

# The Coercive Property and a Priori Error Estimation of the Finite Element Method for Linearly Distributed Time Order Fractional Telegraph Equation with Restricted Initial Conditions

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## Abstract

Finite Element Method (FEM), when applied to solve problems, has faced some challenges over the years, such as time consumption and the complexity of assumptions. In particular, the making of assumptions has had a significant influence on the accuracy of the method, making it mandatory to carry out sensitivity analysis. The sensitivity analysis helps to identify the level of impact the assumptions have on the method. However, sensitivity analysis via FEM can be very challenging. A priori error estimation, an integral part of FEM, is a basic mathematical tool for predicting the accuracy of numerical solutions. By understanding the relationship between the mesh size, the order of basis functions, and the resulting error, practitioners can effectively design and apply FEM to solve complex Partial Differential Equations (PDEs) with confidence in the reliability of their results. Thus, the coercive property and A priori error estimation based on the L1 formula on a mesh in time and the Mamadu-Njoseh basis functions in space are investigated for a linearly distributed time-order fractional telegraph equation with restricted initial conditions. For this purpose, we constructed a mathematical proof of the coercive property for the fully discretized scheme. Also, we stated and proved a cardinal theorem for a priori error estimation of the approximate solution for the fully discretized scheme. We noticed the role of the restricted initial conditions imposed on the solution in the analysis of a priori error estimation.

## Keywords

Coercivity, Finite Element Method, Mamadu-Njoseh Polynomials, A Priori Error Estimation, Cauchy-Schwarz Inequality, Mean Value Theorem

## 1. Introduction

Mathematical modeling of many real-life phenomena, such as viscoelasticity, finance, medicine, signal processing, anomalous diffusion, and many others, can be processed through the notion of fractional differential equations [1]-[7]. In particular, the hereditary and memory properties of many known materials can be best described via fractional derivatives. Many researchers have studied the approximate solutions of fractional partial differential equations in time-space [8]-[12].

In this paper, we examine linearly distributed time-order fractional telegraph equations with restricted initial conditions of the form

$$D_t^{\tau,\alpha} w(x,t) + w_t(x,t) - w_{xx}(x,t) = g(x,t), \quad (x,t) \in [-1,1] \times [0,T], \quad (1)$$

with conditions

$$\left. \begin{aligned} w(x,0) &= w_0(x) \\ w(0,t) &= w(t,+\infty) = 0 \end{aligned} \right\}, \quad x \in [-1,1], t \in [0,T]; \quad (2)$$

where  $D_t^{\tau,\alpha} w(x,t)$  refers to the time-order linearly distributed fractional derivative, which is dependent on time, given as [13]

$$D_t^{\tau,\alpha} w(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \int_0^t \tau(\beta) (t-s)^{-\alpha} w'(x,s) ds d\beta, \quad (3)$$

which can also be written as

$$D_t^{\tau,\alpha} w(x,t) = \int_0^\alpha \tau(\beta) {}_0^c D_t^\beta w(x,t) d\beta, \quad (4)$$

where

$${}_0^c D_t^\beta w(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} w'(x,s) ds, \quad (5)$$

with  $\alpha \in (0,1]$  being a constant, and  ${}_0^c D_t^\beta w(x,t)$  is the Caputo fractional derivative of order  $\beta \in (0,1)$ . The term  $g(x,t)$  is a continuous homogeneous term in  $[-1,1] \times [0,T]$ .

The transmission of electrical signals in marine transmits cables by Oliver Heaviside in 1876 led to the creation of the telegraph equation, which is given as [7] [13]

$$w_{tt}(x,t) + \alpha w_t(x,t) + \beta w(x,t) - \gamma w_{xx}(x,t) = 0, \quad (6)$$

with  $\alpha > 0$ ,  $\beta > 0$ , and  $\gamma \geq 1$ . A general overview of a few applications of (6) can be seen in the reference [14]. In recent years, the theory of fractional telegraph

equation has found its way into electromagnetic waves, the transmission of digital and analog signal processing, modeling of dispersive and dissipative processes, modeling of stochastic processes such as in random walks, biological sciences such as genetic variability, and among many others [15]-[20].

The term Finite Element Method (FEM) was first coined in 1960, when Cough used it in a paper centered on elasticity problems in planes [21]. According to Logan [22], the originality of FEM can be traced back in time to the work of Courant in 1943, when the notion of triangular domains was established for piecewise continuous functions. Specifically, Courant established the idea that a functional can be minimized through a linear approximation over a given sub-domain, with values defined at discrete points that eventually become the node points of the mesh of elements. As FEM gained popularity as an efficient and reliable technique for resolving problems in physics and engineering, mathematicians began to develop the core mathematical principles of the method. As such, the stability, error estimation, and convergence analysis of the method were investigated. For instance, Mamadu *et al.* [23] considered the convergence analysis of FEM for fractional telegraph equations. Li and Reed [24] studied the convergence analysis of elasticity linear problems through the finite element method. Wan *et al.* [25] explored the convergence analysis of the Freidlin-Wentzell functional problem in a dynamic system perturbed by white noise. Mamadu *et al.* [26] equally addressed the issue of error analysis of the time-fractional telegraph equation through the finite element method. Tang [27] studied the convergence and super-convergence of fully discrete finite elements for time fractional optimal control problems finite element method.

FEM, when applied to solve problems, has faced some challenges over the years, such as time consumption and the complexity of assumptions. In particular, the making of assumptions has had a significant influence on the accuracy of the method, making it mandatory to carry out sensitivity analysis. The sensitivity analysis helps to identify the level of impact the assumptions have on the method. However, sensitivity analysis via FEM can be very challenging. On this note, FEM has undergone a lot of modifications over the years by various researchers. For instance, Wang *et al.* [28] employed the H-Galerkin mixed finite element method for the solution of the advection-diffusion equation in space. In like manner, the Galerkin finite element method was considered by Hao *et al.* [29] for a diffusion equation with a one-dimensional variable coefficient. Mamadu *et al.* [7] employed the least squares finite element method for fractional time telegraph equations with the Mamadu-Njoseh basis function. It is well established that all the above-mentioned modified FEMs have significant improvements over the standard FEM, as seen in the reference therein.

In this study, the coercivity property and a priori error estimation of the finite element method for a linearly distributed time-order fractional telegraph equation with restricted initial conditions are considered. Here, we coupled FEM with Mamadu-Njoseh basis functions in space and the L1 formula on a mesh in time

to develop a fully discrete scheme. The coercivity property and the a priori error estimation are derived and proven for the scheme.

## 2. Preliminaries

i. The L1 formulation on the mesh in this paper is given as

$$\begin{aligned}
 D_N^{\beta_i} W_n &= \frac{(t_n - t_{n-1})^{1-\beta_i}}{\Gamma(2 - \beta_i)} W_n - \sum_{j=1}^{n-1} \left( \frac{(t_n - t_{n-j})^{1-\beta_i} - (t_n - t_{n-j+1})^{1-\beta_i}}{t_{n-j+1} - t_{n-j}} \right. \\
 &\quad \left. - \frac{(t_n - t_{n-j+1})^{1-\beta_i} - (t_n - t_{n-j+2})^{1-\beta_i}}{t_{n-j+2} - t_{n-j+1}} \right) W_{n,j} \\
 &\quad - \frac{(t_n - t_0)^{1-\beta_i} - (t_n - t_1)^{1-\beta_i}}{\Gamma(2 - \beta_i)(t_2 - t_1)} W_0, \quad i = 0(1)j,
 \end{aligned} \tag{7}$$

where

$$t_n = \left(\frac{n}{R}\right)^q, \quad n = 0(1), R > 0, q \geq 1.$$

ii. We refer  $[-1, 1] = \mathbb{R}^*$  with Mamadu-Njoseh weight functions  $w_i(x) = 1 + x^{2i}$ ,  $i > -1$  [30]. Let

$$\varphi_{w_i}^2(\mathbb{R}^*) = \int_{\mathbb{R}^*} f g w_i(x) dx < \infty,$$

and

$$\|f\|_{w_i} = \sqrt{\int_{\mathbb{R}^*} f^2 w_i(x) dx},$$

then  $f$  is measurable.

iii. Let  $H^1(\mathbb{R}^*) = \{f \in \varphi_{w_i}^2(\mathbb{R}^*) \text{ such that } \frac{\partial f}{\partial x} \in \varphi_{w_i}^2(\mathbb{R}^*)\}$ , then the norm below exist

$$\|f\|_1 = \sqrt{\left\| \frac{\partial f}{\partial x} \right\| + \|f\|}.$$

Similarly,  $H_0^1(\mathbb{R}^*) = \{f \in H^1(\mathbb{R}^*) \text{ such that } f(0) = 0\}$ .

iv. Mamadu-Njoseh polynomials are defined as follows:

$$\hat{\varphi}_n(x) = \varphi_n(x)(1 + x^2), \quad x \in \mathbb{R}^*;$$

where  $\varphi_n(x)$  are Mamadu-Njoseh polynomials of degree  $n$  [30]. These polynomials are orthogonal on  $\mathbb{R}^*$  with respect to weight function  $w_0(x) = (1 + x^2)$ .

v. For any vectors  $u$  and  $v$  in an inner product space, the Cauchy-Schwarz inequality is stated as

$$|\langle u, v \rangle| \leq \|u\| \|v\|,$$

where  $\|u\|$  is the norm of  $u$ ,  $\|v\|$  is the norm of  $v$ , and  $\langle u, v \rangle$  is the inner product of  $u$  and  $v$ .

vi. Let denote

$$\mathbb{R}_N(\mathbb{R}^*) = \text{span}\{\hat{\varphi}_n(x), n = 0(1)N\},$$

which implies  $\mathbb{R}_N(\mathbb{R}^*)$  is the space of all functions that can be written as a linear combination of the functions  $\hat{\varphi}_n(x)$  for  $n = 0(1)N$ , and

$$\mathbb{R}_0(\mathbb{R}^*) = \text{span}\{f \in \mathbb{R}_N(\mathbb{R}^*) : f(0) = 0\},$$

$\mathbb{R}_0(\mathbb{R}^*)$  is a subspace of  $\mathbb{R}_N(\mathbb{R}^*)$ , consisting of all functions  $f \in \mathbb{R}_N(\mathbb{R}^*)$  that satisfy the condition  $f(0) = 0$ . This introduces a boundary condition into the space, essentially filtering out functions from  $\mathbb{R}_N(\mathbb{R}^*)$  that do not vanish at  $x = 0$ .

Also, for  $f \in H_0^1(\mathbb{R}^*)$ , we have the projection

$$S : f \in H_0^1(\mathbb{R}^*) \rightarrow \mathbb{R}_N(\mathbb{R}^*),$$

with the computational property presented as **Lemma 1** below:

**Lemma 1.** Given any  $g \in H_0^1(\mathbb{R}^*)$ , there exist

$$\left(\frac{\partial}{\partial x}(g - Sg), \frac{\partial u}{\partial x}\right) + \frac{3}{5}(g - Sg, u) = 0, \text{ for all } u \in \mathbb{R}_N(\mathbb{R}^*).$$

Similarly, for  $g \in H_0^1(\mathbb{R}^*)$  and  $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} \in \varphi_{w_i}^2(\mathbb{R}^*)$ , then

$$\|Sg - g\|_1 \leq pM^{0.5(1-k)} \left\| \frac{\partial g}{\partial x} \right\|_{\varphi_{w_i}^2(\mathbb{R}^*)}, \quad k \in [1, M + 1],$$

where  $p$  is a constant and is independent of  $M, k$  and  $g$ .

By **Lemma 1**, we deduce that

$$S = \sum_{i=0}^n a_i \varphi_i(x),$$

with the constants  $a_i, i = 0(1)n$ , satisfying the relation

$$\begin{aligned} & \frac{3}{5}a_m + \sum_{i=0}^N a_i \int_{\mathbb{R}^*} \varphi_n(x) \frac{\partial}{\partial x} \varphi_m(x) dx \\ & = \frac{3}{5} \int_{\mathbb{R}^*} a(x) \varphi_m(x) dx + \int_{\mathbb{R}^*} \frac{\partial}{\partial x} a(x) \frac{\partial}{\partial x} \varphi_m(x) dx, \quad m = 0(1)N. \end{aligned}$$

### 3. Coercivity Property of the L1 Formulation

To do this, we consider the Lemma below.

**Lemma 2.** Let  $p(\beta) \in C^2[0, \alpha]$  with  $\alpha = n\Delta\beta$  and  $\beta_i = \frac{1}{2}\Delta\beta(2i - 1)$  for  $i = 0(1)n$ . Also, let  $[0, \alpha]$  be divided into  $n$  subintervals. Then

$$\int_0^\alpha p(\beta) d\beta = \alpha \sum_{i=1}^n p(\beta_i) + \frac{\alpha^3}{(5n)^2} p''(\tau), \quad \tau \in [0, \alpha].$$

Hence, for  $u(\beta)$  and  ${}_0^c D_t^\beta w(x, t)$  in  $C^2[0, \alpha]$ , then we can rewrite (1) as

$$\Delta\beta \sum_{i=1}^n u(\beta_i) {}_0^c D_t^{\beta_i} w(x, t) + w_t(x, t) - w_{xx}(x, t) = g(x, t) + O\Delta(\beta^2), \quad (8)$$

Based on **Lemma 2**, and from applying the mean value problem,

$$\frac{(t_n - t_{n-j+1})^{1-\beta_i} - (t_n - t_{n-j+2})^{1-\beta_i}}{t_{n-j+2} - t_{n-j+1}} \leq \frac{(t_n - t_{n-j})^{1-\beta_i} - (t_n - t_{n-j+1})^{1-\beta_i}}{t_{n-j+1} - t_{n-j}}, \quad (9)$$

$j \in [0, n-1] \times [R-1]$

Again, let denote

$$\begin{aligned} \varphi_i^\beta W_n &= \Delta\beta \sum_{i=1}^n p(\beta_i) {}_0^c D_t^{\beta_i} w_n \\ &= \frac{(t_n - t_{n-1})^{1-\beta_i}}{\Gamma(2-\beta_i)} W_n - \sum_{j=1}^{n-1} \left( \frac{(t_n - t_{n-j})^{1-\beta_i} - (t_n - t_{n-j+1})^{1-\beta_i}}{t_{n-j+1} - t_{n-j}} \right. \\ &\quad \left. - \frac{(t_n - t_{n-j+1})^{1-\beta_i} - (t_n - t_{n-j+2})^{1-\beta_i}}{t_{n-j+2} - t_{n-j+1}} \right) W_{n,j} \\ &\quad - \frac{(t_n - t_0)^{1-\beta_i} - (t_n - t_1)^{1-\beta_i}}{\Gamma(2-\beta_i)(t_2 - t_1)} W_0, \quad i = 0(1)j. \end{aligned} \quad (10)$$

**Theorem 1.** Let  $W_n \in \varphi^2(\Psi)$ ,  $n = 1(2)R$ , then

$$\left( D_t^{\beta_i} \|W_n\| \|W_n\| \right) \leq \left( D_t^{\beta_i} W_n, W_n \right), \quad n = 1(2)R.$$

**Proof:** Let  $\Delta\beta > 0$ ,  $u(\beta_i) \geq 0$  for  $i \in [1, j]$ , using (8) we have

$$\frac{(t_n - t_{n-j})^{1-\beta_i} - (t_n - t_{n-j+1})^{1-\beta_i}}{t_{n-j+1} - t_{n-j}} \leq \frac{(t_n - t_{n-j+1})^{1-\beta_i} - (t_n - t_{n-j+2})^{1-\beta_i}}{t_{n-j+2} - t_{n-j+1}}, \quad (11)$$

$j \in [0, n-1] \times [R-1]$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left( D_t^{\beta_i} W_n, W_n \right) &= \frac{(t_n - t_{n-1})^{1-\beta_i}}{\Gamma(2-\beta_i)} \|W_n\|^2 - \sum_{j=1}^{n-1} \left( \frac{(t_n - t_{n-j})^{1-\beta_i} - (t_n - t_{n-j+1})^{1-\beta_i}}{t_{n-j+1} - t_{n-j}} \right. \\ &\quad \left. - \frac{(t_n - t_{n-j+1})^{1-\beta_i} - (t_n - t_{n-j+2})^{1-\beta_i}}{t_{n-j+2} - t_{n-j+1}} \right) \|W_{n-j}\| \|W_n\| \\ &\quad - \frac{(t_n - t_0)^{1-\beta_i} - (t_n - t_1)^{1-\beta_i}}{\Gamma(2-\beta_i)(t_2 - t_1)} \|W_0\| \|W_n\| \\ &\leq \frac{(t_n - t_{n-j})^{1-\beta_i} - (t_n - t_{n-j+1})^{1-\beta_i}}{t_{n-j+1} - t_{n-j}} (W_n, W_n) \\ &\quad - \sum_{j=1}^{n-1} \left( \frac{(t_n - t_{n-j})^{1-\beta_i} - (t_n - t_{n-j+1})^{1-\beta_i}}{t_{n-j+1} - t_{n-j}} \right. \\ &\quad \left. - \frac{(t_n - t_{n-j+1})^{1-\beta_i} - (t_n - t_{n-j+2})^{1-\beta_i}}{t_{n-j+2} - t_{n-j+1}} \right) (W_{n-j}, W_n) \\ &\quad - \frac{(t_n - t_0)^{1-\beta_i} - (t_n - t_1)^{1-\beta_i}}{\Gamma(2-\beta_i)(t_2 - t_1)} (W_0, W_n) \\ &= \left( D_t^{\beta_i} W_n \right) \|W_n\|. \end{aligned}$$

### 4. Finite Element Prior Error Estimate of the Approximate Solution

The aim here is to find  $w_{n,N} \in \mathbb{R}_0(\mathbb{R}^*)$  for  $n = 1(2)R$  such that

$$\left(\varphi_t^\beta w_{n,N}, u_n\right) + \left(\frac{\partial}{\partial t} w_{n,N}, u_n, \frac{\partial}{\partial t} u_n\right) - \left(\frac{\partial^2}{\partial x^2} w_{n,N}, u_n\right) = (g_n, u_n) \tag{12}$$

for all  $u_n \in \mathbb{R}_0(\mathbb{R}^*)$

with  $w_{0,N} = S_{N,w_0}$ .

Let  $\mathfrak{S}_{n,j}$  for  $n = 1, 2, \dots, R$  and  $j = 1, 2, \dots, n-1$ , be real such that

$$\mathfrak{S}_{n,j} = \sum_{r=1}^{n-j} \delta_{n-r}^\beta \left( \frac{(t_n - t_{n-r})^{1-\beta_i} - (t_n - t_{n-r+1})^{1-\beta_i}}{t_{n-r+1} - t_{n-r}} - \frac{(t_n - t_{n-r+1})^{1-\beta_i} - (t_n - t_{n-r+2})^{1-\beta_i}}{t_{n-r+2} - t_{n-r+1}} \right) \mathfrak{S}_{n-r,j},$$

with  $\mathfrak{S}_{n,n} = 1$ .

**Remark:** The value of  $n$  depends on the dimension of the space  $\mathbb{R}_0(\mathbb{R}^*)$ . If  $\mathbb{R}_0(\mathbb{R}^*)$  is spanned by  $N + 1$  basis functions  $\hat{\varphi}_n(x)$ , then  $n$  would typically range from 1 to  $N$ .

We hence consider the following Lemmas.

**Lemma 3.** Let  $\theta_n > 0$  and  $\mathcal{G}_n > 0$ , for  $n = 1(2)\infty$ , respectively. Suppose that the mesh function  $W_n, n = 0(1)R$  with  $W_0 \geq 0$ , satisfies

$$W_n \varphi_t^\beta W_n \leq \theta_n W_n + (\mathcal{G}_n)^2, n = 1(2)R,$$

then,

$$W_n \leq W_0 + (t_n - t_{n-1})^{-\beta} \sum_{r=1}^{n-j} \delta_{n-r}^\beta \left( \frac{(t_n - t_{n-r})^{1-\beta_i} - (t_n - t_{n-r+1})^{1-\beta_i}}{t_{n-r+1} - t_{n-r}} - \frac{(t_n - t_{n-r+1})^{1-\beta_i} - (t_n - t_{n-r+2})^{1-\beta_i}}{t_{n-r+2} - t_{n-r+1}} \right) \mathfrak{S}_{n-r,j} + (\theta_n + \mathcal{G}_n) + \max_{1 \leq r \leq n} \{\mathcal{G}_r\}, n = 1(2)R.$$

**Lemma 4.** For  $n = 1(2)R$ , it satisfies

$$\begin{aligned} & (t_n - t_{n-1})^{-\beta} \sum_{r=1}^{n-j} \delta_{n-r}^\beta \left( \frac{(t_n - t_{n-r})^{1-\beta_i} - (t_n - t_{n-r+1})^{1-\beta_i}}{t_{n-r+1} - t_{n-r}} - \frac{(t_n - t_{n-r+1})^{1-\beta_i} - (t_n - t_{n-r+2})^{1-\beta_i}}{t_{n-r+2} - t_{n-r+1}} \right) \mathfrak{S}_{n-r,j} \\ & \leq \sum_{k=0}^N \frac{t_n^{\beta_k}}{\frac{\partial}{\partial t} \beta \mu(\beta_k) \Gamma(1 + \beta_k)}. \end{aligned}$$

**Theorem 2.** Let  $w_{n,N}$  for  $n = 0(1)R$ , be the approximate solution of (12), then we have

$$\|w_{n,N}\| \leq \|w_{0,N}\| + \sum_{k=1}^N \frac{t_n^{\beta_k}}{\frac{\partial}{\partial t} \beta \mu(\beta_k) \Gamma(1 + \beta_k)} \max_{1 \leq r \leq n} \{g_r\}, \quad n = 1(2)R.$$

**Proof.** Let  $u_n = w_{n,N}$  in (12), then

$$\left(\varphi_t^\beta w_{n,N}, w_{n,N}\right) + \left\| \frac{\partial}{\partial t} w_{n,N} \right\|^2 - \left\| \frac{\partial^2}{\partial x^2} w_{n,N} \right\|^2 = (g_n, w_{n,N}). \tag{13}$$

By Theorem 1, we have that

$$\left(\varphi_t^\beta \|w_{n,N}\|\right) \|w_{n,N}\| \leq (\varphi_t^\beta w_{n,N}, w_{n,N}). \tag{14}$$

Since  $\theta_n > 0$  and  $g_n > 0$  as in Lemma 3, we have that

$$\theta_n \left\| \frac{\partial}{\partial t} w_{n,N} \right\|^2 > 0, \quad g_n \left\| \frac{\partial^2}{\partial x^2} w_{n,N} \right\|^2 > 0, \quad n = 1(2)R. \tag{15}$$

Applying the Cauchy-Schwarz inequality on the RHS of (13), we have

$$(g_n, w_{n,N}) \leq \|g_n\| \|w_{n,N}\|. \tag{16}$$

$$\left(\varphi_t^\beta \|w_{n,N}\|\right) \|w_{n,N}\| \leq \|g_n\| \|w_{n,N}\|. \tag{17}$$

Using the Lemmas 3 and 4, we arrive at

$$\begin{aligned} \|w_{n,N}\| &\leq \|w_{0,N}\| + (t_n - t_{n-1})^{-\beta} \sum_{r=1}^{n-j} \delta_{n-r}^\beta \left( \frac{(t_n - t_{n-r})^{1-\beta_i} - (t_n - t_{n-r+1})^{1-\beta_i}}{t_{n-r+1} - t_{n-r}} \right. \\ &\quad \left. - \frac{(t_n - t_{n-r+1})^{1-\beta_i} - (t_n - t_{n-r+2})^{1-\beta_i}}{t_{n-r+2} - t_{n-r+1}} \right) \mathfrak{S}_{n-r,j} \|g_r\| \\ &\leq \|w_{0,N}\| + (t_n - t_{n-1})^{-\beta} \sum_{r=1}^{n-j} \delta_{n-r}^\beta \left( \frac{(t_n - t_{n-r})^{1-\beta_i} - (t_n - t_{n-r+1})^{1-\beta_i}}{t_{n-r+1} - t_{n-r}} \right. \\ &\quad \left. - \frac{(t_n - t_{n-r+1})^{1-\beta_i} - (t_n - t_{n-r+2})^{1-\beta_i}}{t_{n-r+2} - t_{n-r+1}} \right) \mathfrak{S}_{n-r,j} \max_{1 \leq r \leq n} \|g_r\| \\ &\leq \|w_{0,N}\| + \sum_{k=1}^N \frac{t_n^{\beta_k}}{\frac{\partial}{\partial t} \beta \mu(\beta_k) \Gamma(1 + \beta_k)} \max_{1 \leq r \leq n} \|g_r\|. \end{aligned}$$

This completes the proof.

### 5. Conclusion

We have analyzed a fully discrete scheme for the coercive property and Priori error estimation based on L1 formulation on a mesh in time and the FEM coupled with Mamadu-Njoseh basis functions in space. We have established the coercive property and Priori error estimation of the approximate solution of the fully discrete scheme based on the L1 formula. It is imperative to note that the coercive property and Priori error estimates were made realistic due to the imposition of restricted conditions on the solution.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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