American Journal of Computational Mathematics, 2024, 14, 391-400
https://www.scirp.org/journal/ajcm
ISSN Online: 2161-1211
ISSN Print: 2161-1203

NK-Labeling of Graphs

Nasreen Almohanna, Khawlah Alhulwah

Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia
Email: nimohanna@imamu.edu.sa, khalhulwah@imamu.edu.sa

How to cite this paper: Almohanna, N. and Alhulwah, K. (2024) *NK*-Labeling of Graphs. *American Journal of Computational Mathematics*, **14**, 391-400.
https://doi.org/10.4236/ajcm.2024.144020

Received: September 3, 2024
Accepted: October 18, 2024
Published: October 21, 2024

Abstract

A graph labeling is the assigning of labels to the vertices, edges, or both (usually non-negative integers), often satisfying some prescribed requirements. This terminology has become standard. A graph Gs edges can be colored by assigning a different color to each of its edges. The edge coloring is appropriate if adjacent edges are given different colors. In this work, we introduce a new labeling called *NK*-labeling. Let $c: E(G) \to \mathbb{N}$ be a proper edge coloring of G which induces a proper vertex coloring $c': V(G) \to \mathbb{Z}_n$ defined by

$$c'(v) \equiv \sum_{e \in E_v} c(e) \bmod n$$

Such that E_v is the set of edges incident with v in *G*. The minimum positive integer for which the graph G has *NK*-labeling called *NK*-chromatic index and denoted by $\chi'_{NK}(G)$. We study the *NK*-labeling of several well-known classes of graphs. It is shown that the *NK*-chromatic of the path P_n for $n \geq 4$ is three and for odd n, the *NK*-chromatic of the complete graph K_n is n. Other results dealing with the *NK*-labeling are also presented.

Keywords

Graph, Edge Coloring, *NK*-Labeling, Label, Path, Cycle, Wheel, Complete Graph

1. Introduction

For terms and symbols related to graph theory that are not covered in this paper, please see the book [1]. In 1968, Alexander Rosa received his Ph.D from Slovak Academy of Science under Anton Kotzig after he published his paper titled "On certain valuations of the vertices of a graph," which is the source of many labelings. The most attention of his labeling is β-valuation, which recalled a graceful labeling by Solomon W. Golomb [2]. Graph coloring is one of the most useful models in graph theory. Many problems relating to computer registers allocation, electronic bandwidth allocation, and school scheduling have all been resolved using it.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A labeling $c:E(G)\to\{1,2,\cdots,k\}$ such that $c(e)\neq c(f)$ for every two adjacent edges e and f in G is called proper edge coloring.

We frequently have an interest in proper edge coloring that use a minimum number of colors which called chromatic index (edge chromatic number) $\chi'(G)$ of a graph G. A graph G is k-colorable, if $\chi'(G)\le k$, and G is k-chromatic, if $\chi'(G)=k$.

Maheo and Sacle in [3] studied the irregular-sum chromatic index for connected graph G with vertex set $V(G)$ and edge set $E(G)$ of order $n\ge 3$. A proper edge coloring $c:E(G)\to\{1,2,\cdots,k\}$ for some integer $k\ge 2$ is called an irregular-sum chromatic coloring of G if the induced vertex coloring $c':V(G)\to\mathbb{N}$ defined by

$$c'(v)=\sum_{e\in E_v}c(e)$$

is irregular (vertex-distinguishing). The minimum positive integer for which G has such an irregular-sum chromatic coloring is called an irregular-sum chromatic index of G and is denoted by $\chi'_{is}(G)$.

In [4] [5], Ping Zhang introduced the labeling for a graph G of order n. Let $c:E(G)\to\mathbb{Z}_k$, $k\ge n$ be an unrestricted edge coloring (where the adjacent edges may be colored the same). The edge coloring c induces a vertex coloring $c':V(G)\to\mathbb{Z}_k$ defined by

$$c'(v)=\sum_{e\in E_v}c(e)$$

where the sum is computed in $\mathbb{Z}_k$. For some positive integer k, such an edge coloring c is always present. The modular edge-gracefulness $meg(G)$ of G is the minimum k for which such a vertex-distinguishing edge coloring of G exists. Thus, $meg(G)\ge n$ for every connected graph G of order $n\ge 3$. If $meg(G)=n$, then G is called a modular edge-graceful graph and a vertex-distinguishing edge coloring $c:E(G)\to\mathbb{Z}_n$ is called a modular edge-graceful labeling as well as a modular edge-graceful coloring of G.

Based on established graph coloring ideas and inspired by irregular-sum and modular edge-graceful labeling we present a new concept for coloring graphs focusing on proper coloring in this paper.

Let $\mathbb{N}$ be the set of all positive integers and let E_v denoted the set of edges incident with v in G. For a connected simple graph $G(V,E)$ of order $n\ge 3$, A proper edge coloring $c:E(G)\to\mathbb{N}$ induces a proper vertex coloring $c':V(G)\to\mathbb{Z}_n$, defined by

$$c'(v)\equiv\sum_{e\in E_v}c(e)\bmod n$$

The minimum positive integer for which G has NK-labeling is called NK-chromatic index and denoted by $\chi'_{NK}(G)$. A graph G with chromatic number

$\chi'_{NK}(G) \le k$ is called a *k*-*NK*colorable graph. If no such *k*, the graph *G* is not *NK*-colorable.

2. Preliminaries

Proposition 2.1 *For any connected graph G*

$$\Delta(G) \le \chi'(G) \le \chi'_{NK}(G)$$

where $\Delta(G)$ is the maximum degree of *G*.

Proposition 2.2 *If G is a connected graph of order at least* 3 *such that G contains two adjacent vertices of maximum degree, then* $\chi'_{NK}(G) \ge \Delta(G)+1$.

An illustration of the *NK*-chromatic index. We show in Figure 1, $\chi'_{NK}(G) \le 4$ and from preposition 2.2, $\chi'_{NK}(G) \ge 4$. Hence, $\chi'_{NK}(G) = 4$.

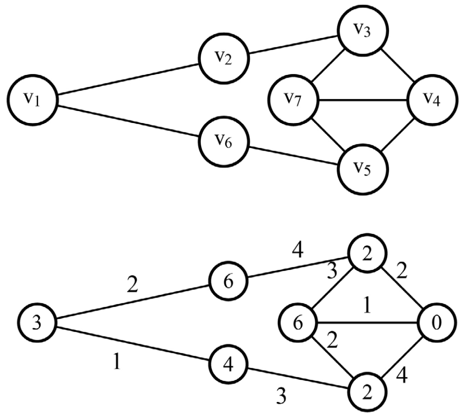

Figure 1. An illustration of *NK*-labeling.

3. Well Known Classes of Graphs

We now turn our attention to two other well-known classes of graphs, namely path and cycles.

Theorem 3.1 *Let n be an integer greater than* 4, *the NK-chromatic index of the path* P_n *of order n is three.*

Proof. By preposition 2.2, $\chi'_{NK}(P_n) \ge 3$. To prove $\chi'_{NK}(P_n) \le 3$, for $1 \le i \le n-1$, assign the edge $v_i v_{i+1}$ by a color *c* as following.

$$c(v_i v_{i+1}) = \begin{cases} 1 & \text{if } i \equiv 1 \bmod 3 \\ 2 & \text{if } i \equiv 2 \bmod 3 \\ 3 & \text{if } i \equiv 0 \bmod 3 \end{cases}$$

Case 1: $n \equiv 0 \bmod 3$, the previous edge color induces the following proper vertex coloring for $v_j, 1 \le j \le n$ given by

$$c'(v_j) = \begin{cases} 1 & \text{if } j = 1 \\ 2 & \text{if } j = n \\ 3 & \text{if } j \equiv 2 \bmod 3 \\ 4 & \text{if } j \equiv 1 \bmod 3,\ j \ne 1 \\ 5 & \text{if } j \equiv 0 \bmod 3,\ j \ne n \end{cases}$$

Hence, $\chi_{NK}(P_n)=3$.

Case 2: $n\equiv 1 \bmod 3$. For $n=4$, we show the *NK*-labeling of P_4 in **Figure 2**, so $\chi'_{NK}(P_4)=3$.

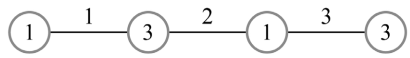

Figure 2. *NK*-labeling of P_4.

For $n\geq 7$, the previous edge color induces the following proper vertex coloring for $v_j, 1\leq j\leq n$ given by

$$c'(v_j)=\begin{cases}1 & \text{if } j=1\\ 3 & \text{if } j\equiv 2 \bmod 3,\ j=n\\ 4 & \text{if } j\equiv 1 \bmod 3,\ j\neq 1,n\\ 5 & \text{if } j\equiv 0 \bmod 3\end{cases}$$

Hence, $\chi'_{NK}(P_n)=3$.

Case 3: $n\equiv 2 \bmod 3$. For $n=5$, we show the *NK*-labeling of P_5 in **Figure 3**, so $\chi'_{NK}(P_5)=3$.

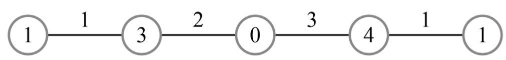

Figure 3. *NK*-labeling of P_5.

For $n\geq 8$, the previous edge color induces the following proper vertex coloring for $v_j, 1\leq j\leq n$ given by

$$c'(v_j)=\begin{cases}1 & \text{if } j=1,n\\ 3 & \text{if } j\equiv 2 \bmod 3,\ j\neq n\\ 4 & \text{if } j\equiv 1 \bmod 3,\ j\neq 1\\ 5 & \text{if } j\equiv 0 \bmod 3\end{cases}$$

Hence, $\chi'_{NK}(P_5)=3$. Therefore, for $n\geq 4$ the *NK*-chromatic index of P_n is 3.

Theorem 3.2 *Let* n *be an integer greater than* 3, *the NK-chromatic index of the cycle* C_n *of order* n *is*

$$\chi'_{NK}(C_n)=\begin{cases}3 & \text{if } n\equiv 0 \bmod 3\\ 4 & \text{if } n\equiv 1 \bmod 3\\ 5 & \text{if } n\equiv 2 \bmod 3\end{cases}$$

Proof. Case 1: $n\equiv 0 \bmod 3$.

Trivially we can show that $\chi'_{NK}(C_3)=3$. Now consider the cycle $C_n=(v_1,v_2,\cdots,v_n,v_1)$. Clearly, $3\leq \chi'_{NK}(C_n)$ by using the Proposition 2.2. We want to prove that $\chi'_{NK}(C_n)\leq 3$. For $1\leq i\leq n-1$, the edge v_iv_{i+1} assigned by a color c as following

$$c(v_iv_{i+1})=\begin{cases}1 & \text{if } i\equiv 1 \bmod 3\\ 2 & \text{if } i\equiv 2 \bmod 3\\ 3 & \text{if } i\equiv 0 \bmod 3\end{cases}\quad c(v_nv_1)=3$$

So, we get a proper vertex coloring as following, for $1 \le j \le n$,

$$c'(v_j) = \begin{cases} 3 & \text{if } n \equiv 2 \bmod 3 \\ 4 & \text{if } n \equiv 1 \bmod 3 \\ 5 & \text{if } n \equiv 0 \bmod 3 \end{cases}$$

Therefore, $\chi'_{NK}(C_n) = 3$.

Case 2: $n \equiv 1 \bmod 3$

Clearly for any three consecutive edges in C_n, they cannot color by *i*, *j*, *i*, otherwise we get improper vertex coloring. Hence we start to color v_1v_2 by 1, v_2v_3 by 2, v_3v_4 by 3, v_4v_5 by 1, v_5v_6 by 2 and so on. Since $n \equiv 1 \bmod 3$, we end with $c(v_{n-1}v_n) = 3$. Then we need a new color 4 for the edge v_nv_1, otherwise we get improper vertex coloring. Therefore $\chi'_{NK}(C_n) \ge 4$.

To prove that $\chi'_{NK}(C_n) \le 4$, for C_4 we show in Figure 4, $\chi'_{NK}(C_4) \le 4$.

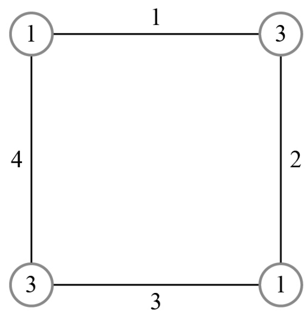

Figure 4. *NK*-labeling of C_4.

For $n \ge 7$, for $1 \le i \le n-1$, we did as case 1 with

$$c(v_iv_{i+1}) = \begin{cases} 1 & \text{if } i \equiv 1 \bmod 3 \\ 2 & \text{if } i \equiv 2 \bmod 3 \\ 3 & \text{if } i \equiv 0 \bmod 3 \end{cases} \quad c(v_nv_1) = 4$$

Then we get a proper vertex coloring as following, for $1 \le j \le n$,

$$c'(v_j) = \begin{cases} 3 & \text{if } j \equiv 2 \bmod 3 \\ 4 & \text{if } j \equiv 1 \bmod 3,\ j \ne 1, n \\ 5 & \text{if } j \equiv 0 \bmod 3,\ j = 1 \\ 7 \bmod n & \text{if } j = n \end{cases}$$

Hence, $\chi'_{NK}(C_n) = 4$.

Case 3: $n \equiv 2 \bmod 3$.

We start to color v_1v_2 by 1, v_2v_3 by 2, v_3v_4 by 3, v_4v_5 by 1, v_5v_6 by 2 and so on. Since $n \equiv 2 \bmod 3$, we end with $c(v_{n-2}v_{n-1}) = 3$. For the edges $v_{n-1}v_n$ and v_nv_1 we have to use new colors 4, 5 respectively, otherwise we get improper vertex coloring. Hence $\chi'_{NK}(C_n) \ge 5$. To show $\chi'_{NK}(C_n) \le 5$. For C_5, we show in Figure 5, $\chi'_{NK}(C_5) \le 5$.

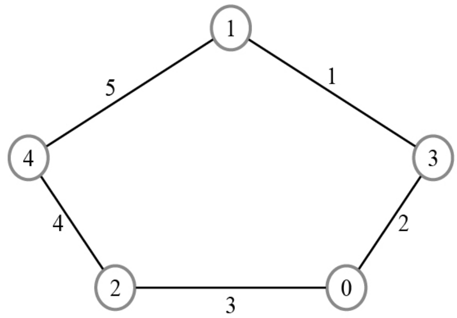

Figure 5. *NK*-labeling of C_5.

For $n \geq 8$, for $1 \leq i \leq n-1$,

$$c(v_i v_{i+1}) = \begin{cases} 1 & \text{if } i \equiv 1 \bmod 3, i \neq n-1 \\ 2 & \text{if } i \equiv 2 \bmod 3 \\ 3 & \text{if } i \equiv 0 \bmod 3 \\ 4 & \text{if } i = n-1 \end{cases} \qquad c(v_n v_1) = 5$$

Then we get a proper vertex coloring as following, for $1 \leq j \leq n$,

$$c'(v_j) = \begin{cases} 3 & \text{if } j \equiv 2 \bmod 3, \ j \neq n \\ 4 & \text{if } j \equiv 1 \bmod 3, \ j \neq 1, n-1 \\ 5 & \text{if } j \equiv 0 \bmod 3, \ j = 1 \\ 6 & \text{if } j = 1 \\ 7 & \text{if } j = n-1 \\ 9 \bmod n & \text{if } j = n \end{cases}$$

Now we will focus on some well-known classes of graph with odd order.

Theorem 3.3 *Let S be a star of odd order* $n \geq 3$, *the NK-chromatic index of S is* $n-1$.

Proof. Let $V(S) = \{v, v_1, v_2, \cdots, v_{n-1}\}$ where $\deg(v) = n-1$, $\deg(v_i) = 1$, $1 \leq i \leq n-1$. Color the edge vv_i by i, which induces a proper vertex coloring

$$c'(v_i) = i \text{ and } c'(v) \equiv \sum_{i=1}^{n-1} i \bmod n \equiv \frac{(n-1)n}{2} \bmod n \equiv 0 \bmod n$$

Hence, $\chi'_{NK}(S) \leq n-1$. Moreover, by preposition 2.1, we have $\chi'_{NK}(S) \geq n-1$ because $\chi'(S) = n-1$.

Theorem 3.4 *For odd integer* n. *Let* W_n *be a wheel of order* $n \geq 5$, *the NK-chromatic index of* W_n *is* $n-1$

Proof. Let $V(W_n) = \{v, v_1, v_2, \cdots, v_{n-1}\}$ where $\deg(v) = n-1$, $\deg(v_i) = 3$, $1 \leq i \leq n-1$. By proposition 2.1, $n-1 \leq \chi'_{NK}(W_n)$. It is reminded to prove that $\chi'_{NK}(W_n) \leq n-1$.

Case 1: $n \not\equiv 0 \bmod 3$. Assign the colors of the edge e as following

$$c(e) = \begin{cases} 1 & \text{for } e = v_{n-2} v_{n-1} \\ 2 & \text{for } e = v_{n-1} v_1 \\ i & \text{for } e = vv_i \\ i+2 & \text{for } e = v_j v_{j+1}, 1 \leq j \leq n-3 \end{cases}$$

We get three types of 3-cycle

1- $C_3 = (v, v_i, v_{i+1}, v)$, $1 \le i \le n-3$,

2- $C_3' = (v, v_{n-2}, v_{n-1}, v)$,

3- $C_3'' = (v, v_{n-1}, v_1, v)$,

First, suppose that there exist two vertices of C_3 have same color. Thus $c'(v_i) = c'(v_{i+1})$ or $c'(v) = c'(v_i)$. Assume that

$$c'(v_i) = c'(v_{i+1})$$

$$i + (i+1) + (i+2) - k_1 n = (i+1) + (i+2) + (i+3) - k_2 n \quad k_1, k_2 \text{ are integers}$$

$$3(i+2) - 3(i+1) = k_2 n - k_1 n$$

$$3 = (k_2 - k_1) n$$

Which impossible since $n \ge 5$. So, we may assume that

$$c'(v) = c'(v_i)$$

$$0 - k_1 n = i + (i+1) + (i+2) - k_2 n \quad k_1, k_2 \text{ are integers}$$

$$3(i+1) = (k_2 - k_1) n$$

Since 3 is prime, 3 divides $k_2 - k_1$ which implies that $i+1$ is a multiple of n, contradiction. Hence $C_3 = (v, v_i, v_{i+1}, v)$, $1 \le i \le n-3$ has proper vertex-coloring. Moreover, according on the edge coloring $c'(v_{n-2}) = n-2$, $c'(v_{n-1}) = 2$, $c'(v_1) = 6 \bmod n$. Thus, C_3' and C_3'' have a proper vertex-coloring. This implies $\chi'_{NK}(W_n) \le n-1$.

Next, if $n \equiv 0 \bmod 3$, then assign the colors of the edge e as following

$$c(e) = \begin{cases} 1 & \text{for } e = v_{n-3} v_{n-2} \\ 2 & \text{for } e = v_{n-2} v_{n-1} \\ 3 & \text{for } e = v_{n-1} v_1 \\ i & \text{for } e = v v_i \\ i+3 & \text{for } e = v_j v_{j+1}, 1 \le j \le n-4 \end{cases}$$

We get four types of 3-cycle

1- $C_3 = (v, v_i, v_{i+1}, v)$, $1 \le i \le n-4$,

2- $C_3' = (v, v_{n-3}, v_{n-2}, v)$,

3- $C_3'' = (v, v_{n-2}, v_{n-1}, v)$,

4- $C_3''' = (v, v_{n-1}, v_1, v)$,

As above we can prove that C_3 has proper vertex coloring. First, suppose that there exist two vertices of C_3 have same color. Thus $c'(v_i) = c'(v_{i+1})$ or $c'(v) = c'(v_i)$. Assume that

$$c'(v_i) = c'(v_{i+1})$$

$$i + (i+2) + (i+3) - k_1 n = (i+1) + (i+3) + (i+4) - k_2 n \quad k_1, k_2 \text{ are integers}$$

$$3 = (k_1 - k_2) n$$

Which impossible since $n \ge 9$. So, we may assume that

$$c'(v) = c'(v_i)$$

$$0 - k_1 n = i + (i+2) + (i+3) - k_2 n \quad k_1, k_2 \text{ are integers}$$

$$5 + 3i = (k_1 - k_2) n$$

Since $n \equiv 0 \bmod 3$, there is integer k such that $n = 3k$ so

$$5 + 3i = 3(k_1 - k_2) k$$

which implies that 5 is a multiple of 3, contradiction. Hence $C_3 = (v, v_i, v_{i+1},, v), 1 \le i \le n-3$ has proper vertex-coloring. Moreover, according to the edge coloring $c'(v_{n-3}) = n-3$, $c'(v_{n-2}) = 1$, $c'(v_{n-1}) = 4$ and $c'(v_1) = 8 \bmod n$ .Thus C_3', C_3'' and C_3''' have proper vertex coloring. Hence, $\chi'_{NK}(W_n) = n-1$.

Theorem 3.5 *For odd integer* n, *the NK-chromatic index of the complete graph* K_n *is* n.

Proof. The graph K_n is $(n-1)$-regular graph with size $\frac{n(n-1)}{2}$. Moreover, for odd order *n*, $\chi'(K_n) = n$. From Proposition 2.1, $n \le \chi'_{NK}(K_n)$. We will color the edges of K_n by proper edge coloring using this *n* colors. Therefore, every vertex in K_n incident with $n-1$ different colors. We will denote the vertex that incident with the colors $V_i = \{1, 2, 3, i-1, i+1, n\}$ by $v_i, 1 \le i \le n$. Each color i belong to all V_j such that $i \ne j$, otherwise $deg(v_i) < n-1$. Since $i \notin V_i$ and from the definition of *NK*-labeling, $c'(v_i) = \sum_{j=1}^{n} j - i = n - i$. It is clear that for $i \ne j$, $c'(v_i) \ne c'(v_j)$.

An example of odd complete graph, Figure 6 illustrate $\chi'_{NK}(K_7) \le 7$ so *NK*-chromatic index of K_7 is 7.

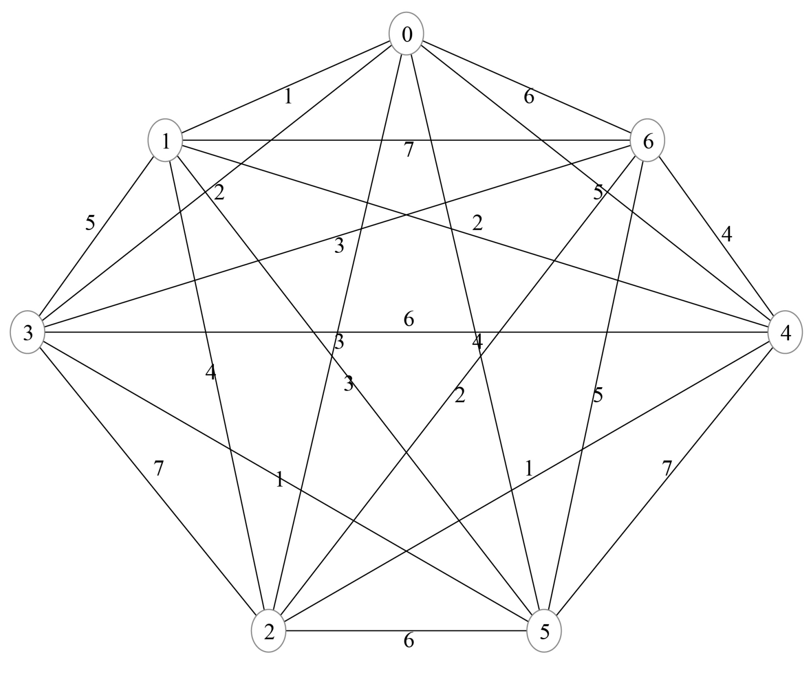

Figure 6. The *NK*-labeling of K_7.

Proposition 3.6 *If G is a graph of size* $m \geq 1$, *then*

$$\chi'_{NK}(G) \geq \frac{m}{\alpha'(G)}$$

where $\alpha'(G)$ is the maximum number of edges in an independent set of edges of a graph *G*.

Proof. Suppose that $\chi'_{NK}(G) = k$ and that $E_1, E_2, \cdots, E_k$ are the edge color classes in *k*-edge coloring of *G*. Thus $|E_i| \leq \alpha'(G)$ for each i ($1 \leq i \leq k$). Moreover, from the definition of *NK*-labeling, it is clear that $\chi'(G) \leq \chi'_{NK}(G)$. Therefore if $F_1, F_2, \cdots, F_l$, $l \geq k$ are the color classes of $\chi'_{NK}(G)$ such that $\chi'_{NK}(G) = l$, then $|F_j| \leq |E_i| \leq \alpha'(G)$ for all i, j. Hence $m = |E(G)| = \sum_{j=1}^{l} |F_j| \leq \chi'_{NK}(G) \cdot \alpha'(G)$.

Some of the complete graphs K_{2n}, $n \geq 3$ are *NK*-colorable, $\chi'_{NK}(K_4) = 5$. From **Figure 7**, $\chi'_{NK}(G) \leq 5$. Now, suppose that K_4 can be color by using 4 colors. We may assume that $c(v_1 v_2) = 1$ in **Figure 2**. Since *c* is a proper edge coloring, none of the edges $v_1 v_3$, $v_2 v_4$, $v_2 v_3$ and $v_1 v_4$ can be colored by the color 1. Hence two of these four edges must be assigned the same color and the remaining two edges must be assigned different colors, otherwise $c'(v_1) = c'(v_2)$ which is impossible. Assume $c(v_1 v_3) = c(v_2 v_4) = 2$, $c(v_2 v_3) = 3$ and $c(v_1 v_4) = 4$. The last edge should be colored by the color 1, but in this case $c'(v_2) = c'(v_3)$, contradiction. Therefore $\chi'_{NK}(K_4) \geq 5$. Hence, $\chi'_{NK}(K_4) = 5$.

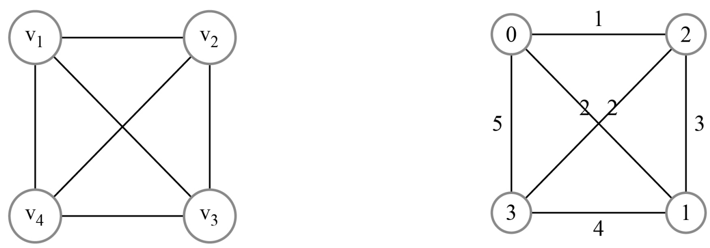

Figure 7. The *NK*-labeling of K_4.

Problem: For even integer n, determine the *NK*-chromatic index of the complete graph K_n if it exists.

4. Conclusion

Graph labeling is one of the main topics of graph theory, and it is used in a variety of fields, such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing, and database management. In this article, we introduced a new concept of graph labeling and found the *NK*-labeling of special graphs such as path, cycle, wheel, star, and complete graph for some *n*. Furthermore, we find the lower bound of our labeling.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Chartrand, G., Lesniak, L. and Zhang, P. (2010) Graphs and Digraphs. 5th Edition, Chapman and Hall/CRC. https://doi.org/10.1201/b14892

[2] Golomb, S. (1972) How to Number a Graph. *Graph Theory and Computing*, 23-27. https://doi.org/10.1016/B978-1-4832-3187-7.50008-8

[3] Mahéo, M. and Saclé, J.F. (2008) Some Results in (Σ,p,g)-Valuation of Connected Graphs. Report de Recherche 1497. *Universte'* de Paris-Sud, Center d'Orsay.

[4] Zhang, P. (2015) Color-Induced Graph Colorings. Springer. https://doi.org/10.1007/978-3-319-20394-2

[5] Zhang, P. (2015) A Kaleidoscopic View of Graph Colorings. Springer. https://doi.org/10.1007/978-3-319-30518-9