumber $\kappa_{d,1,m,m}$ and uniform pulson $h_{d,y,2,m,m}$ for $k = 1, 2,$ and each m. Eight-tuple $\boldsymbol{f}_{d,c,e,x}$ comprises four sine waves $f_{d,x,4k-3,m,n}$ with wavenumber $\kappa_{d,1,m,n}$ and four cosine waves $f_{d,x,4k,m,n}$ with wavenumber $\kappa_{d,2,m,n}$ for $k = 1, 2, 3, 4$, and each m, n.

If all wavenumbers of the dck-x pulson are distinct, then the dck-x pulson is visualized by a M^2-w pulson. For any frozen $y = y_0, z = z_0, t = t_0$, the dck-$x$ pulson is represented by the M^2-w, deterministic, supercritical pulson in x, which is produced by M 4-tuples $\boldsymbol{f}_{d,c,i,x}$ with wavenumber $\kappa_{d,1,m,m}$ for all m and $M(M-1)/2$ 8-tuples $\boldsymbol{f}_{d,c,e,x}$ with wavenumbers $\kappa_{d,1,m,n}, \kappa_{d,2,m,n}$ for all m, n because

$$\begin{aligned}
K_{e,d,d} = \rho_c \Bigg\{ & \sum_{m=1}^{M} ez_{d,m}^2 \Big[\mu_{d,m}^2 \left(Av_{d,m}^2 + Bv_{d,m}^2 + Cv_{d,m}^2 + Dv_{d,m}^2 \right) \\
& + \lambda_{d,m}^2 \left(Q_{d,y,m}^2 g_{d,x,1,m,m} + R_{d,y,m}^2 g_{d,x,5,m,m} \right) + 2\kappa_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,2,m,m} \Big] \\
& + \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} ez_{d,m} ez_{d,n} \Big[\Lambda_{d,m,d,n} \left(Q_{d,y,m} Q_{d,y,n} f_{d,x,1,m,n} + R_{d,y,m} R_{d,y,n} f_{d,x,13,m,n} \right) \\
& \qquad - \mathrm{N}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,5,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,9,m,n} \right) \\
& \qquad + \mathrm{M}_{d,m,d,n} \left(Q_{d,y,m} Q_{d,y,n} f_{d,x,4,m,n} + R_{d,y,m} R_{d,y,n} f_{d,x,16,m,n} \right) \\
& \qquad + \mathrm{K}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,8,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,12,m,n} \right) \Big] \Bigg\}.
\end{aligned} \tag{296}$$

If N wavenumbers are repeated, then the number of independent modes of the dck-x pulson diminishes to $M^2 - N$ due to the wavenumber resonance.

The wavelength of the dck-x pulson is given by

$$L_{e,d,x} = \mathrm{LCM}\left(L_{e,d,x,i}, L_{e,d,x,d}, L_{e,d,x,e} \right) = k_{e,d,x,i} L_{e,d,x,i} = k_{e,d,x,d} L_{e,d,x,d} = k_{e,d,x,e} L_{e,d,x,e}, \tag{297}$$

where $k_{e,d,x,i}, k_{e,d,x,d}, k_{e,d,x,e}$ are integers.

Combining the average of the dk-x pulson and the dik-x, ddk-x, dek-x oscillons, the average of the dck-x pulson over $L_{e,d,x}$ becomes

$$\begin{aligned}
\frac{1}{L_{e,d,x}} \int_0^{L_{e,d,x}} K_{e,d,d} \mathrm{d}x = \rho_c \sum_{m=1}^{M} ez_{d,m}^2 \Big[& \mu_{d,m}^2 \left(Av_{d,m}^2 + Bv_{d,m}^2 + Cv_{d,m}^2 + Dv_{d,m}^2 \right) \\
& + 2\kappa_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,2,m,m} \Big].
\end{aligned} \tag{298}$$

Since

$$K_{e,d,d}\left(x,y_0,z_0,t_0\right)=\frac{\rho_c}{2}\left(u_{d,x}^2+u_{d,y}^2+u_{d,z}^2\right)\left(x,y_0,z_0,t_0\right), \tag{299}$$

where $u_{d,x},u_{d,y},u_{d,z}$ are *x*-, *y*-, *z*-components of deterministic velocity $\boldsymbol{u}_d$, the dck-*x* pulson remains positive for all *x* and transfers a positive amount of the kinetic energy along the *x*-axis.

8.6. The RCK Pulson

The random, cumulative, kinetic-energy pulson along the *x*-axis (the rck-*x* pulson for fastness), which is constructed as the superposition of the rk-*x* pulson and the rik-*x*, rdk-*x*, rek-*x* oscillons,

$$\begin{aligned}K_{e,r,r}&=K_{e,r,i,m,r,i,m}+K_{e,r,i,m,r,j,m}+K_{e,r,i,m,r,i,n}+K_{e,r,i,m,r,j,n}\\&=K_{e,r,r}\left(\boldsymbol{f}_{r,c,i,x},\boldsymbol{f}_{r,c,e,x}\right)\end{aligned} \tag{300}$$

is produced by 4-tuple of the random, cumulative, internal interaction in the *x*-direction

$$\boldsymbol{f}_{r,c,i,x}=\left\{\boldsymbol{f}_{r,g,x},\boldsymbol{f}_{r,i,g,x}\right\}=\left\{1,g_{r,x,4k-3,m,m},h_{r,y,2,m,m}\right\} \tag{301}$$

and 8-tuple of the random, cumulative, external interaction along the *x*-axis

$$\boldsymbol{f}_{r,c,e,x}=\left\{\boldsymbol{f}_{r,d,g,x},\boldsymbol{f}_{r,e,g,x}\right\}=\left\{f_{r,x,4k-3,m,n},f_{r,x,4k,m,n}\right\}. \tag{302}$$

Four-tuple $\boldsymbol{f}_{r,c,i,x}$ is composed of the unit pulson, two sine waves $g_{r,x,4k-3,m,m}$ with wavenumber $\kappa_{r,1,m,m}$ and uniform pulson $h_{r,y,2,m,m}$ for $k=1,2$, and each *m*. Eight-tuple $\boldsymbol{f}_{r,c,e,x}$ includes four sine waves $f_{r,x,4k-3,m,n}$ with wavenumber $\kappa_{r,1,m,n}$ and four cosine waves $f_{r,x,4k,m,n}$ with wavenumber $\kappa_{r,2,m,n}$ for $k=1,2,3,4$, and each *m*, *n*.

If all wavenumbers of the rck-*x* pulson are distinct, then the rck-*x* pulson is exposed as a *M*²-w pulson. For any frozen $y=y_0,z=z_0,t=t_0$, the rck-*x* pulson is exhibited by the *M*²-w, random, supercritical pulson in *x*, which is established by *M* 4-tuples $\boldsymbol{f}_{r,c,i,x}$ with wavenumber $\kappa_{r,1,m,m}$ for all *m*, *Re* and $M(M-1)/2$ 8-tuples $\boldsymbol{f}_{r,c,e,x}$ with wavenumbers $\kappa_{r,1,m,n},\kappa_{r,2,m,n}$ for all *m*, *n*, *Re* since

$$\begin{aligned}K_{e,r,r}=\rho_c&\left\{\sum_{m=1}^{M}ez_{r,m}^2\left[\mu_{r,m}^2\left(Av_{r,m}^2+Bv_{r,m}^2+Cv_{r,m}^2+Dv_{r,m}^2\right)\right.\right.\\&+\left.\lambda_{r,m}^2\left(Q_{r,y,m}^2g_{r,x,1,m,m}+R_{r,y,m}^2g_{r,x,5,m,m}\right)+2\kappa_{r,m}^2Q_{r,y,m}R_{r,y,m}h_{r,y,2,m,m}\right]\\&+\sum_{m=1}^{M-1}\sum_{n=m+1}^{M}ez_{r,m}ez_{r,n}\left[\Lambda_{r,m,r,n}\left(Q_{r,y,m}Q_{r,y,n}f_{r,x,1,m,n}+R_{r,y,m}R_{r,y,n}f_{r,x,13,m,n}\right)\right.\\&\qquad-\mathrm{N}_{r,m,r,n}\left(Q_{r,y,m}R_{r,y,n}f_{r,x,5,m,n}+R_{r,y,m}Q_{r,y,n}f_{r,x,9,m,n}\right)\\&\qquad+\mathrm{M}_{r,m,r,n}\left(Q_{r,y,m}Q_{r,y,n}f_{r,x,4,m,n}+R_{r,y,m}R_{r,y,n}f_{r,x,16,m,n}\right)\\&\qquad\left.\left.+\mathrm{K}_{r,m,r,n}\left(Q_{r,y,m}R_{r,y,n}f_{r,x,8,m,n}+R_{r,y,m}Q_{r,y,n}f_{r,x,12,m,n}\right)\right]\right\}.\end{aligned} \tag{303}$$

If N wavenumbers are repeated, then the number of independent modes of the rck-*x* pulson diminishes to M^2-N due to the wavenumber resonance.

The wavelength of rck-*x* pulson is computed by

$$L_{e,r,x} = \mathrm{LCM}\left(L_{e,r,x,i}, L_{e,r,x,d}, L_{e,r,x,e}\right) = k_{e,r,x,i} L_{e,r,x,i} = k_{e,r,x,d} L_{e,r,x,d} = k_{e,r,x,e} L_{e,r,x,e}, \tag{304}$$

where $k_{e,r,x,i}, k_{e,r,x,d}, k_{e,r,x,e}$ are integers.

We combine the averages of the rk-*x* pulson and the rik-*x*, rdk-*x*, rek-*x* oscillons to compute the average of the rck-*x* pulson over $L_{e,r,x}$ as follows:

$$\begin{aligned}\frac{1}{L_{e,r,x}} \int_0^{L_{e,r,x}} K_{e,r,r} \mathrm{d}x = \rho_c \sum_{m=1}^{M} e z_{r,m}^2 \Big[\mu_{r,m}^2 \left(A v_{r,m}^2 + B v_{r,m}^2 + C v_{r,m}^2 + D v_{r,m}^2 \right) \\ + 2\kappa_{r,m}^2 Q_{r,y,m} R_{r,y,m} h_{r,y,2,m,m} \Big].\end{aligned} \tag{305}$$

As

$$K_{e,r,r}\left(x, y_0, z_0, t_0\right) = \frac{\rho_c}{2}\left(u_{r,x}^2 + u_{r,y}^2 + u_{r,z}^2\right)\left(x, y_0, z_0, t_0\right), \tag{306}$$

where $u_{r,x}, u_{r,y}, u_{r,z}$ are *x*-, *y*-, *z*-components of random velocity $\boldsymbol{u}_r$, the rck-*x* pulson is positively defined for all *x* and also transfers a positive amount of the kinetic energy along the *x*-axis.

8.7. The TCK Pulson

The turbulent, cumulative, kinetic-energy pulson in the *x*-direction (the tck-*x* pulson for swiftness), which is the superposition of the dck-*x*, rck-*x* pulsons and the drik-*x*, drek-*x* oscillons (see (1) and (201) of [7]),

$$\begin{aligned}K_{e,t} &= K_{e,d,d} + K_{e,d,i,m,r,j,m} + K_{e,d,i,m,r,j,n} + K_{e,r,r} \\ &= K_{e,t}\left(\boldsymbol{f}_{t,c,i,x}, \boldsymbol{f}_{t,c,e,x}\right)\end{aligned} \tag{307}$$

is formed by 15-tuple of the turbulent, cumulative, internal interaction in the *x*-direction

$$\begin{aligned}\boldsymbol{f}_{t,c,i,x} &= \left\{\boldsymbol{f}_{d,c,i,x}, \boldsymbol{f}_{r,c,i,x}, \boldsymbol{f}_{d,r,i,g,x}\right\} \\ &= \{1, g_{d,x,4l-3,m,m}, g_{r,x,4l-3,m,m}, h_{d,y,2,m,m}, h_{r,y,2,m,m}, \\ &\quad g_{d,r,x,4k-3,m,m}, g_{d,r,x,4k,m,m}\}\end{aligned} \tag{308}$$

and 32-tuple of the turbulent, cumulative, external interaction along the *x*-axis

$$\begin{aligned}\boldsymbol{f}_{t,c,e,x} &= \left\{\boldsymbol{f}_{d,c,e,x}, \boldsymbol{f}_{r,c,e,x}, \boldsymbol{f}_{d,r,e,g,x}\right\} \\ &= \{ f_{d,x,4k-3,m,n}, f_{d,x,4k,m,n}, f_{r,x,4k-3,m,n}, f_{r,x,4k,m,n}, \\ &\quad f_{d,r,x,4k-3,m,n}, f_{d,r,x,4k,m,n}, f_{r,d,x,4k-3,m,n}, f_{r,d,x,4k,m,n}\}.\end{aligned} \tag{309}$$

Fifteen-tuple $\boldsymbol{f}_{t,c,i,x}$ embraces the unit pulson, two deterministic sine waves $g_{d,x,4l-3,m,m}$ with wavenumber $\kappa_{d,1,m,m}$, two random sine waves $g_{r,x,4l-3,m,m}$ with wavenumber $\kappa_{r,1,m,m}$, deterministic uniform pulson $h_{d,y,2,m,m}$, random uniform pulson $h_{r,y,2,m,m}$, four deterministic-random sine waves $g_{d,r,x,4k-3,m,m}$ with wavenumber $\kappa_{d,r,1,m,m}$, and four deterministic-random cosine waves $g_{d,r,x,4k,m,m}$ with wavenumber $\kappa_{d,r,2,m,m}$ for $l = 1, 2$, $k = 1, 2, 3, 4$, and each *m*.

Thirty-two-tuple $\boldsymbol{f}_{t,c,e,x}$ is constructed of four deterministic sine waves $f_{d,x,4k-3,m,n}$ with wavenumber $\kappa_{d,1,m,n}$, four deterministic cosine waves $f_{d,x,4k,m,n}$

with wavenumber $\kappa_{d,2,m,n}$, four random sine waves $f_{r,x,4k-3,m,n}$ with wavenumber $\kappa_{r,1,m,n}$, four random cosine waves $f_{r,x,4k,m,n}$ with wavenumber $\kappa_{r,2,m,n}$, four deterministic-random sine waves $f_{d,r,x,4k-3,m,n}$ with wavenumber $\kappa_{d,r,1,m,n}$, four deterministic-random cosine waves $f_{d,r,x,4k,m,n}$ with wavenumber $\kappa_{d,r,2,m,n}$, four random-deterministic sine waves $f_{r,d,x,4k-3,m,n}$ with wavenumber $\kappa_{r,d,1,m,n}$, and four random-deterministic cosine waves $f_{r,d,x,4k,m,n}$ with wavenumber $\kappa_{r,d,2,m,n}$ for $k = 1,2,3,4$, and each m, n.

If all wavenumbers of the tck-x pulson are distinct, then the tck-x pulson is displayed as a $4M^2$-w pulson. For any frozen $y = y_0, z = z_0, t = t_0$, the dck-$x$ pulson is given by the $4M^2$-w, turbulent, supercritical pulson in x, which depends on M 15-tuples $\boldsymbol{f}_{t,c,i,x}$ with wavenumbers $\kappa_{d,1,m,m}, \kappa_{r,1,m,m}, \kappa_{d,r,l,m,m}$ for $l = 1,2$, all m, Re and $M(M-1)/2$ 32-tuples $\boldsymbol{f}_{t,c,e,x}$ with wavenumbers $\kappa_{d,l,m,n}, \kappa_{r,l,m,n}, \kappa_{d,r,l,m,n}, \kappa_{r,d,l,m,n}$ for $l = 1,2$, all m, n, Re seeing that

$$
\begin{aligned}
K_{e,t} = \rho_c \Bigg(& \sum_{m=1}^{M} \Big\{ ez_{d,m}^2 \Big[\mu_{d,m}^2 \left(Av_{d,m}^2 + Bv_{d,m}^2 + Cv_{d,m}^2 + Dv_{d,m}^2 \right) \\
& + \lambda_{d,m}^2 \left(Q_{d,y,m}^2 g_{d,x,1,m,m} + R_{d,y,m}^2 g_{d,x,5,m,m} \right) + 2\kappa_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,2,m,m} \Big] \\
& \qquad + ez_{r,m}^2 \Big[\mu_{r,m}^2 \left(Av_{r,m}^2 + Bv_{r,m}^2 + Cv_{r,m}^2 + Dv_{r,m}^2 \right) \\
& + \lambda_{r,m}^2 \left(Q_{r,y,m}^2 g_{r,x,1,m,m} + R_{r,y,m}^2 g_{r,x,5,m,m} \right) + 2\kappa_{r,m}^2 Q_{r,y,m} R_{r,y,m} h_{r,y,2,m,m} \Big] \\
& + ez_{d,m} ez_{r,m} \Big[\Lambda_{d,m,r,m} \left(Q_{d,y,m} Q_{r,y,m} g_{d,r,x,1,m,m} + R_{d,y,m} R_{r,y,m} g_{d,r,x,13,m,m} \right) \\
& \qquad - \mathrm{N}_{d,m,r,m} \left(Q_{d,y,m} R_{r,y,m} g_{d,r,x,5,m,m} + R_{d,y,m} Q_{r,y,m} g_{d,r,x,9,m,m} \right) \\
& \qquad + \mathrm{M}_{d,m,r,m} \left(Q_{d,y,m} Q_{r,y,m} g_{d,r,x,4,m,m} + R_{d,y,m} R_{r,y,m} g_{d,r,x,16,m,m} \right) \\
& \qquad + \mathrm{K}_{d,m,r,m} \left(Q_{d,y,m} R_{r,y,m} g_{d,r,x,8,m,m} + R_{d,y,m} Q_{r,y,m} g_{d,r,x,12,m,m} \right) \Big] \Big\} \\
& + \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} \Big\{ ez_{d,m} ez_{d,n} \Big[\Lambda_{d,m,d,n} \left(Q_{d,y,m} Q_{d,y,n} f_{d,x,1,m,n} + R_{d,y,m} R_{d,y,n} f_{d,x,13,m,n} \right) \\
& \qquad - \mathrm{N}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,5,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,9,m,n} \right) \\
& \qquad + \mathrm{M}_{d,m,d,n} \left(Q_{d,y,m} Q_{d,y,n} f_{d,x,4,m,n} + R_{d,y,m} R_{d,y,n} f_{d,x,16,m,n} \right) \\
& \qquad + \mathrm{K}_{d,m,d,n} \left(Q_{d,y,m} R_{d,y,n} f_{d,x,8,m,n} + R_{d,y,m} Q_{d,y,n} f_{d,x,12,m,n} \right) \Big] \\
& \quad + ez_{r,m} ez_{r,n} \Big[\Lambda_{r,m,r,n} \left(Q_{r,y,m} Q_{r,y,n} f_{r,x,1,m,n} + R_{r,y,m} R_{r,y,n} f_{r,x,13,m,n} \right) \\
& \qquad - \mathrm{N}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,5,m,n} + R_{r,y,m} Q_{r,y,n} f_{r,x,9,m,n} \right) \\
& \qquad + \mathrm{M}_{r,m,r,n} \left(Q_{r,y,m} Q_{r,y,n} f_{r,x,4,m,n} + R_{r,y,m} R_{r,y,n} f_{r,x,16,m,n} \right) \\
& \qquad + \mathrm{K}_{r,m,r,n} \left(Q_{r,y,m} R_{r,y,n} f_{r,x,8,m,n} + R_{r,y,m} Q_{r,y,n} f_{r,x,12,m,n} \right) \Big] \\
& \quad + ez_{d,m} ez_{r,n} \Big[\Lambda_{d,m,r,n} \left(Q_{d,y,m} Q_{r,y,n} f_{d,r,x,1,m,n} + R_{d,y,m} R_{r,y,n} f_{d,r,x,13,m,n} \right) \\
& \qquad - \mathrm{N}_{d,m,r,n} \left(Q_{d,y,m} R_{r,y,n} f_{d,r,x,5,m,n} + R_{d,y,m} Q_{r,y,n} f_{d,r,x,9,m,n} \right) \\
& \qquad + \mathrm{M}_{d,m,r,n} \left(Q_{d,y,m} Q_{r,y,n} f_{d,r,x,4,m,n} + R_{d,y,m} R_{r,y,n} f_{d,r,x,16,m,n} \right) \\
& \qquad + \mathrm{K}_{d,m,r,n} \left(Q_{d,y,m} R_{r,y,n} f_{d,r,x,8,m,n} + R_{d,y,m} Q_{r,y,n} f_{d,r,x,12,m,n} \right) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ ez_{r,m} ez_{d,n} \Big[\Lambda_{r,m,d,n} \left(Q_{r,y,m}\, Q_{d,y,n}\, f_{r,d,x,1,m,n} + R_{r,y,m}\, R_{d,y,n}\, f_{r,d,x,13,m,n} \right) \\
&\quad - \mathrm{N}_{r,m,d,n} \left(Q_{r,y,m}\, R_{d,y,n}\, f_{r,d,x,5,m,n} + R_{r,y,m}\, Q_{d,y,n}\, f_{r,d,x,9,m,n} \right) \\
&\quad + \mathrm{M}_{r,m,d,n} \left(Q_{r,y,m}\, Q_{d,y,n}\, f_{r,d,x,4,m,n} + R_{r,y,m}\, R_{d,y,n}\, f_{r,d,x,16,m,n} \right) \\
&\quad + \mathrm{K}_{r,m,d,n} \left(Q_{r,y,m}\, R_{d,y,n}\, f_{r,d,x,8,m,n} + R_{r,y,m}\, Q_{d,y,n}\, f_{r,d,x,12,m,n} \right) \Big] \Big\} \Big).
\end{aligned}
\tag{310}
$$

If N wavenumbers are repeated, then the number of independent modes of the tck-x pulson diminishes to $4M^2 - N$ due to the wavenumber resonance.

The wavelength of the tck-x pulson is provided by

$$
\begin{aligned}
L_{e,t,x} &= \mathrm{LCM}\left(L_{e,d,x}, L_{e,r,x}, L_{e,d,r,x,i}, L_{e,d,r,x,e} \right) \\
&= k_{e,d,x} L_{e,d,x} = k_{e,r,x} L_{e,r,x} = k_{e,d,r,x,i} L_{e,d,r,x,i} = k_{e,d,r,x,e} L_{e,d,r,x,e},
\end{aligned}
\tag{312}
$$

where $k_{e,d,x}, k_{e,r,x}, k_{e,d,r,x,i}, k_{e,d,r,x,e}$ are integers.

Finally, we use the averages of the dck-x, rck-x pulsons and the drik-x, drek-x oscillons to find the average of the tck-x pulson over $L_{e,t,x}$ in the following form:

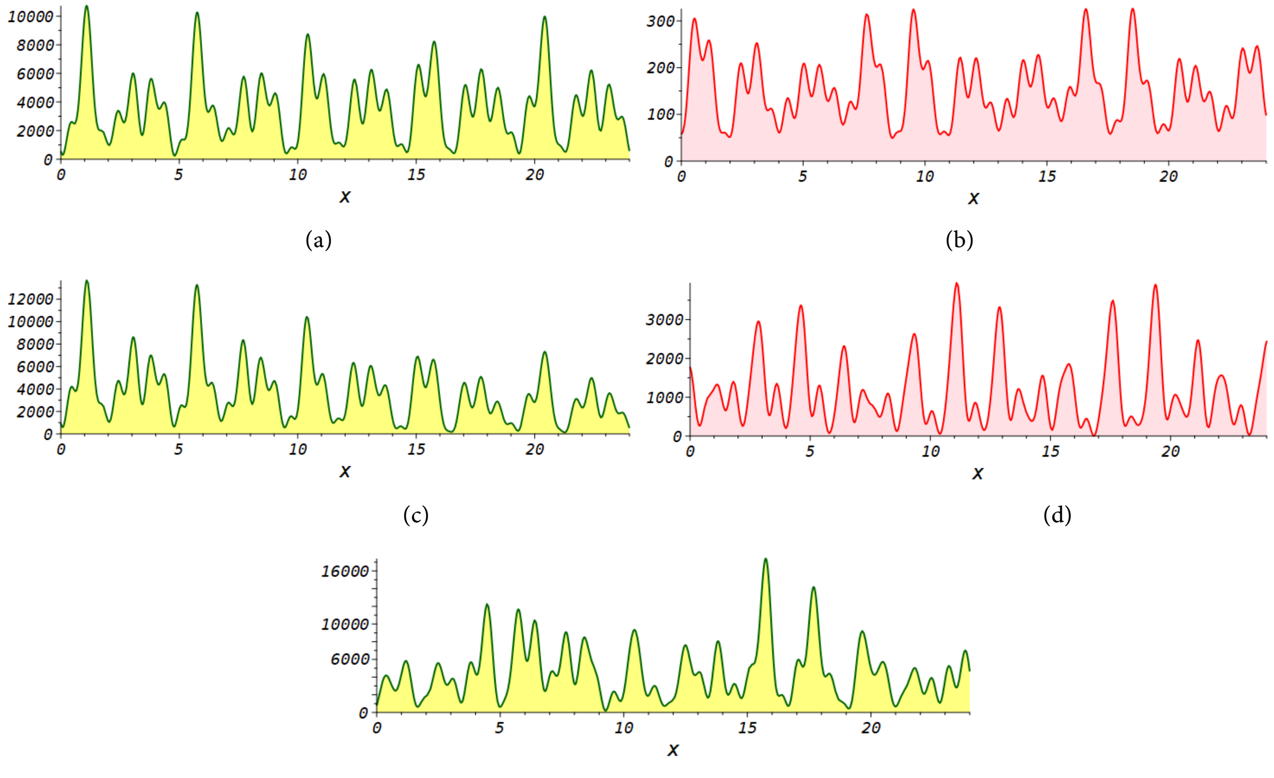
$$
\begin{aligned}
\frac{1}{L_{e,t,x}} \int_0^{L_{e,t,x}} K_{e,t} \mathrm{d}x = \rho_c \sum_{m=1}^{M} \Big\{ & ez_{d,m}^2 \Big[\mu_{d,m}^2 \left(Av_{d,m}^2 + Bv_{d,m}^2 + Cv_{d,m}^2 + Dv_{d,m}^2 \right) \\
& \quad + 2\kappa_{d,m}^2 Q_{d,y,m} R_{d,y,m} h_{d,y,2,m,m} \Big] \\
& + ez_{r,m}^2 \Big[\mu_{r,m}^2 \left(Av_{r,m}^2 + Bv_{r,m}^2 + Cv_{r,m}^2 + Dv_{r,m}^2 \right) \\
& \quad + 2\kappa_{r,m}^2 Q_{r,y,m} R_{r,y,m} h_{r,y,2,m,m} \Big] \Big\}.
\end{aligned}
\tag{313}
$$

Because

$$
K_{e,t}\left(x, y_0, z_0, t_0 \right) = \frac{\rho_c}{2} \left(u_{t,x}^2 + u_{t,y}^2 + u_{t,z}^2 \right)\left(x, y_0, z_0, t_0 \right),
\tag{314}
$$

where $u_{t,x}, u_{t,y}, u_{t,z}$ are x-, y-, z-components of turbulent velocity $\boldsymbol{u}_t$, the tck-x pulson remains positive for all x and transfers a positive amount of the kinetic energy along the x-axis, as well.

The dck-x, rck-x, and tck-x pulsons for $y = y_0, z = z_0, t = t_0, Re = 10^3, Re = 10^5$, and wave parameters (140), (145), (146) are compared on $L_{e,d,x,e}$ in **Figure 6**. Although range of the 9-w, random, supercritical rck-x pulson in **Figure 6(b)** is $[0, 330]$, the 36-w, turbulent, supercritical tck-x pulson at $Re = 10^3$ **Figure 6(c)** changes significantly compared with the 9-w, deterministic, supercritical dck-x pulson in **Figure 6(a)** due to range $[-1500, 1400]$ of the 12-w, deterministic-random, random-deterministic, neutral drek-x oscillon in **Figure 1(a)** and range $[-1600, 1550]$ of the 6-w, deterministic-random, neutral drik-x oscillon in **Figure 2(a)**. Shape of the 36-w, turbulent, supercritical tck-x oscillon at $Re = 10^5$ in **Figure 6(e)** is unrecognizable in comparison with the 9-w, deterministic, supercritical dck-x pulson in **Figure 6(a)** because of comparable ranges $[0, 3900]$ of the 9-w, random, supercritical rck-x pulson in **Figure 6(d)**, $[-5000, 5300]$ of the 12-w, deterministic-random, random-deterministic, neutral drek-x oscillon in **Figure 1(b)**, and $[-3600, 3700]$ of the 6-w, deterministic-random, neutral drik-x oscillon in Figure 2(b).



(a) (b)

(c) (d)

(e)

Figure 6. The dck-*x*, rck-*x*, and tck-*x* pulsons: (a) $-K_{e,d,d}(x)$, (b) $-K_{e,r,r}(x)$ for $Re = 10^3$, (c) $-K_{e,t}(x)$ for $Re = 10^3$, (d) $-K_{e,r,r}(x)$ for $Re = 10^5$, (e) $-K_{e,t}(x)$ for $Re = 10^5$.

9. Conclusions

The method of inhomogeneous Fourier expansions, which was originally developed for deterministic *x*-, *y*-, *t*-eigenfunctions in [5] [6], has been successfully extended in this paper to the deterministic-random, random-deterministic, random, external, and internal *x*-eigenfunctions. The previous results [7] on theoretical quantization in experimental DDSD, DRSD, RDSD, and RRSD structures have been confirmed, analyzed, and visualized in the present work using experimental quantization in the novel eigenfunctions.

It was discovered that the exact solution for the quantized oscillons and pulsons are grouped into the vector, deterministic-random, elementary, external oscillons, into the vector, random-deterministic, elementary, external oscillons, and into the vector, deterministic-random, elementary, internal oscillons with two components. The vector, deterministic-random, random-deterministic, wave, external oscillons and the vector, deterministic-random, wave, internal oscillons have four components. The vector, turbulent, elementary, external, internal, and diagonal oscillons, the vector, turbulent, wave, external, internal, and diagonal oscillons, and the vector, turbulent, elementary, and wave pulsons also include two components. From the mathematical point of view, components of all vector oscillons and pulsons are invariant structures constructed on the corresponding tuples with various amplitudes.

The vector and scalar oscillons and pulsons depend on 1-, 2-, 3-, 4-, 5-, 6-, 8-, 12-, 15-, 16, and 32-tuples of the relevant x-eigenfunctions. Namely, the vector, deterministic-random, elementary, external oscillons on eight 2-tuples $\boldsymbol{f}_{d,r,e,e,x,q}$, the vector, random-deterministic, elementary, external oscillons on eight 2-tuples $\boldsymbol{f}_{r,d,e,e,x,q}$, the vector, deterministic-random, random-deterministic, wave, external oscillons on four 8-tuples $\boldsymbol{f}_{d,r,e,w,x,q}$, the vector, deterministic-random, elementary, internal oscillons on eight 2-tuples $\boldsymbol{f}_{d,r,i,e,x,q}$, and the vector, deterministic-random, wave, internal oscillons on four 4-tuples $\boldsymbol{f}_{d,r,i,w,x,q}$. The vector, turbulent, elementary and wave, external oscillons on four-tuple $\boldsymbol{f}_{t,e,e,x,1}$ and two 8-tuples $\boldsymbol{f}_{t,e,e,x,q}$, the vector, turbulent, elementary, diagonal oscillons on two 4-tuples $\boldsymbol{f}_{t,d,e,x,q}$, the vector, turbulent, wave, diagonal oscillons on 8-tuple $\boldsymbol{f}_{t,d,w,x}$, the vector, turbulent, elementary, internal oscillons on three 4-tuples $\boldsymbol{f}_{t,i,e,x,q}$, the vector, turbulent, wave, internal oscillons on 4-tuple $\boldsymbol{f}_{t,i,w,x,1}$ and two 2-tuples $\boldsymbol{f}_{t,i,w,x,q}$, the vector, turbulent, elementary, pulsons on two 3-tuples $\boldsymbol{f}_{t,e,x,q}$, and the vector, turbulent, wave pulsons on 5-tuple $\boldsymbol{f}_{t,w,x}$.

Independent random parameters (145), (146), of the deterministic-random, random-deterministic, and turbulent oscillons and pulsons have been computed for all m using the random oscillatory cn-noise [8] for $Re = 10^3$ and $Re = 10^5$. Although the deterministic oscillons and pulsons do not depend on the Reynolds number, the empirical scales (144) result in a strong dependence of the quantized oscillons and pulsons on Re. So, the 36-w, turbulent, supercritical tck-x pulson at $Re = 10^3$ in **Figure 6(c)** changes significantly compared with the 9-w, deterministic, supercritical dck-x pulson in **Figure 6(a)** due to the 12-w, deterministic-random, random-deterministic, neutral drek-x oscillon and the 6-w, deterministic-random, neutral drik-x oscillon. The 36-w, turbulent, supercritical tck-x pulson at $Re = 10^5$ in **Figure 6(e)** is unrecognizable in comparison with the 9-w, deterministic, supercritical dck-x pulson because of the 9-w, random, supercritical rck-x pulson, the 12-w, deterministic-random, random-deterministic, neutral drek-x oscillon, and the 6-w, deterministic-random, neutral drik-x oscillon.

As regards a list of open problems, we may mention spatial experimental quantization of the exact wave turbulence in the y-direction and temporal experimental quantization of the exact wave turbulence.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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