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Fractional-Order Financial System Using Efficient Numerical Methods

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Abstract

This work focuses on the development and analysis of a financial system using advanced mathematical modeling techniques. Starting from an ordinary financial system, we extend it to a fractional-order framework by incorporating the Caputo fractional-order operator. The fractional differential equations are solved both analytically and numerically. Analytical solutions are derived using the Elzaki transform method, providing deeper insights into the system's dynamics. Stability analysis is performed to identify equilibrium points and derive precise conditions for system stability. Furthermore, we explore chaotic behavior within the system and propose effective control strategies using the feedback control method to regulate its dynamics. The results offer significant contributions to understanding and managing complex financial systems, enabling improved decision-making in financial analysis and policy design.

Keywords

Fractional-Order Financial System, Caputo Fractional Derivative, Elzaki Transform Method, Stability Analysis, Equilibrium Points, Chaos in Financial Systems, Feedback Control Method

1. Introduction

Fractional-order systems have emerged as a powerful tool in modeling complex real-world dynamics, particularly in fields such as finance, physics, engineering, and biology. The nonlocal properties and memory effects inherent in fractional derivatives provide a more comprehensive framework for capturing system behaviors compared to traditional integer-order models [1]-[4]. In financial systems, these attributes enable the modeling of intricate economic interactions, revealing insights into market dynamics and stability under varying conditions [5]-[7].

The Caputo derivative has been widely adopted in fractional-order studies due to its ability to handle initial value problems effectively. Numerous studies have employed this derivative to explore financial dynamics, highlighting its flexibility and mathematical rigor [3] [8]. For instance, Diethelm *et al.* [8] provided a detailed analysis of fractional-order systems using the Caputo derivative, while Podlubny [3] established its foundational role in fractional calculus applications.

The Elzaki transform, another powerful mathematical tool, has been used to simplify and solve fractional differential equations (FDEs), proving particularly effective in handling complex fractional-order systems with memory-dependent behavior [9]-[11]. This method has been instrumental in reducing computational complexity, as demonstrated in applications across engineering and finance [9] [12]. The combination of these methods has led to breakthroughs in understanding and controlling chaotic systems in finance [5] [13].

In the realm of numerical solutions, the Predictor-Corrector Adams-Bashforth-Moulton (PC-ABM) method has gained prominence for its efficiency and accuracy in solving FDEs. This method enables researchers to obtain high-fidelity solutions, even for systems with strong nonlinearities [7] [8] [14]. Garrappa [14] extended the method's application to fractional chaotic systems, demonstrating its robustness in predicting long-term dynamics. Recent studies have applied this method to fractional financial systems, offering precise insights into the stability and behavior of such systems under varying parameters [7] [15].

Synchronization and control of fractional-order financial systems remain active areas of research. Fractional feedback control, in particular, has been shown to be an effective method for stabilizing chaotic systems and guiding them toward desired trajectories. This approach leverages the memory-dependent nature of fractional derivatives to achieve enhanced control precision [4] [16]. Additionally, nonlinear control strategies have been developed to tackle the unique challenges posed by chaotic financial systems, enabling the stabilization and synchronization of these systems under uncertain conditions [10] [15].

Driven by the advancements in fractional calculus and its applications in finance, this study develops a comprehensive approach to analyze and control a fractional financial system. Key contributions of this research include modeling the financial system using the Caputo derivative, solving the system using the Elzaki transform and PC-ABM method, and applying fractional feedback and nonlinear control to stabilize the system. Furthermore, this work delves into the stability analysis of the system.

2. Preliminaries, Algorithms and Modeling

2.1. Preliminaries

Fractional-order calculus offers a mathematical framework to describe systems with memory effects, making it suitable for modeling physical and engineering processes. The definitions and tools used in this study are as follows:

Definition 1 [3] *The Caputo derivative is widely used in initial value problems* *due to its compatibility with physical conditions. It is defined as*:

$$D_c^{\alpha} f(x)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,\mathrm{d}t, & \text{if } n-1\le\alpha<n,\ n\in\mathbb{N},\\[2ex] \dfrac{\mathrm{d}^n}{\mathrm{d}x^n} f(x), & \text{if } \alpha=n,\ n\in\mathbb{N},\end{cases}\tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function, $f^{(n)}(t)$ is the *n*th derivative of $f(t)$, and n is a positive integer.

Definition 2 [17] *The Elzaki transform is a powerful tool for solving differential equations. It is defined as*:

$$E\{f(t)\}=T(v)=v\int_0^{\infty} f(t)\mathrm{e}^{-\frac{t}{v}}\mathrm{d}t,\ t>0,\ v\in\mathbb{R}.\tag{2}$$

Definition 3 [12] *The Differential Transform* (*DT*) *simplifies the analysis of differential equations by converting them into algebraic equations. It is defined as*:

$$T_k=\frac{1}{k!}\frac{\mathrm{d}^k f(t)}{\mathrm{d}t^k}\bigg|_{t=t_0},\tag{3}$$

where T_k represents the transformed coefficients of $f(t)$, and t_0 is the initial point. The original function $f(t)$ can then be reconstructed as:

$$f(t)=\sum_{k=0}^{\infty} T_k (t-t_0)^k.\tag{4}$$

Definition 4 [18] *For the Caputo fractional derivative* $D_{\vartheta}^{u}(\Psi(\vartheta))$, *its Elzaki transform is expressed as*:

$$E\left\{D_{\vartheta}^{u}(\Psi(\vartheta))\right\}(s)=s^{-u}\tilde{U}(s)-\sum_{k=0}^{m-1} s^{2-u+k}\Psi^{(k)}(0),\tag{5}$$

where $m-1<u<m$.

Proposition 1 [8] *Let* $x=x^*$ *be an equilibrium point of a fractional nonlinear system*:

$$D^{\alpha}x=f(x),\ 0<\alpha<1.\tag{6}$$

If the eigenvalues of the Jacobian matrix $J=\frac{\partial f}{\partial x}\Big|_{x^*}$ satisfy:

$$\min\left|\arg(\lambda_i)\right|>\frac{\alpha\pi}{2},\ i=1,2,\cdots,n,\tag{7}$$

then the system is asymptotically stable at the equilibrium point $x=x^*$.

2.2. Algorithms

This study employs three main methodologies for analyzing fractional-order systems: Predictor-Corrector Adams-Bashforth-Moulton (PC-ABM) method, Elzaki transform method, and fractional feedback control.

Algorithm 1: Predictor-Corrector Adams-Bashforth-Moulton (PC-ABM)

Method [8]

The PC-ABM method solves fractional differential equations numerically, involving two main steps:

1) **Predictor Step:** Approximates the fractional derivative as:

$$D^{\alpha} y(t) \approx \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(t)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \tag{8}$$

using the Adams-Bashforth predictor.

2) **Corrector Step:** Refines predictions using the Adams-Moulton corrector, ensuring stability and accuracy.

Algorithm 2: Elzaki Transform Method [9] [12]

The Elzaki transform method simplifies solving fractional differential equations by converting them into algebraic equations in the transform domain. Nonlinear terms are effectively handled using the Differential Transform (DT) method, which is particularly efficient for nonlinear fractional systems.

Algorithm 3: Fractional Feedback Control [16]

Feedback control for fractional-order systems aims to stabilize or guide the system's trajectory. The dynamic equation is given by:

$$D^{\alpha} x(t) = f(x, u, t), \tag{9}$$

where $u(t)$ is the control input. The control law $g(x,t)$ is designed to stabilize the system or achieve trajectory tracking:

$$D^{\alpha} x(t) = f(x, g(x,t), t). \tag{10}$$

Fractional feedback control enhances stability and performance for systems with memory-dependent behaviors.

2.3. Modeling

The financial system models the nonlinear dynamics among critical economic variables, such as interest rates, investment demand, and price indices. A widely studied three-dimensional integer-order financial model introduced by Chen *et al.* [5] is given as:

$$\begin{cases} x'(t) = z + (y-a)x, \\ y'(t) = 2 - by - x^2, \\ z'(t) = xy - x - cz, \end{cases} \tag{11}$$

where $x(t)$, $y(t)$, and $z(t)$ represent the interest rate, investment demand, and price index, respectively. Here, a is the saving amount, b is the cost per investment, and c is the elasticity of the demand of the commercial markets, and all three constants are nonnegative.

Inspired by a four-dimensional extension discussed in Xia Lu's work [15], which was originally derived from prior studies, the model was expanded by adding the variable $u(t)$, representing the average profit margin. The resulting system is:

$$\begin{cases} x'(t) = z + (y-a)x + u, \\ y'(t) = 2 - by - x^2, \\ z'(t) = xy - x - cz, \\ u'(t) = -dxy - gu, \end{cases} \tag{12}$$

where $d, g > 0$ are parameters modeling the coupling and dissipation effects of $u(t)$.

To account for memory and hereditary properties, the integer-order derivatives are replaced with the Caputo fractional derivatives, leading to the fractional-order system:

$$\begin{cases} {}_0^C D^q x(t) = z + (y-a)x + u, \\ {}_0^C D^q y(t) = 2 - by - x^2, \\ {}_0^C D^q z(t) = xy - x - cz, \\ {}_0^C D^q u(t) = -dxy - gu, \end{cases} \tag{13}$$

where $q \in (0,1]$ is the fractional order of the derivative. The fractional formulation generalizes the classical system ($q = 1$) and better captures the dynamics of financial systems by introducing memory effects and nonlocal interactions [3].

3. Numerical Solution of Fractional System

This section presents the numerical solution of the fractional differential equation (FDE) system (13) using MATLAB. The Predictor-Corrector Adams-Bashforth-Moulton (PECE) scheme [7] is employed due to its efficiency in solving fractional systems and accounting for memory effects (**Figure 1**).

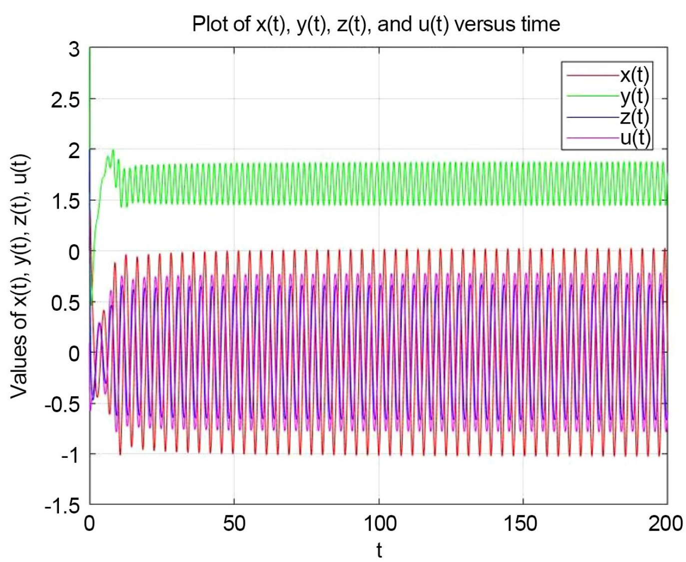


Figure 1. Numerical solution of the FDE system.

The key behaviors observed are as follows:

- $x(t)$**:** Initially oscillates significantly but stabilizes at lower amplitudes over time due to fractional damping.
- $y(t)$**:** Rapidly stabilizes to a steady value with no oscillations.
- $z(t)$ **and** $u(t)$**:** Exhibit oscillations similar to $x(t)$ but stabilize more quickly.

In contrast, integer-order systems lack these stabilizing memory effects and exhibit persistent oscillations across all variables.

4. Analytical Solution of Fractional System

In this section, the Elzaki transform method [9] is applied to solve the fractional differential equation (FDE) system (13) analytically. This approach converts the system into algebraic equations, facilitating efficient series solutions.

Applying the Elzaki transform with the initial conditions $x(0)=x_0$, $y(0)=y_0$, $z(0)=z_0$, and $u(0)=u_0$, the transformed system is obtained as:

$$\begin{cases} x(t)=E^{-1}\left\{v^q E\left\{z+(y-a)x+u\right\}\right\}, \\ y(t)=E^{-1}\left\{v^q E\left\{1-by-x^2\right\}\right\}, \\ z(t)=E^{-1}\left\{v^q E\left\{-x-cz\right\}\right\}, \\ u(t)=E^{-1}\left\{v^q E\left\{-dxy-gu\right\}\right\}. \end{cases} \tag{14}$$

Since nonlinear terms pose a challenge for the Elzaki transform, the Differential Transform Method (DTM) is employed to handle them effectively and compute the solution coefficients recursively [7]. This hybrid approach ensures both accuracy and efficiency in solving fractional nonlinear systems.

The final solution is expressed as a series:

$$\begin{cases} x(t)=x_0+\sum_{n=1}^{\infty} c_n^{(x)} t^{nq}, \\ y(t)=y_0+\sum_{n=1}^{\infty} c_n^{(y)} t^{nq}, \\ z(t)=z_0+\sum_{n=1}^{\infty} c_n^{(z)} t^{nq}, \\ u(t)=u_0+\sum_{n=1}^{\infty} c_n^{(u)} t^{nq}, \end{cases}$$

where $c_n^{(x)}$, $c_n^{(y)}$, $c_n^{(z)}$, and $c_n^{(u)}$ are determined recursively using DTM.

Convergence and Validity of the Solution

To ensure the accuracy and reliability of the series solution, its convergence properties must be examined. Previous studies [19] [20] have established that power series solutions for fractional differential equations converge when the coefficients $c_n^{(x)}, c_n^{(y)}, c_n^{(z)}, c_n^{(u)}$ decrease sufficiently as $n \to \infty$.

For our system, convergence is ensured if:

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} t^q < 1. \tag{15}$$

This condition guarantees that the series remains bounded for all t. The convergence behavior depends on the fractional order q and system parameters

a,b,c,d,g.

Since the Elzaki transform is closely related to the Laplace transform, it ensures the well-posedness of the transformed system. Additionally, numerical verification confirms the rapid convergence of the series for suitable parameter choices. Thus, the combined Elzaki-DTM approach provides a robust and accurate solution framework for fractional nonlinear systems.

5. Stability Analysis of Fractional System

The stability of the fractional-order financial system (13) is analyzed through equilibrium points and eigenvalue conditions. Using numerical and analytical methods, we determine the equilibrium points and evaluate stability criteria based on Proposition (1).

Equilibrium Points Analysis

The equilibrium points are obtained by setting $D^q x(t) = D^q y(t) = D^q z(t) = D^q u(t) = 0$. Solving the resulting equations yields three equilibrium points:

$$E_1 = \left(0, \frac{1}{b}, 0, 0\right),\ E_2 = \left(\gamma, \frac{g+acg}{c(g-d)}, \frac{-\gamma}{c}, \frac{d\gamma(1+ac)}{c(d-g)}\right),$$

$$E_3 = \left(-\gamma, \frac{g+acg}{c(g-d)}, \frac{\gamma}{c}, \frac{-d\gamma(1+ac)}{c(d-g)}\right).$$

Here, $\gamma = \pm\sqrt{1+\frac{gb+acgb}{c(d-g)}}$.

Stability Condition

To analyze the stability of the system (13), we compute the Jacobian matrix:

$$J = \begin{bmatrix} y-a & x & 1 & 1 \\ -2x & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ -dy & -dx & 0 & -g \end{bmatrix}.$$

Substituting $E_1 = \left(0, \frac{1}{b}, 0, 0\right)$, the Jacobian becomes:

$$J(E_1) = \begin{bmatrix} \frac{1}{b}-a & 0 & 1 & 1 \\ 0 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ -\frac{d}{b} & 0 & 0 & -g \end{bmatrix}.$$

To attain stability, we calculate the eigenvalues and adjust them to be either negative real numbers or complex numbers, ensuring stability.

We aim to find suitable values for a, b, c, d, and g such that:

- λ_1 and λ_2 are negative.
- λ_3 and λ_4 are complex.

This ensures that the system is stable at the first equilibrium point using Proposition (1).

Eigenvalue Analysis

The eigenvalues of $J(E_1)$ are:

$$\lambda_1 = -b,\ \lambda_2 = \frac{\sigma_2}{\sigma_1} + \sigma_1 - \frac{\sigma_4}{3b},\ \lambda_{3,4} = -\frac{\sigma_2}{2\sigma_1} - \frac{\sigma_1}{2} - \frac{\sigma_4}{3b} \pm i\frac{\sqrt{3}\left(\frac{\sigma_2}{\sigma_1} - \sigma_1\right)}{2}.$$

where

$$\sigma_1 = \left(\sqrt{\left(\frac{\sigma_4^3}{27b^3} + \frac{cd + bg - cg + abcg}{2b} - \sigma_3\right)^2 - \sigma_2^3}\right)^{1/3},$$

$$\sigma_2 = \frac{\sigma_4^2}{9b^2} - \frac{b - c + d - g + abc + abg + bcg}{3b},$$

$$\sigma_3 = \frac{\sigma_4\left(b - c + d - g + abc + abg + bcg\right)}{6b^2},$$

$$\sigma_4 = ab + bc + bg - 1.$$

Stability Criteria

1) **Condition for** λ_1**:** $\lambda_1 = -b < 0$, which is satisfied for $b > 0$.

2) **Condition for** λ_2**:** To ensure $\lambda_2 < 0$, the following inequality must hold:

$$\frac{\sigma_2}{\sigma_1} + \sigma_1 < \frac{\sigma_4}{3b},$$

3) **Condition for** $\lambda_{3,4}$**:** For $\lambda_{3,4}$ to be complex, $\sigma_2 < 0$ ensures the square root remains valid:

$$\sigma_4^2 < 3b\left(b - c + d - g + abc + abg + bcg\right).$$

$$b^2\left(3 + g(a + c + g) - (a + c)^2\right) < b(c - 3d + g - 2a) + 1.$$

Combining the conditions, the stability criteria become:

$$\frac{\sigma_2}{\sigma_1} + \sigma_1 < \frac{\sigma_4}{3b} \tag{16}$$

$$b^2\left(3 + g(a + c + g) - (a + c)^2\right) < b(c - 3d + g - 2a) + 1 \tag{17}$$

Numerical Validation

The coefficients are chosen as follows:

$$a = 0.1,\ b = 0.3,\ c = 0.1,\ d = 0.9,\ g = 0.6.$$

With these values, the equilibrium points are calculated as:

$$E_1 = (0, 3.3333, 0, 0),\ E_2 = (2.6833, -20.2000, -26.8328, 81.3034),$$

$$E_3 = (-2.6833, -20.2000, 26.8328, -81.3034).$$

Stability Test for E_1

The eigenvalues are:

$$\lambda_1 = -0.3,\ \lambda_2 = -0.2749,\ \lambda_3 = 1.4041 + 0.7727i,\ \lambda_4 = 1.4041 - 0.7727i.$$

Based on Proposition (1), E_1 is asymptotically stable when:

$$0 < q < 0.3203.$$

Stability Test for E_2 and E_3

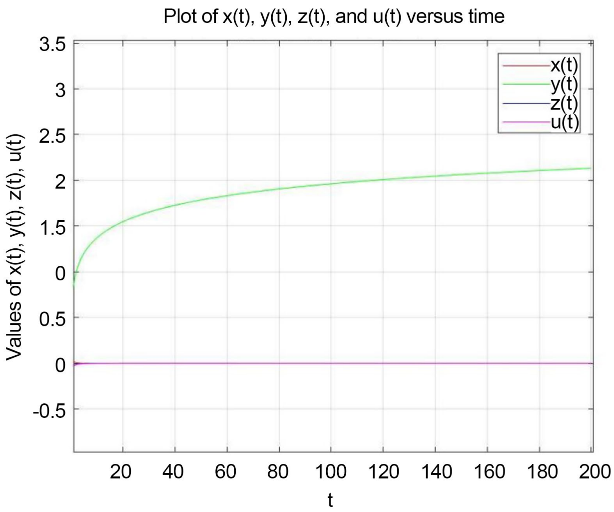
The eigenvalues for both E_2 and E_3 are:

$$\lambda_1 = -20.4198,\ \lambda_2 = -1.0368,\ \lambda_3 = 0.2412,\ \lambda_4 = -0.0846.$$

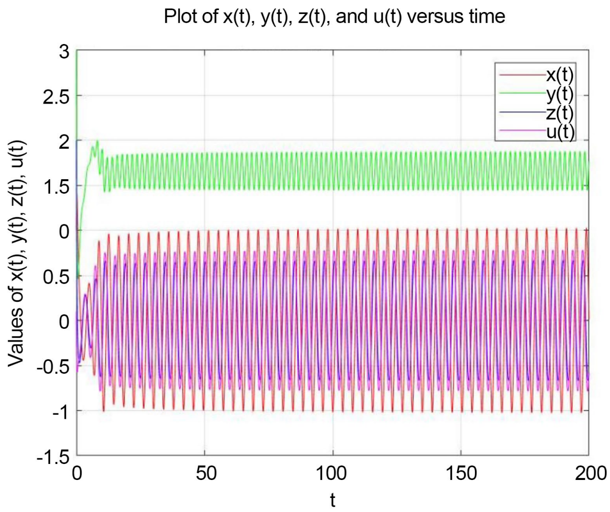
Both points are unstable as they fail to satisfy the conditions in Proposition (1). Additionally, these points do not meet the derived stability criteria (16) and (17).

Numerical Results

Figure 2 illustrates the system's behavior for $q < 0.3203$ (stable) and $q > 0.3203$ (unstable).



Stable ($q < 0.3203$)



Unstable ($q > 0.3203$)

Figure 2. Stability behavior at E_1.

- **Stable ($q < 0.3203$):** The system converges to equilibrium, with all state variables stabilizing over time. Oscillations are absent, confirming system stability.
- **Unstable ($q > 0.3203$):** Persistent oscillations occur, and the system fails to reach a steady state, indicating instability.

Stability Analysis of E_2 and E_3

The stability of a fixed point in a dynamical system is determined by the eigenvalues (λ) of the Jacobian matrix. A point is stable if:

1) All eigenvalues are either complex or purely negative real numbers.

2) If any eigenvalue is a positive real numbers, the system becomes unstable.

Here, we analyze the stability of the second equilibrium point (E_2) and demonstrate its instability under the given conditions. The same reasoning applies to E_3.

Key Analysis Steps

1) **Condition** $d > g$**:** This ensures that γ remains real (not complex), which is crucial for analyzing system stability.

2) **Analysis of** σ_2**:** If σ_2 is complex, then both eigenvalues λ_1 and λ_2 must also be complex.

3) **Analysis of** σ_1**:** If σ_2 is complex, it is impossible for σ_1 to also be complex. This ensures that λ_3 and λ_4 remain real.

4) **Analysis of** λ_3 **and** λ_4**:** It is impossible for both λ_3 and λ_4 to be negative under the given conditions. At least one eigenvalue must be positive, rendering the second equilibrium point unstable.

Jacobian Matrix at E_2

The Jacobian matrix at E_2 is expressed as:

$$J(E_2) = \begin{bmatrix} \frac{g+acg}{c(g-d)} - a & \gamma & 1 & 1 \\ -2\gamma & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ -\frac{d(g+acg)}{c(g-d)} & -d\gamma & 0 & -g \end{bmatrix}.$$

The eigenvalues are:

$$\lambda_1 = -\sigma_3 - \frac{\sigma_{17}}{4\sigma_{19}} - \sigma_2,\ \lambda_2 = -\sigma_3 - \frac{\sigma_{17}}{4\sigma_{19}} + \sigma_2,$$

$$\lambda_3 = \sigma_3 - \frac{\sigma_{17}}{4\sigma_{19}} - \sigma_1,\ \lambda_4 = \sigma_3 - \frac{\sigma_{17}}{4\sigma_{19}} + \sigma_1.$$

Proof of Instability:

We analyze the stability of E_2 using the eigenvalues of the Jacobian matrix. The steps are as follows:

1) **Condition** $d > g$**:** Ensures positivity of key parameters:

$$\sigma_{19} = c(d-g),\ \sigma_{17} = g + c^2 d - c\left(g^2 + g\right) + acd + bcd - bcg + cdg.$$

- Since $d > g$ and $c \neq 0$, $\sigma_{19} > 0$.
- The dominant terms involving d ensure $\sigma_{17} > 0$.

This guarantees no undefined or negative values in subsequent eigenvalue analysis.

2) **If** σ_2 **is complex,** λ_1 **and** λ_2 **are also complex:**

The condition for σ_2 to be complex is:

$$12\sigma_{12}\sqrt{\sigma_9} - \sigma_8 - \sigma_7 - \sigma_6 + \sigma_4 < 0.$$

Since λ_1 and λ_2 depend on σ_2:

$$\lambda_1 = -\sigma_3 - \frac{\sigma_{17}}{4\sigma_{19}} - \sigma_2,\ \lambda_2 = -\sigma_3 - \frac{\sigma_{17}}{4\sigma_{19}} + \sigma_2,$$

the imaginary component of σ_2 makes both eigenvalues complex.

3) **If** σ_2 **is complex,** σ_1 **cannot be complex, and** λ_3 **and** λ_4 **remain real:**

The condition for σ_1 to be complex is:

$$12\sigma_{12}\sqrt{\sigma_9}-\sigma_8-\sigma_7-\sigma_6-\sigma_4<0.$$

The key difference lies in the sign of σ_4: For σ_2, positive σ_4 pushes the argument towards negativity. For σ_1, negative σ_4 pushes the argument towards positivity.

Thus, σ_1 and σ_2 cannot both be complex, ensuring λ_3 and λ_4 remain real.

4) **It is impossible for both λ_3 and λ_4 to be negative:**

The expressions for λ_3 and λ_4 are:

$$\lambda_3=\sigma_3-\frac{\sigma_{17}}{4\sigma_{19}}-\sigma_1,\ \lambda_4=\sigma_3-\frac{\sigma_{17}}{4\sigma_{19}}+\sigma_1.$$

For λ_3 to be negative:

$$\sigma_3-\sigma_1<\frac{\sigma_{17}}{4\sigma_{19}}.$$

For λ_4 to be negative:

$$\sigma_3+\sigma_1<\frac{\sigma_{17}}{4\sigma_{19}}.$$

Both conditions cannot hold simultaneously because: A larger σ_1 making λ_3 negative causes λ_4 to become positive. A smaller σ_1 making λ_4 negative causes λ_3 to become positive.

At the end, it is impossible for both λ_3 and λ_4 to be negative; at least one must be positive.

We may conclude that, at least one eigenvalue for E_2 is positive, violating the stability conditions that require all eigenvalues to be complex or purely negative real numbers. Therefore, E_2 is unstable. The same analysis confirms that E_3 is also unstable.

6. Control of the Fractional System

This section explores the application of nonlinear feedback control to stabilize a fractional-order financial system (13). Control inputs $w(t)=\left[w_1(t),w_2(t),w_3(t),w_4(t)\right]^{\mathrm{T}}$ are introduced to drive the system's state variables (x,y,z,u) towards a desired equilibrium $(\tilde{x},\tilde{y},\tilde{z},\tilde{u})$ [13].

Controlled System and Error Dynamics

The controlled system is expressed as:

$$\begin{cases} {}_{t_0}^{C}\mathrm{D}_t^q x(t)=z+(y-a)x+u+w_1, \\ {}_{t_0}^{C}\mathrm{D}_t^q y(t)=1-by-x^2+w_2, \\ {}_{t_0}^{C}\mathrm{D}_t^q z(t)=-x-cz+w_3, \\ {}_{t_0}^{C}\mathrm{D}_t^q u(t)=-dxy-gu+w_4. \end{cases} \tag{18}$$

Define the error terms e_i = current state – desired state, such that:

$$e_1=x-\tilde{x},\ e_2=y-\tilde{y},\ e_3=z-\tilde{z},\ e_4=u-\tilde{u}.$$

The error dynamics evolve as:

$$\begin{cases} D^q e_1 = e_3 + e_4 - ae_1 + yx - \tilde{y}\tilde{x} + w_1, \\ D^q e_2 = -be_2 - x^2 + \tilde{x}^2 + w_2, \\ D^q e_3 = -e_1 - ce_3 + w_3, \\ D^q e_4 = -ge_4 - dxy + d\tilde{x}\tilde{y} + w_4. \end{cases} \tag{19}$$

Control Law Design

To cancel nonlinear terms and stabilize the system, the control inputs are defined as:

$$\begin{cases} w_1(t) = \tilde{y}\tilde{x} - yx, \\ w_2(t) = x^2 - \tilde{x}^2, \\ w_3(t) = e_1, \\ w_4(t) = dxy - d\tilde{x}\tilde{y}. \end{cases} \tag{20}$$

Substituting these control laws simplifies the error dynamics to:

$$\begin{cases} D^q e_1 = e_3 + e_4 - ae_1, \\ D^q e_2 = -be_2, \\ D^q e_3 = -ce_3, \\ D^q e_4 = -ge_4. \end{cases} \tag{21}$$

The controlled system is as follows:

$$\begin{cases} {}_{t_0}^{C}D_t^q x(t) = z - ax + u + \tilde{y}\tilde{x}, \\ {}_{t_0}^{C}D_t^q y(t) = 1 - by - \tilde{x}^2, \\ {}_{t_0}^{C}D_t^q z(t) = -cz - \tilde{x}, \\ {}_{t_0}^{C}D_t^q u(t) = -gu - d\tilde{x}\tilde{y}. \end{cases} \tag{22}$$

Results and Observations

Figure 3 illustrates the system behavior before and after applying control.

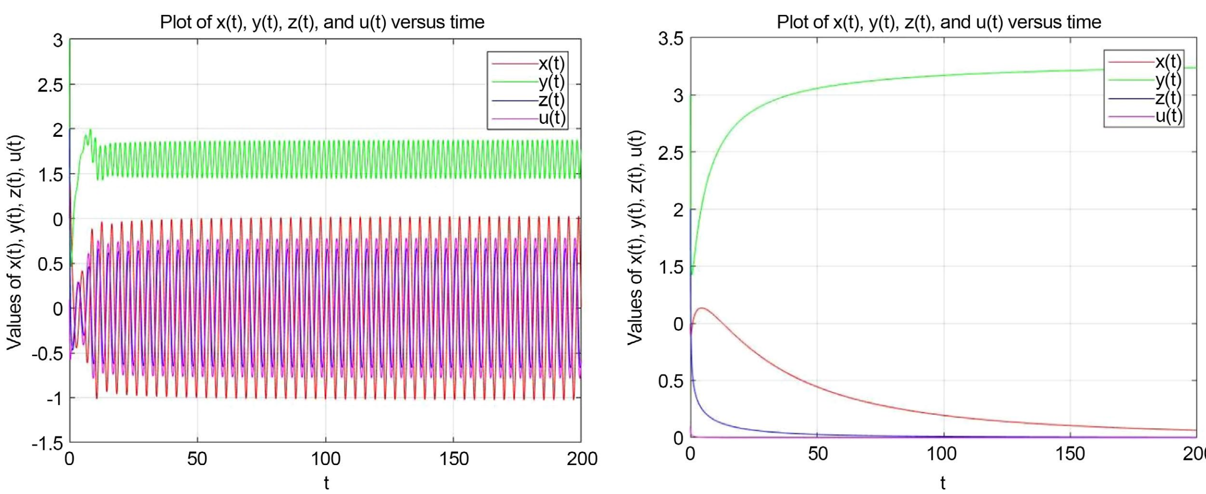


Figure 3. Controlled system.

- **Before control:** The system demonstrates instability with persistent oscillations in all variables.
- **After control:** All variables converge to steady states, with oscillations effectively dampened, indicating stability.
- **Significance:** The control mechanism improves stability, reduces volatility, and ensures predictable performance, making the system suitable for practical applications.

6.1. Challenges and Limitations

While the proposed control strategy stabilizes the system, several challenges remain:

- **Model Uncertainty:** Financial systems experience unpredictable disturbances and structural changes, requiring adaptive or robust control approaches.
- **Parameter Sensitivity:** Control effectiveness depends on accurate parameter estimation (a,b,c,d,g), with deviations potentially causing instability.
- **Computational Complexity:** Fractional-order control demands numerical approximations, increasing computational load, which may hinder real-time implementation.
- **Practical Constraints:** Some financial variables (e.g., investor sentiment) cannot be directly controlled, raising feasibility concerns for external interventions.

6.2. Enhancing the Control Strategy

The current approach primarily cancels nonlinear terms, effectively linearizing the system. To improve novelty:

- **Leverage Fractional-Order Benefits:** Fractional models better capture memory effects and long-term dependencies, enhancing financial system representation.
- Explore Advanced Control Techniques: Adaptive control or fractional sliding mode control can improve robustness and performance under uncertainties.

These refinements align the control strategy with real-world financial dynamics, enhancing both accuracy and applicability.

7. Conclusions

This study investigated the fractional-order financial system, focusing on its stability and control dynamics. Using the Caputo fractional derivative framework, the system's equilibrium points were derived, and their stability was rigorously analyzed. For the first equilibrium point E_1, the stability was proven under specific conditions. Numerical simulations verified the theoretical results, demonstrating that the system transitions to stability when these conditions are met.

For the second and third equilibrium points (E_2 and E_3), the analysis showed that at least one eigenvalue always has a positive real part, rendering these points inherently unstable. These results were validated through eigenvalue computations and verified by numerical simulations, aligning with the theoretical framework presented in [3] [7].

The study further introduced a nonlinear feedback control strategy to stabilize the fractional financial system. The control inputs were designed to suppress chaotic behavior, ensuring convergence of the system states to a desired equilibrium. The proposed control method effectively dampened oscillations and stabilized the system, as validated by numerical experiments. This demonstrates the practical applicability of fractional-order modeling and control in financial systems [13] [16].

In conclusion, this work provides a robust framework for analyzing and controlling fractional-order financial systems. Future research can extend these findings by exploring adaptive control strategies to account for parameter uncertainties and dynamic changes in financial systems. Additionally, further studies could investigate the implications of fractional-order dynamics on financial market predictability and risk assessment [5] [15].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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