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Some Properties of Fuzzy Sets: Graphical Representations and Practical Applications

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Abstract

Fuzzy set theory, an extension of classical set theory, provides a mathematical framework for handling uncertainty and imprecision. This paper provides some key properties of fuzzy sets, emphasizing their graphical representations and practical applications to enhance visualization and understanding. Fundamental concepts such as membership functions, core, support, and α-cuts are analyzed alongside essential operations, including complement, intersection, and union. Additionally, the study explores triangular and trapezoidal fuzzy numbers, as well as triangular norms (t-norms) and conorms (t-conorms) that facilitate fuzzy set operations. Beyond theoretical insights, this study highlights the practical applications of fuzzy set theory in real-world scenarios. We demonstrate how fuzzy logic is applied in finance (profit optimization), meteorology (rainfall prediction), and medical science (diabetes classification). Through these applications, the paper underscores the effectiveness of fuzzy sets in modeling uncertainty and improving decision-making processes.

Keywords

Fuzzy Set, Fuzzy Membership Function, Triangular Norms, Fuzzy Numbers, Fuzzy Equation

1. Introduction

In the realm of mathematics and logic, traditional sets are bounded by clear and unambiguous membership criteria, allowing elements to either belong or not belong to a set. This binary approach, while effective in many circumstances, often falls short when dealing with uncertainty and imprecision that characterize real-world scenarios. Herein lies the value of fuzzy sets, a concept introduced by Lotfi A. Zadeh [1] in 1965. Fuzzy sets extend the traditional framework by allowing for degrees of membership, thereby providing a more nuanced means of capturing the complexity of various phenomena.

The fundamental property of fuzzy sets is that elements can have varying levels of membership, expressed through membership functions. These functions assign a value between 0 and 1 to each element, indicating its degree of belonging to a fuzzy set [2]. This flexibility opens up new avenues for modeling and solving problems across a diverse range of fields, including control systems [3], artificial intelligence [4], decision-making processes [5], and information retrieval [6].

This paper delves into the essential properties of fuzzy sets, including their graphical representations using MATLAB, which facilitate a deeper understanding of their behavior and characteristics. By illustrating these properties with clear graphs, we aim to highlight the distinct advantages fuzzy sets offer over classical sets. Additionally, we will explore practical applications that benefit from the incorporation of fuzzy logic, demonstrating the relevance and transformative potential of this approach in addressing complex problems that traditional methods struggle to resolve.

By examining these aspects, our objective extends beyond mere dissemination of information. We aspire to inspire further exploration and adoption of fuzzy set theory across various disciplines, encouraging academics and practitioners alike to consider its applications and implications more seriously. As we navigate through the properties and practicalities of fuzzy sets, we invite readers to recognize their value in advancing understanding in complex systems and to consider how fuzzy logic might enhance their work in diverse fields. This discussion aims to contribute to a broader appreciation of fuzzy sets as a powerful tool for modeling uncertainty and facilitating decision-making in an increasingly complex world.

2. Fuzzy Set

In traditional set theory, an element is either in or not in a set A, that is $x \in A$ or $x \notin A$. This kind of set is called a crisp set. A fuzzy set is a set that is characterized by a fuzzy membership function $\mu_A(x) \in [0,1]$. If $\mu_A(x) = 0$, it implies that $x \notin A$. On the other hand, if $\mu_A(x) = 1$, then $x \in A$ [7].

A membership function for a fuzzy set "A" on the universe of discourse X is defined as $\mu_A : X \to [0,1]$, where each element of X is mapped to a value between 0 and 1 [2]. A function $\mu_A : X \to [0,1]$ is called a fuzzy set on X where X is a nonempty set of objects called referential set and [0, 1] (the unit interval) is called valuation set and $\forall x \in X$; $\mu_A(x)$ represents the grade of membership of x [2].

Let X be a universal set. Then a fuzzy set A can be defined as the set of ordered pairs such that $A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0,1]\}$ where $\mu_A(x)$ is called the membership function or grade of membership of x [8].

Different techniques to represent fuzzy set:

Let $X = \{a,b,c,d\}$ is a referential set and $\mu : X \to [0,1]$ is a fuzzy set defined on X then

$$\mu(a)=0.9,\ \mu(b)=0.4,\ \mu(c)=1,\ \mu(d)=0$$

$$\mu=\left(\frac{0.9}{a},\frac{0.4}{b},\frac{1}{c},\frac{0}{d}\right)$$

$$\mu=[0.9,0.4,1,0]$$

$$\mu=\{(a,0.9),(b,0.4),(c,1),(d,0)\}$$

$$\mu=\frac{0.9}{a}+\frac{0.4}{b}+\frac{0.1}{c}+\frac{0}{d}$$

Examples: Suppose $S:H\to[0,1]$, $M:H\to[0,1]$, and $T:H\to[0,1]$ define the fuzzy sets of Short people, Medium people, and Tall people respectively, where H is the set of height of human beings in inch, such that:

$$\text{Short people,}\quad Y(h)=\begin{cases}1; & h\le 50\\ \dfrac{55-h}{5}; & 50<h<55\\ 0; & h\ge 55\end{cases}$$

$$\text{Medium people,}\quad M(h)=\begin{cases}\dfrac{h-50}{5}; & 50<h<55\\ 1; & 55\le h\le 65\\ \dfrac{70-h}{5}; & 65<h<70\\ 0; & \text{otherwise}\end{cases}$$

$$\text{Tall people,}\quad T(h)=\begin{cases}0; & h\le 65\\ \dfrac{h-65}{5}; & 65<h<70\\ 1; & h\ge 70\end{cases}$$

The membership function shown into **Figure 1**.

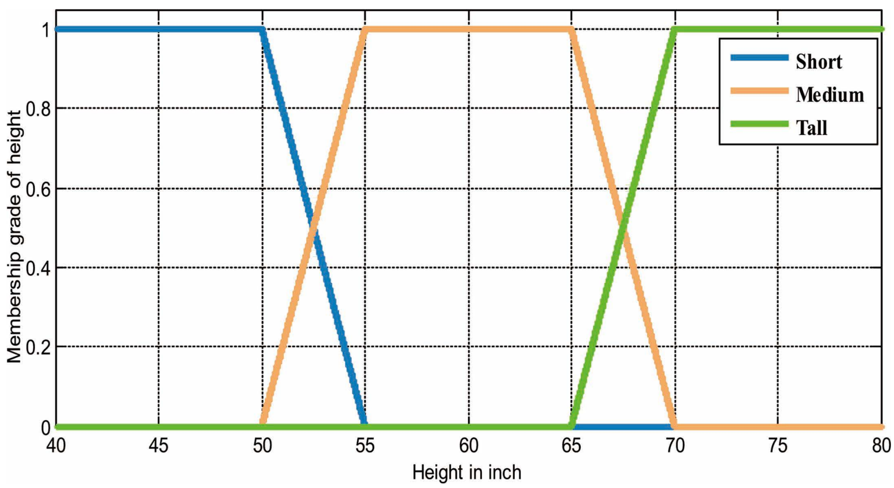


Figure 1. Membership function of the height of the people.

2.1. Range

Suppose $f:X\to[0,1]$ is a fuzzy set, then the range of f is defined as

$f(x)=\{f(x):x\in X\}$ [2].

Example: Suppose $X=\{1,2,3,4,5,6,7,8,9,10\}$ and $f(x)=\frac{1}{3x}$ is a fuzzy set on X, then the range of f is given in the following table:

X	1	2	3	4	5	6	7	8	9	10
$f(x)$	0.33	0.17	0.11	0.08	0.07	0.06	0.05	0.04	0.04	0.03

2.2. Height

Suppose $A:X\to[0,1]$ is a fuzzy set, then the height of A is denoted by $h(A)$ and is defined as $h(A)=\vee A(x)=\text{Sup}\{\mu_A(x):x\in X\}$. This is the highest membership grade obtained by any element in that set [2].

2.3. Support

Suppose $A:X\to[0,1]$ is a fuzzy set. Then the "Support of A" is the ordinary subset of X, that is the support of fuzzy set A within a universal set X is the crisp set that contains all the elements of X that have nonzero membership grades in A. Symbolically, $\text{supp } A=\{x\in X:\mu_A(x)>0\}$ [2] [3].

2.4. Core

Suppose $A:X\to[0,1]$ is a fuzzy set. Then "Core of A" is denoted by $\text{Core}(A)$ and defined as follows: $\text{Core}(A)=\{x\in X:\mu_A(x)=1\}$ [2]. Core is shown into Figure 2.

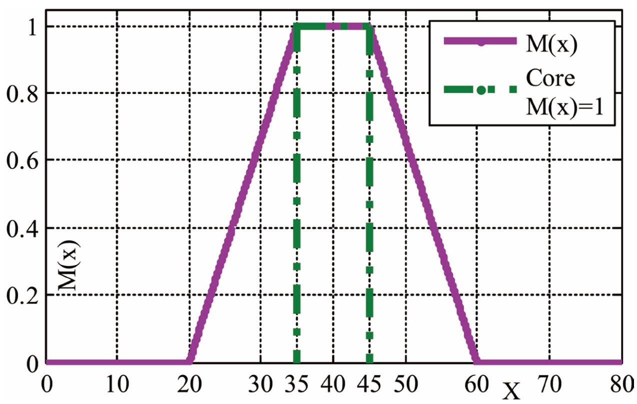


Figure 2. Core of fuzzy set.

2.5. Normal Fuzzy Set

Suppose $A:X\to[0,1]$ is a fuzzy set. Then A is called a normal fuzzy set if and only if $h(A)=\vee A(x)=\text{Sup}\{\mu_A(x)\}=1$ [2].

Example: Suppose $X=[3,50]$ and $A:X\to[0,1]$ is a fuzzy set, where $\mu_A(x)=\frac{3}{x}$. Then $\mu_A(x)=[0.06,1]$. Therefore $h(A)=1$, hence A is a normal fuzzy set.

2.6. Sub-Normal Fuzzy Set

Suppose $B: X \to [0,1]$ is a fuzzy set. Then B is called a sub normal fuzzy set if an only if $h(B) = \vee B(x) = \text{Sup}\{\mu_B(x)\} < 1$ [2]. Normal and Subnormal fuzzy set is shown into Figure 3.

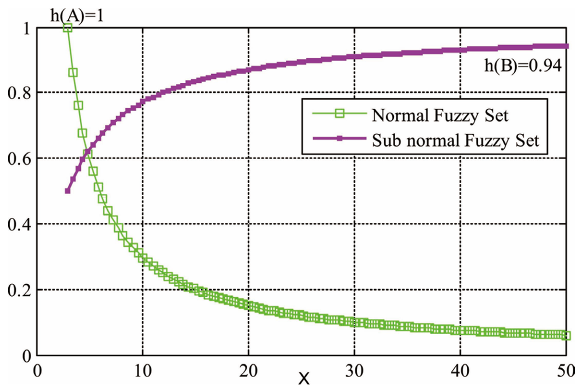


Figure 3. Normal and subnormal fuzzy set.

Example: Suppose $X = [3,50]$ and B is a fuzzy set where $\mu_B(x) = \dfrac{x}{x+3}$. Then $\mu_B(x) = [0.5, 0.94]$. Therefore $h(B) = 0.94$, hence B is a sub normal fuzzy set.

2.7. Alpha Cut of Fuzzy Set

One of the most important concepts of fuzzy sets is the concept of α-cut and a strong α-cut. Given a fuzzy set A defined on X and any number $\alpha \in [0,1]$, the α-cut, ${}^{\alpha}A$, and the strong α-cut, ${}^{\alpha+}A$, are the crisp sets

${}^{\alpha}A = \{x : A(x) \geq \alpha\}$ and ${}^{\alpha+}A = \{x : A(x) > \alpha\}$ [2]. That is, the α-cut (or the strong α-cut) of a fuzzy set A is the crisp set ${}^{\alpha}A$ (or the crisp set ${}^{\alpha+}A$) that contains all the elements of the universal set X whose membership grades in A are greater than or equal to (or only greater than) the specified value of α. It is shown into Figure 4.

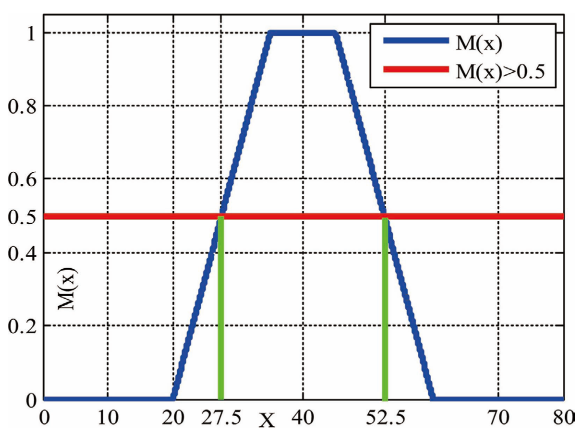


Figure 4. α-cut of fuzzy set.

Example: Suppose $M : X \to [0,1]$ defines a fuzzy set of "Middle aged people" where, X is the set of ages of human beings, such that

$$M(x) = \begin{cases} 0; & x \le 20 \\ \frac{x-20}{15}; & 20 < x < 35 \\ 1; & 35 \le x \le 45 \\ \frac{60-x}{15}; & 45 < x < 60 \\ 0; & x \ge 60 \end{cases}$$

Let $\alpha = 0.5$ then ${}^{0.5}M = [27.5, 52.5]$ and ${}^{0.5+}M = (27.5, 52.5)$.

2.8. Complement of Fuzzy Set

The standard complement $\bar{A}$, of fuzzy set A with respect to the universal set X is defined for all $x \in X$ by the equation $\bar{A}(x) = 1 - A(x)$ [2].

Example: Suppose $X = [0,3]$ and $A : X \to [0,1]$ is a fuzzy set, where $A(x) = \frac{1}{1+10x}$. Then the membership function of complement of A is given as $\bar{A}(x) = 1 - \frac{1}{1+10x} = \frac{10x}{1+10x}$. Complement of fuzzy set is shown into **Figure 5**.

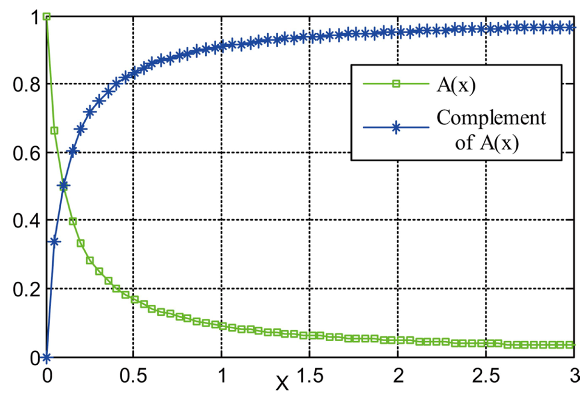


Figure 5. Complement of fuzzy set.

2.9. Intersection and Union of Fuzzy Set

Given two fuzzy sets A and B defined on the universal set X. Then the "standard intersection" of A and B is denoted by $A \cap B$ and defined by $(A \cap B)(x) = \min\{A(x), B(x)\}; \forall x \in X$ [2].

On the other hand, the "standard union" of two fuzzy sets A and B is denoted by $A \cup B$ and defined by $(A \cup B)(x) = \max\{A(x), B(x)\}; \forall x \in X$ [2].

Example: Consider the fuzzy sets A and B defined on the interval $X = [0,10]$ of real numbers by the membership grade functions $A(x) = \frac{x}{x+2}$ and $B(x) = 2^{-x}$. The graphs of the membership grade functions of $(A \cap B)(x)$ and $(A \cup B)(x)$ is shown in **Figure 6**.

2.10. Subset of Fuzzy Set

Suppose $A: X \to [0,1]$ and $B: X \to [0,1]$ are two fuzzy sets, then we say that B is a fuzzy subset of A and write $B \subseteq A$, if and only if $B(x) \le A(x)$; $\forall x \in X$ [2].

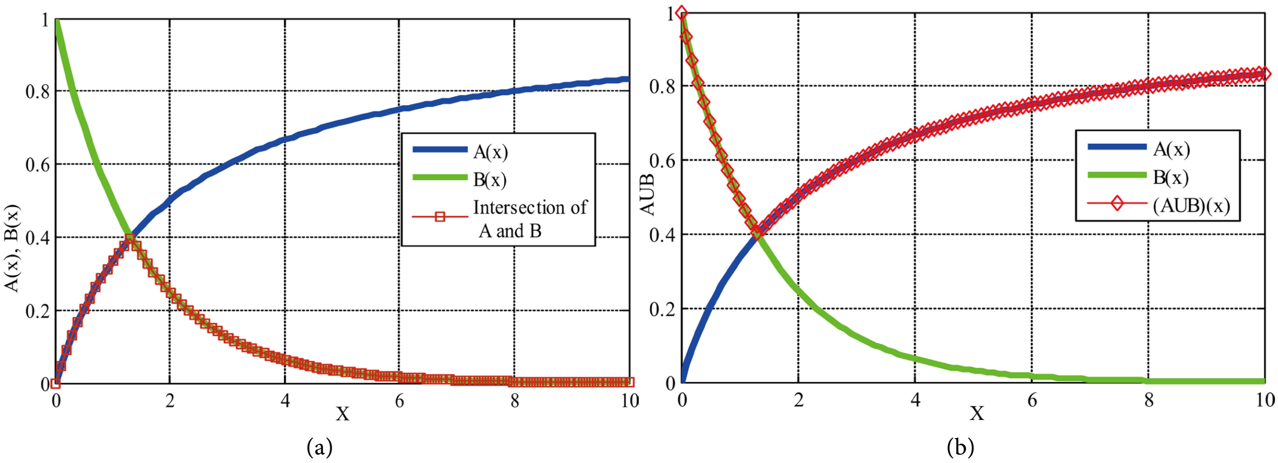


Figure 6. Intersection and Union of fuzzy.

Example: Suppose $X = \{1,2,3,4,5\}$ and $A: X \to [0,1]$, $B: X \to [0,1]$ are two fuzzy sets on X, where $A(x) = \frac{1}{x}$ and $B(x) = \frac{1}{x^2}$, then

X	1	2	3	4	5
$A(x)$	1	0.5	0.33	0.25	0.20
$B(x)$	1	0.25	0.11	0.062	0.04

Clearly, $\forall x \in X; B(x) \le A(x)$ Therefore $B \subset A$.

2.11. Convex Fuzzy Set

A fuzzy set A is convex if and only if it's α-cuts are convex [2] [3]. An equivalent definition of convexity is that the fuzzy set A on $\mathbb{R}$ is convex if and only if $\forall x_1, x_2 \in X$ and $\forall \lambda \in [0,1]$, $\mu_A(\lambda x_1 + (1-\lambda)x_2) \ge \min[\mu_A(x_1), \mu_A(x_2)]$ [3].

Example: Suppose $X = [0,5]$ and $\mu_A: X \to [0,1]$ is a fuzzy set, where $\mu_A(x) = \frac{1}{1+10(x-2)^2}$. Then it is a convex fuzzy set shown in Figure 7.

3. Triangular Norms and Conorms

In Mathematics, a *t*-norm (*T*- norm or triangular norm) is a kind of binary operation used in the frame work of probabilistic metric spaces and in multi-valued logic, specifically in fuzzy logic. In this section, some *t*-norms are discussed along with their surface plot by using computer package MATLAB. The functions used for intersection of fuzzy sets $t(a,b)$ are called *t*-norms and those used for union $u(a,b)$ are called t-conorms.

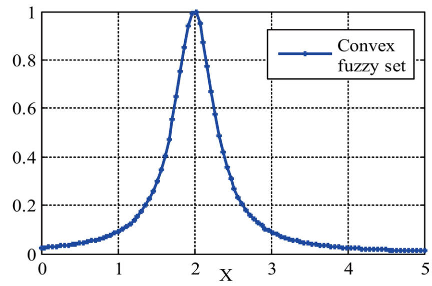


Figure 7. Covex fuzzy set.

3.1. Fuzzy Intersections: *t*-Norms

A function $t: I \times I \to I$ is called *t*-norm if it satisfies the following properties:

$[t_1]: a \le b, c \le d \Rightarrow t(a,c) \le t(b,d)$ [Monotonicity]

$[t_2]: t(a,b) = t(b,a)$ [Commutativity]

$[t_3]: t(a,t(b,c)) = t(t(a,b),c)$ [Associativity]

$[t_4]: t(a,1) = a$ [Boundary condition]

$\forall a,b,c,d \in I$; Where $I = [0,1]$ and set of all *t*-norms will be dented by tN.

All the *t*-norms are presented in Figure 8.

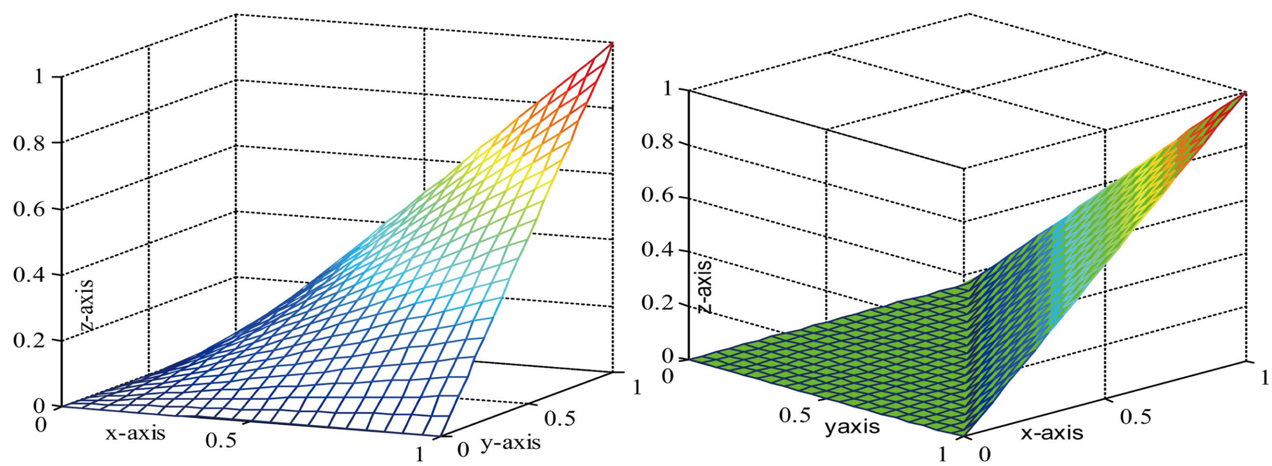
3.1.1. Algebraic Product

The *t*-norm Algebraic product will be denoted by t_a and defined by $t_a(x,y) = xy$; $\forall (x,y) \in I \times I$.

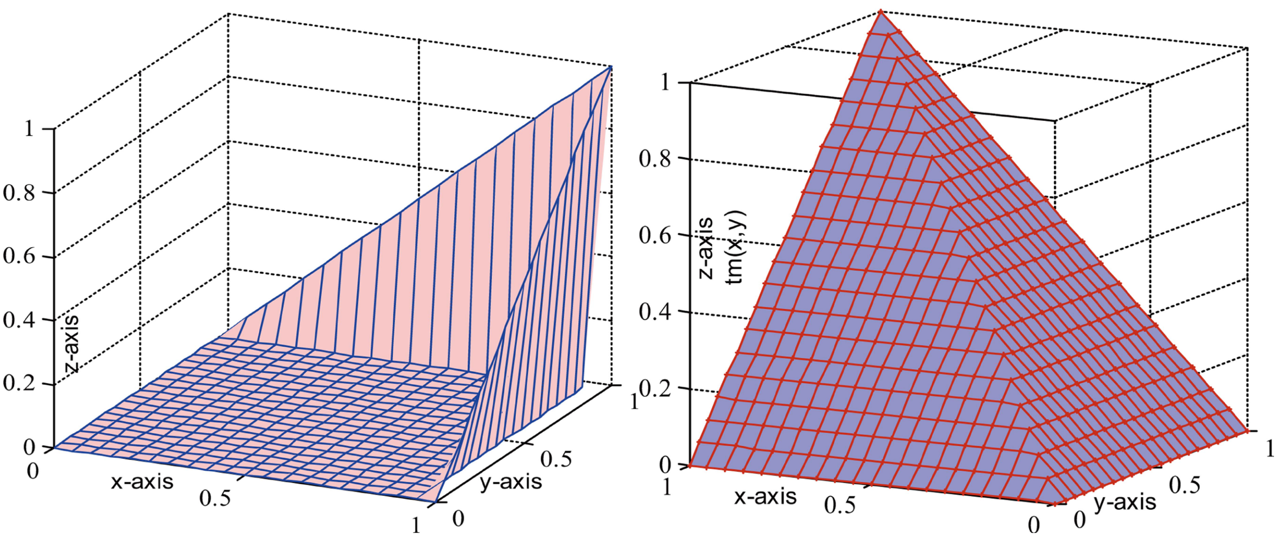
3.1.2. Bounded Product

The *t*-norm Bounded product will be denoted by t_b and defined by $\forall (x,y) \in I \times I$;

$$t_b(x,y) = 0 \vee (x+y-1) = \begin{cases} x+y-1; & x+y \ge 1 \\ 0; & x+y \le 1 \end{cases}$$



(a) (b)



(c) (d)

Figure 8. Fuzzy intersections: *t*-Norms.

3.1.3. Drastic Product

The *t*-norm drastic intersection or drastic product will be denoted by t_d and define by $\forall(x,y)\in I\times I$; $t_d(x,y)=\begin{cases} x; & y=1 \\ y; & x=1 \\ 0; & \text{otherwise} \end{cases}$.

3.1.4. Standard Intersection

The *t*-norm standard intersection or Min will be denoted by t_m and defined by $\forall(x,y)\in I\times I$; $t_m(x,y)=\min(x,y)=x\wedge y=\begin{cases} x; & x\le y \\ y; & y\le x \end{cases}$.

3.2. Fuzzy Unions: *t*-Conorms

A function $u: I\times I\to I$ is called *t*-conorm **if it** satisfies the following properties:

$[u_1]: a\le b, c\le d \Rightarrow u(a,c)\le u(b,d)$ [Monotonicity]

$[u_2]: u(a,b)=u(b,a)$ [Commutativity]

$[u_3]: u(a,u(b,c))=u(u(a,b),c)$ [Associativity]

$[u_4]: u(a,0)=a$ [Boundary condition]

$\forall a,b,c,d\in I$; where $I=[0,1]$. All the *t*-conorms are presented in Figure 9.

3.2.1. Algebraic Sum

The *t*-conorm Algebraic sum will be denoted by u_a and defined by $u_a(x,y)=x+y-xy$; $\forall(x,y)\in I\times I$.

3.2.2. Bounded Sum

The *t*-conorm Bounded sum will be denoted by u_b and defined by

$\forall(x,y)\in I\times I$; $u_b(x,y)=\min(1,x+y)=1\wedge(x+y)=\begin{cases} x+y; & x+y\le 1 \\ 1; & x+y\ge 1 \end{cases}$.

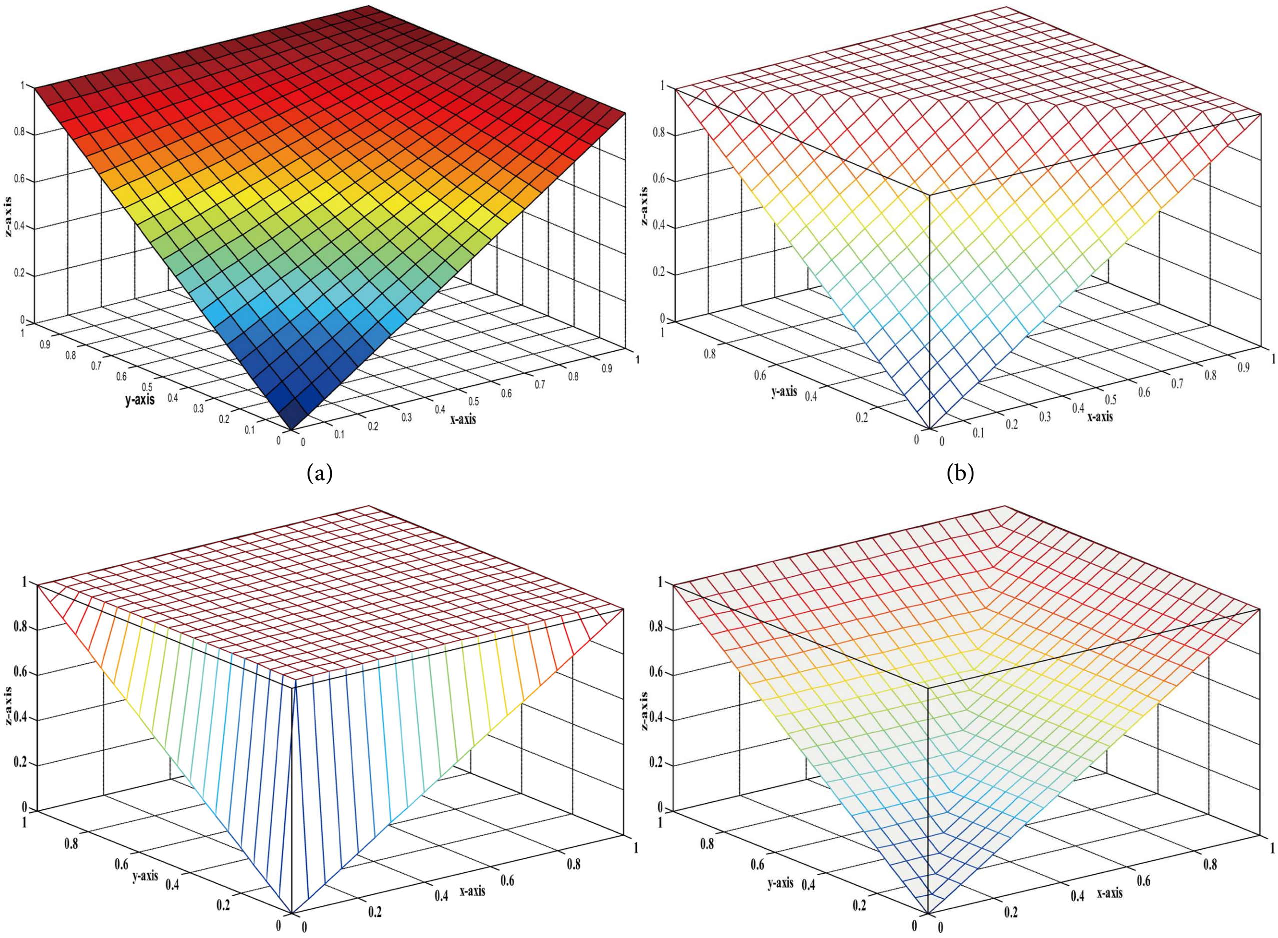


Figure 9. Fuzzy unions: t-conorms.

3.2.3. Drastic Union

The t-conorm drastic union or drastic sum will be denoted by u_d and define by

$$\forall (x,y) \in I \times I\ ;\ u_d(x,y) = \begin{cases} x; & y = 0 \\ y; & x = 0 \\ 1; & \text{otherwise} \end{cases}.$$

3.2.4. Standard Union

The t-conorm standard union or Max will be denoted by u_m and defined by

$$\forall (x,y) \in I \times I\ ;\ u_m(x,y) = \max(x,y) = x \vee y = \begin{cases} x; & x \ge y \\ y; & x \le y \end{cases}$$

4. Fuzzy Number

A fuzzy set A defined on the set of real number $\mathbb{R}$ is called a fuzzy number if and only if A is a normal fuzzy set, ${}^{\alpha}A$ must be closed interval for every $\alpha \in (0,1]$ and support of A, ${}^{0+}A$ must be bounded. In other words, if a fuzzy set A is convex and normalized, also its membership function is defined as piecewise continuous, then A is called fuzzy number [2] [9].

4.1. Triangular Fuzzy Number

Among the various shapes of fuzzy number, triangular fuzzy number (TFN) is the most popular one [9]. It is a fuzzy number represented with three points $A=(a_1,a_2,a_3)$ where the fuzzy membership function (fmf) of A is given by,

$$\mu_A(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1}; & a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2}; & a_2\le x\le a_3\\ 0\,; & \text{otherwise}\end{cases}$$

The α-cut is written as ${}^{\alpha}A=\left[(a_2-a_1)\alpha+a_1,-(a_3-a_2)\alpha+a_3\right];\forall\alpha\in[0,1]$ for Triangular fuzzy number. Triangular fuzzy number is shown in **Figure 10**.

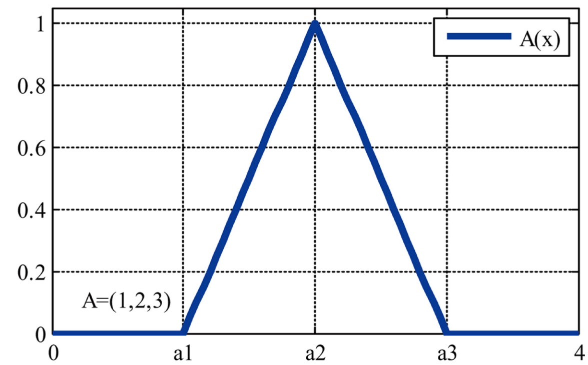


Figure 10. Triangular fuzzy number.

4.2. Trapezoidal Fuzzy Number

Another shape of fuzzy number is trapezoidal fuzzy number [9]. This shape is originated from the fact that there are four points whose membership degree is maximum ($\alpha=1$). $A=(a_1,a_2,a_3,a_4)$ is called a trapezoidal fuzzy number with the fuzzy membership function (fmf) given as follows:

$$\mu_A(x)=\begin{cases}0; & x<a_1\\ \dfrac{x-a_1}{a_2-a_1}; & a_1\le x\le a_2\\ 1; & a_2\le x\le a_3\\ \dfrac{a_4-x}{a_4-a_3}; & a_3\le x\le a_4\\ 0; & x>a_4\end{cases}$$

The α-cut is written as ${}^{\alpha}A=\left[(a_2-a_1)\alpha+a_1,-(a_4-a_3)\alpha+a_4\right];\forall\alpha\in[0,1]$ for Trapezoidal fuzzy number. Trapezoidal fuzzy number is shown in **Figure 11**.

5. Fuzzy Equation

Fuzzy set theory involves fuzzy numbers and their arithmetic operations, particularly in the context of fuzzy equations. These equations involve fuzzy numbers

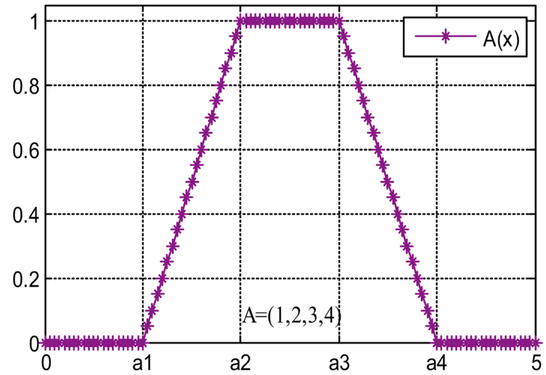


Figure 11. Trapezoidal fuzzy number.

as coefficients and unknowns, with formulas constructed using fuzzy arithmetic operations [2]. Such equations are currently being used to solve problems in many sectors. The most common fuzzy equations are linear fuzzy equation $\bar{A}\cdot\bar{X}+\bar{B}=\bar{C}$ and fully fuzzy quadratic equation $\bar{A}\bar{X}^2+\bar{B}\bar{X}+\bar{C}=\bar{D}$, where the coefficient $\bar{A},\bar{B},\bar{C},\bar{D}$ and the variable $\bar{X}$ all are fuzzy numbers. The solution procedure of fuzzy equations are not similar to crisp equation. Many researcher are working on finding the solution process of fuzzy equations [10]-[13]. Research is underway to invent new methods to solve these fuzzy equations. A solution process of fuzzy linear equation $\bar{A}\bar{X}+\bar{B}=\bar{C}$ is discussed here with the application.

5.1. Arithmetic Operations on Intervals

Fuzzy arithmetic is based on two properties of fuzzy numbers: (a) each fuzzy set, and thus also each fuzzy number, can fully and uniquely be represented by it's α -cut; and (b) α -cut of each fuzzy number are closed intervals of real numbers for all $\alpha\in(0,1]$. These properties enable us to define arithmetic operations on fuzzy numbers in terms of arithmetic operations on their α -cut [2]. Let "$*$" denote any of the four arithmetic operations on closed intervals: addition (+), subtraction (−), multiplication (·) and division (/). Then,

$[a,b]*[d,e]=\{f*g:a\le f\le b,d\le g\le e\}$ is a general property of all arithmetic operations on closed intervals [2]. The four arithmetic operations on closed intervals are defined as follows:

1) Addition: $[a,b]+[d,e]=[a+d,b+e]$

2) Subtraction: $[a,b]-[d,e]=[a-e,b-d]$

3) Multiplication: $[a,b]\cdot[d,e]=[\min(ad,ae,bd,be),\max(ad,ae,bd,be)]$

4) Division
$$[a,b]/[d,e]=[a,b]\cdot\left[\frac{1}{e},\frac{1}{d}\right]$$
$$=[\min(a/d,a/e,b/d,b/e),\max(a/d,a/e,b/d,b/e)]$$

5.2. Solution of $\bar{A}\cdot\bar{X}+\bar{B}=\bar{C}$

In the crisp equation $ax+b=c$, we obtain the solution $x=\frac{(c-b)}{a}$, if $a\neq 0$

considering the facts $(b-b)=0$ and $\left(\frac{1}{a}\right)a=1$. But if we are trying to apply the same approach with the fuzzy equation $\overline{A}\cdot\overline{X}+\overline{B}=\overline{C}$, we get $\left(\frac{1}{\overline{A}}\right)\left(\overline{A}\overline{X}+\left(\overline{B}-\overline{B}\right)\right)=\left(\frac{1}{\overline{A}}\right)\left(\overline{C}-\overline{B}\right)$, from where the solution is not possible since $\left(\overline{B}-\overline{B}\right)\neq 0$ and $\left(\frac{1}{\overline{A}}\right)\left(\overline{A}\right)\neq 1$.

One way to solve the fuzzy linear equation $\overline{A}\cdot\overline{X}+\overline{B}=\overline{C}$ is to simply fuzzify the crisp solution $\frac{(c-b)}{a}, a\neq 0$. The fuzzified crisp solution is $\frac{\overline{C}-\overline{B}}{\overline{A}}$, where assuming that zero does not belong to the support of $\overline{A}$.

If $\overline{X}_e$ is the value of $\frac{\overline{C}-\overline{B}}{\overline{A}}$ then by using extension principle [2], we get $\overline{X}_e=\max\left\{\pi(a,b,c)\,|\,\frac{(c-b)}{a}=x\right\}$ where

$\pi(a,b,c)=\min\{A(a),B(b),C(c)\}$

Since the expression $\frac{c-b}{a}, a\neq 0$, is continuous in a,b,c we know how to find α – cut of $\overline{X}_e$.

$$x_{e1}(\alpha)=\min\left\{\frac{c-b}{a}\,|\,a\in{}^{\alpha}\overline{A}, b\in{}^{\alpha}\overline{B}, c\in{}^{\alpha}\overline{C}\right\}$$

$$x_{e2}(\alpha)=\max\left\{\frac{c-b}{a}\,|\,a\in{}^{\alpha}\overline{A}, b\in{}^{\alpha}\overline{B}, c\in{}^{\alpha}\overline{C}\right\}$$

where ${}^{\alpha}\overline{X}_e=\left[x_{e1}(\alpha), x_{e2}(\alpha)\right]$ and $0\leq\alpha\leq 1$.

6. Application of Fuzzy Set

In this section, we explored two real-life problems where vagueness can be effectively addressed using fuzzy sets. One key issue is related to business profit, which is effectively solved using a fuzzy equation. Another critical issue involves accurately estimating rainfall through a fuzzy membership function.

6.1. Application in Finance

"DARAZ"[1] bought 10,000 thermal scanners at tk. 5000 each. He wishes to sell around 40% of them for a profit of approximately 25%. The problem is that at what price should the dealer sell the rest if he wants his average profit for all the items to be about 40%?

Solution. If sell percent, $a=40\%=0.40$; percentages of profit, $b=25\%=0.25$; percentages of average profit, $c=40\%=0.40$ and setting price of rest of the item $=p$, then the crisp equation of this problem be like $(10000a)(5000)b+(10000-10000a)(p-150)=(10000)5000c$.

[1](2022) Daraz Online Shopping in Bangladesh. https://buyer-helpcenter.daraz.com.bd/s/page.

$$\Rightarrow p = 5000\left(1+\frac{c-ab}{1-a}\right) = 7500 .$$

That means the dealer should set a fixed selling price tk. 7500 for each of the rest of the items. But in reality it is very difficult to determine such a specific price for rest of the products. Now we construct the fuzzy equation of the given problem by considering three triangular fuzzy numbers $\overline{A} = (0.30/0.40/0.50)$, $\overline{B} = (0.15/0.25/0.35)$ and $\overline{C} = (0.35/0.40/0.45)$. Where $\overline{A}, \overline{B}, \overline{C}$ represent the sell percentage, percentages of profit and percentages of average profit respectively. Hence the fuzzy equation to solve is

$$\left(10000\overline{A}\right)\left(5000\overline{B}\right) + \left(10000 - 10000\overline{A}\right)\left(\overline{P} - 5000\right) = (10000)(5000)\overline{C} \tag{1}$$

Here $\overline{P}$ is the expected sell price for the rest of products. We look forward to the solution $\overline{P}$ by using the method discussed above.

Let ${}^{\alpha}\overline{A} = [0.30 + 0.10\alpha, 0.50 - 0.10\alpha]$, ${}^{\alpha}\overline{B} = [0.15 + 0.10\alpha, 0.35 - 0.10\alpha]$ and ${}^{\alpha}\overline{C} = [0.35 + 0.05\alpha, 0.45 - 0.05\alpha]$ where $\alpha \in [0,1]$; also assume that $\overline{P} > 5000$. The α-cuts of $\overline{P}$ are:

$p_1(\alpha) = \min\left\{p : a \in {}^{\alpha}\overline{A}, b \in {}^{\alpha}\overline{B}, c \in {}^{\alpha}\overline{C}\right\}$ and

$p_2(\alpha) = \max\left\{p : a \in {}^{\alpha}\overline{A}, b \in {}^{\alpha}\overline{B}, c \in {}^{\alpha}\overline{C}\right\}$

${}^{\alpha}\overline{P} = 5000 + 5000\,{}^{\alpha}I = 5000\left(1 + {}^{\alpha}I\right)$; where ${}^{\alpha}I = \left[i_1(\alpha), i_2(\alpha)\right]$.

$$i_1(\alpha) = \frac{(0.35 + 0.05\alpha) - (0.30 + 0.10\alpha)(0.35 - 0.10\alpha)}{0.70 - 0.10\alpha}$$

$$i_2(\alpha) = \frac{(0.45 - 0.05\alpha) - (0.50 - 0.10\alpha)(0.15 + 0.10\alpha)}{0.50 + 0.10\alpha}$$

Now defuzzify this solution by putting the values of $\alpha = 0$ and $\alpha = 1$ we get, $\overline{P} = (6750/7500/8750)$ is also a triangular fuzzy number. Membership function of the solution is given as follows:

$$\overline{P}(x) = \begin{cases} \dfrac{x - 6750}{7500 - 6750}; & 6750 \le x \le 7500 \\ \dfrac{8750 - x}{8750 - 7500}; & 7500 \le x \le 8750 \\ 0; & \text{otherwise} \end{cases}$$

Graphical presentation of this solution is given in **Figure 12**.

6.2. Application in Meteorology

Bangladesh has a warm and humid climate. It experiences heavy rain and tropical cyclones. The average temperature is around 26˚C, but it can range from 15˚C to 34˚C during the year [14]. The warmest months are from April to September, which is also the rainy season. The winter months, from December to February, are cooler and drier. Bangladesh gets about 2200 millimeters (mm) of rain each year. Most areas receive at least 1500 mm, while some northeastern regions get up to 5000 mm of rain annually. Humidity is high all year, especially during the monsoon season from June to October. The Southwest monsoon brings warm and moist air from the Indian Ocean, causing heavy rain. On average, a tropical cyclone classified as a Tropical Storm or stronger hits Bangladesh every two to three years. These storms bring heavy rain, strong winds, and storm surges [14].

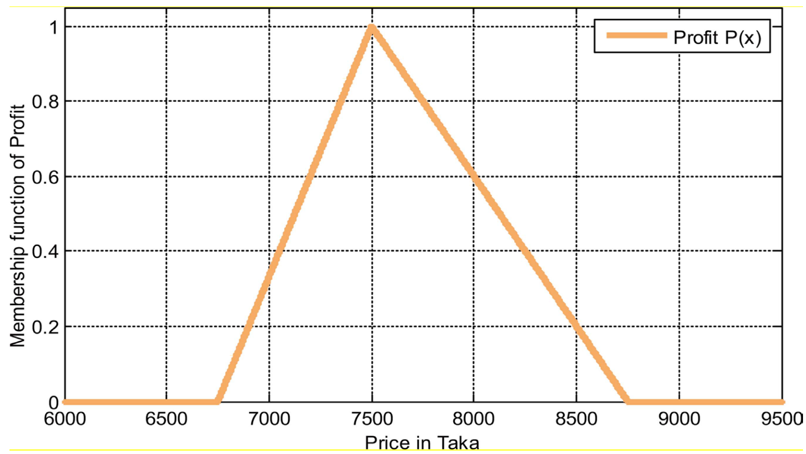


Figure 12. Membership function of profit.

Rainfall Prediction

Weather systems can be influenced by many factors, which can be difficult to model precisely. Fuzzy logic can help categorize conditions into fuzzy sets like "light rain," "moderate rain," and "heavy rain." When certain meteorological indicators (like temperature, humidity, and atmospheric pressure) fall within these fuzzy ranges, forecasts can be generated that express the likelihood of different rainfall levels, accommodating uncertainty. Forecasting rainfall using fuzzy logic involves defining a fuzzy membership function to represent the degree of membership of rainfall data in specific linguistic categories (e.g., "low," "moderate," "high"). We construct a fuzzy membership function for rainfall forecasting, let the rainfall be classified into three categories:

1) Low rainfall (<0.5 mm per hour)

2) Moderate rainfall (<2 mm per hour)

3) High rainfall (>4 mm per hour)

Assume rainfall values range between 0 mm (no rainfall) and 4 mm (very heavy rainfall) per hour. We define Fuzzy Membership Functions are as follows:

1) Low Rainfall (A_1): $$\mu_{A_1}(x)=\begin{cases} 1; & 0 \le x \le 0.5 \\ \dfrac{2-x}{1.5}; & 0.5 < x \le 2 \\ 0; & x > 2 \end{cases}$$

2) Moderate Rainfall (A_2): $$\mu_{A_2}(x)=\begin{cases} 0; & x < 0.5 \text{ or } x > 4 \\ \dfrac{x-0.5}{1.5}; & 0.5 \le x \le 2 \\ \dfrac{4-x}{2}; & 2 < x \le 4 \end{cases}$$

3) High Rainfall (A_3): $\mu_{A_3}(x) = \begin{cases} 0; & x \le 2 \\ \dfrac{x-2}{2}; & 2 < x < 4 \\ 1; & x \ge 4 \end{cases}$

Classification of rainfall is shown in Figure 13.

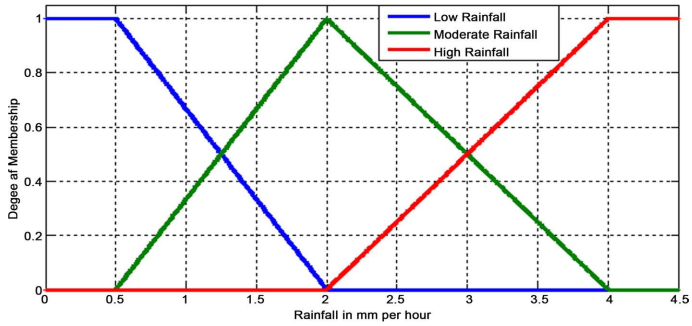


Figure 13. Degree of rainfall.

6.3. Application in Medical Science

Fuzzy set theory allows us to handle imprecise information, making it ideal for medical diagnoses such as diabetes classification. Instead of categorizing patients strictly as "diabetic" or "non-diabetic," fuzzy sets allow for gradual transitions based on membership functions.

Fuzzy Set Representation of Diabetes

Let B be the set of blood sugar levels (in mg/dL). We define a fuzzy set "Diabetic" (D) with a membership function $\mu_D(b)$ that assigns a degree of membership to each blood sugar level. We classify blood sugar levels into three fuzzy sets:

1) Normal Blood Sugar (Non-Diabetic)

2) Pre-diabetic (Borderline Case)

3) Diabetic (High Risk)

Now fuzzy membership function for Diabetes can be defined as follows:

1) Normal Blood Sugar: $\mu_D(b) = 1$ for $b \le 100$, gradually decreasing to 0 for $100 \le b < 125$.

2) Pre-diabetic: $\mu_D(b) = 0$ for $b \le 100$, gradually increasing to 1 at $b = 125$, then decreasing to 0 at $b = 140$.

3) Diabetic: $\mu_D(b) = 0$ for $b \le 125$, gradually increasing to 1 for $b \ge 140$.

Mathematical Representation:

$$\mu_D(b) = \begin{cases} 1; & b \le 100 & (\text{Normal}) \\ \dfrac{125-b}{25}; & 100 < b \le 125 & (\text{Transition from Normal to Pre-diabetic}) \\ \dfrac{b-100}{25}; & 100 < b \le 125 & (\text{Transition from Normal to Pre-diabetic}) \\ \dfrac{140-b}{15}; & 125 < b \le 140 & (\text{Transition from Pre-diabetic to Diabetic}) \\ 1; & b > 140 & (\text{Diabetic}) \end{cases}$$

Graphical representation of this classification is shown in Figure 14.

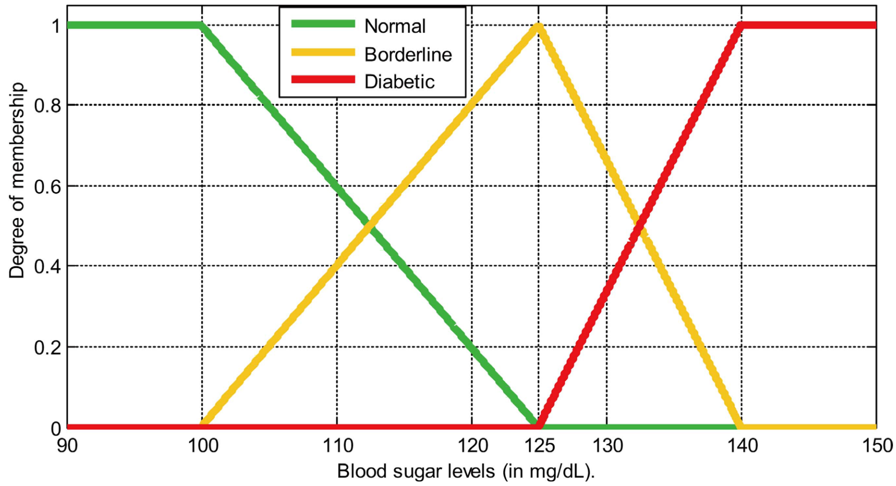


Figure 14. Diabetic classification according to degree of membership.

Sample Calculation: If a patient has a fasting blood sugar level of 130 mg/dL, then

- Normal: $\mu_N(130) = 0$ (not in the normal range)
- Pre-diabetic: $\mu_P(130) = \frac{140-130}{15} = 0.67$
- Diabetic: $\mu_D(130) = \frac{130-125}{15} = 0.33$

Thus, the patient has a 67% membership in the pre-diabetic category and a 33% membership in the diabetic category.

7. Conclusion

Fuzzy set theory serves as a powerful tool for modeling and analyzing uncertainty, offering a flexible alternative to classical set theory. This study explored the fundamental properties of fuzzy sets, including fuzzy numbers, and fuzzy equations. The graphical representation of these properties provided deeper insights into their behavior. The study also examined triangular norms (t-norms) and conorms (t-conorms), which play a crucial role in fuzzy logic operations. Furthermore, solving fuzzy equations, such as linear fuzzy equations, demonstrated the mathematical applications of fuzzy sets in complex problem-solving. Practical applications in finance and meteorology illustrated how fuzzy logic can accurately determine profit margins and predict rainfall, showcasing its relevance in real-world decision-making. Incorporating fuzzy logic into rainfall forecasting significantly enhances the ability to model and understand complex weather patterns. Fuzzy set theory provides a more flexible approach to diagnosing diabetes by allowing a gradual transition between categories instead of a rigid classification. Our model helps in better understanding patient conditions and supports more informed decision-making in medical diagnoses. Overall, this research reaffirms the importance of fuzzy set theory as a versatile tool in mathematics, engineering, and applied sciences. As the field continues to evolve, further advancements in fuzzy mathematics will enhance its integration into emerging technologies, artificial intelligence, and data-driven decision-making. The combination of theoretical insights, graphical representations, and practical applications presented in this study highlights the enduring significance of fuzzy sets in modern problem-solving and uncertainty management.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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