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A New History of Logic: The Laborious Birth of a Formal Pluralism

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Abstract

The paper starts by remarking that the ancient Greek word for "truth" was "alétheia" (unveiling), which is a double negation. But, after Plato the affirmative meaning of the idea of truth has prevailed. The same meaning was reiterated by Romans's word for truth, *veritas*. Not before the year 1968 the double negation law was re-evaluated, since its failure was recognized as representing, more appropriately than the failure of the excluded middle law, the borderline between classical logic and almost all non-classical kinds of logic. Moreover, its failure is easily recognized within a (scientific) text; which therefore can be analyzed in a new logical way. As an example, the analysis of Kolmogorov's 1932 paper about the foundations of the intuitionist logic shows many interesting results, in particular his reasoning through arguments pertaining to the same intuitionist logic. In addition, previous papers have suggested—through a comparative analysis of all scientific theories which are based, like the previous one, on a general problem—that there exists a new model of organization of a theory which is alternative to the deductive-axiomatic model and it is governed by intuitionist logic. Some important logical events pertaining to non-classical logic are recognized by inspecting through both the double negations and this new model of organizing the original texts of past theories; so that the entire history of logic appears as a development along two main lines, one line representing classical logic dominated all others for a very long period of time, although at the origins the logical arguing had pertained to the alternative line. This history of logic confirms both Kuhn's category of a paradigm and Koyré's categories, represented by him through a couple of "characteristic propositions" in mutual conflict; analogous couples of propositions are suggested as representing the categories for adequately interpreting the entire history of logic.

Keywords

Double Negation, Intuitionist Logic, Kolmogorov's 1932 Paper,

Problem-Based Organization of a Scientific Theory, Two Fundamental Dichotomies, Structure of a Theory, Kuhn's Categories, Koyré's Categories

1. Two Dichotomic Meanings of a Same, Crucial Word: *Alétheia*

Let us consider how in the past times the notion of "truth" has been expressed by a word. Romans had the word *veritas*, which came from the Balkan and Slavic areas; it means "faith" (the fact that the wedding ring in Italian is called "vera" confirms this etymology by joining the two meanings: truth ("vera") and faith in the spouse) and therefore represents truth without any need for critical scrutiny. No different meaning pertains to the truth in modern English language: "faithfulness, constancy".

Instead, the ancient Greeks had the word ἀλήθεια. It comes from a privative α and λανθάνω, which means "I am hidden". Therefore, we can translate *alétheia* as "un-veiling"[1]; it has to be regarded not as a simple factual reality or as a hypostatized idea, but as a dynamic act, through which also the recognition and the refutation of errors occur: in sum, not a static and defined thought once and for all, but a revelatory process.

From the history of the early times of Greek thinking we know that the word αλήθεια was intended as a search of σοφός (wisdom); this search was a popular endeavor, connected to every human affair such as politics, ethics, and household management; so much to constitute the practice of a large social group. The meaning of *alétheia*, intended as a double negation, is not the same meaning as of the affirmative meaning of truth. As a consequence the two meanings are not mutually equivalent.

However, in V century B.C. the logical practice suggested by the investigative meaning of the word *alétheia* degenerated into the phenomenon of Sofists's arguing, who programmatically proved both a proposition and its contrary. A confusion in the use of reason followed. To it first Socrates reacted; he proved that a correct reasoning pertains even to a slave (Plato's dialog *Meno*). Then Plato attached the ideas of the beings of real world to a world of notions of idealistic, absolute validity (Olympus of ideas: Plato's *Republic*); later Aristotle built a theoretical system of every subject (cosmology, physics, biology, art, politics, etc.) and moreover a system of reasoning (logic), and even the abstract model of a theory (*Analytica Posteriora*). No more convincing proof could be offered to mankind that human reason can correctly work so well to obtain a reasoned system of the entire reality.

As a consequence of both this reaction to Sophists' arguments and the great influence of Plato's idealistic philosophy, the meaning of Greek word *alétheia* changed into a fixed, metaphysical meaning. Aristotle was immersed within the

[1]In order to make it easier for the reader to recognize a DNP, in the following each negative word inside a doubly negated proposition will be underlined.

new idealistic meaning; he stated an absolute decision about truth, as shown by his principle of strict bivalence, which makes equivalent a double negation with its affirmation: "To say of what is that it is, and of what is not that it is not, is true" (*Metaphysics* 1011b).

This fact means that the history of the Greek civilization was marked by an unrecognized intellectual conflict between two dichotomic meanings of a same word, *alétheia*. After Plato, This conflict was won by the idealistic meaning of truth. After two millennia the German philosopher Martin Heidegger (1927) stressed the difference between the Greek word *aletheia* and the Roman word *veritas*, which Romans represented in a manifestly idealistic way, i.e. as a divinity: a daughter of Saturn, called Κρόνος by the Greeks, the Titan of Time. Hence, from the original notion of truth as an unveiling process Western civilization went to an idealistic notion which moreover excluded the other meaning of this notion. Then the abandon of the heuristic meaning of *alétheia* and the assumption of the idealistic meaning of truth allowed the natural languages of Greek-Roman civilization to introduce an ever greater number of idealistic ideas, i.e. lacking a strict correspondence with facts. The so-called rhetoric use of notions led to no make attention whether a doubly negated term was supported by evidence or not. However, this trend was not shared by Court's language which, according to Roman Law, had to rely upon safe evidence. In sum, a double practice of truth persisted.

No surprise if past debate on the notion of truth, although absorbing the reflections of the major minds of all past centuries, did not achieve a stable and commonly accepted result and also at present time many troubles arise from the confusion of these two conceptions of truth.[2]

2. The Two-Steps Birth of a Specific Formal Logic

What we have considered in the above Section is the two different logical meanings of the word truth within the informal logical thinking. An epochal step was Aristotle's introduction of logical technique: the arguing through syllogisms—each constituted by a chain of three propositions in order to derive the truth of the final proposition from the previous two. He also listed the 24 valid syllogisms. They have been recognized valid by modern logicians. Along a millennium and half this formal logic was the unique instance of the correct reasoning, although some novelties born (modal syllogisms, etc.). An entire philosophical school of the Middle Age, Scolasticism, intensively worked by means of syllogisms along centuries.

First Descartes, wanting to undergo the entire his knowledge to the doubt, supported an inductive, heuristic reasoning, which was elucidated by his aim of clear and distinct ideas.

Also the empiricist F. Bacon detached himself from le Aristotelian tradition. He suggested—as a work which has to be preliminary to logical arguing—an *ad*

[2]An example is the notion of "principle". It is commonly intended as an a priori presupposition. But it means also "principle" in a methodological sense, i.e. as the starting proposition of a new method.

excludendum work, i.e. to eliminate the four main idols constituting the prejudices of a reasoning mind; i.e. a program essentially based on a double negation: eliminating prejudices. In addition, he founded a *Novum Organon* (1620) with respect to the classical Aristotle's book about logic, *Organon*; but his inductive arguing actually constituted no more than an address of research. But the specific research on this problem to define induction was unsuccessful along several centuries. Many scholars considered this fact as by a further proof that Aristotelian logic was unavoidably unique.

Such was Kant's belief: Aristotelian logic was the apex of the search for a formal logic, i.e. a search already completed since longtime.

After Kant, Hegel bravely tried to obtain an alternative kind of logic, but he relied on a philosophical base only. One can see his attempt as a reduction of a syllogism—involving three different propositions, to three variations on a same term, being these variations obtained by a technique of merely introducing two times a negation; in sum, only two basic logical elements: one term and negation. It could seem a genial simplification of the ancient logical technique to a more basic and hence much more powerful technique. Unfortunately, his arguments were so confuse that they are usually recalled as instances of illicit results. Marx charged Hegelian dialectics to be based on its (theological) head. He attempted to try to overturn it on his feet; yet, he was unsuccessful. The same his followers, notwithstanding the seventy-year support they received from a state (USSR), wanting to achieve the logical secret of the "dialectical materialism".

In sum, almost all logicians and philosophers supported the old classical logic as the unique, well-founded formal logic. It is not surprising if in the year 1919 a logician, Jan Lukasiewiecz, in his solemn lecture at Warsaw University called all to "a struggle for the liberation of the human spirit", i.e. for achieving an alternative to the oppressively closed logical world of Aristotelian logic (Lukasiewicz, 1970: p. 83).

3. The Difficult Discovery of a New Formal Logic

An historical fact may be surprising. After Euclides, rigorous reasoning was linked in a strong way to the deductive study of geometry.[3] Yet, the birth of modern logic is attributed to the reflections on algebra (John Venn, George Boole) or analysis (the notion of function: Gottlieb Frege), not geometry. One can give a reason: ancient logic was logic of terms; syllogisms made use of propositions for only linking together these terms. Instead, modern formal logic is different in a radical way; it is logic of propositions, each including terms as its elements; this for this reason Euclidean geometry did not have influence on the development of the formal logic of the ancient times.

However at the end of 18th century there was a scientist who started a formal

[3]The father of the six-year old Pascal hed forbidden him to study Euclidean geometry because he was too passionate to it. The father closed him in a room without any book of geometry. But after a time the father discovered that Pascal had re-constructed the list of the first theorems by the only power of his reason.

way of arguing as based on propositions. At the end of his celebrated book on geometry (1803a) Lazare Carnot interpreted negative numbers in geometry through the "tables of correlation" of a geometrical figure; they are obtained by registering the relations of the values of the basic magnitudes of the figure when one of its elements is moved in a continuous way. He translated all geometrical propositions in a symbolic language of signs (Carnot, 1803: pp. 78-89) and then introduced an algebraic calculus of them through the logical connective "*donne*" (gives). He declared to stop the illustration of his innovations since he was dubious on their acceptation by geometers (Carnot, 1803: pp. 485-489; Drago, 1989). His introduction of the above logical connective together with a computational technique is sufficient to formalize a minimal system of logic (see e.g. Gabbey 2014: p. 48).

It is well-known that later George Boole and Frege founded modern logic as a calculus based on several kinds of connective between propositions. What is not always remarked is that the former one founded logic as an *algebraic* calculus, whereas the latter one founded it in particular the quantifiers on an *analytical* basis, functions. This difference corresponds to a long dichotomic tradition inside the history of Mathematics: the two different approaches alternated in prevailing one on another. It is not a case that the present attitude—strongly linked to the analytical attitude through set theory—attributes to Frege the decisive role for this birth. However, several scholars (e.g. Jean van Heijenoort, 1963) stressed such a dichotomy within the foundations of logic as an essentially element of history of logic.

Not so much years passed before an opposition to this formal logic born, first by Brouwer (1975) (an intuitionist view on logic) and just afterwards by Lewis (he proposed a new formal logic, the modal one).

In particular, Brouwer's disbelieved in the law of the excluded middle of classical logic (LEM) when it is applied to an infinite set. He tried to define the intuitionist negation as implying the absurd (his approach is disputable; see Dummett, 1977: p. 13; van Stigt, 1990: p. 244, pp. 253-254). He left to a second time the definition of the law of double negation also because he rejected logic as an independent language (van Stigt, 1990: pp. 257-260). Also the historians of intuitionism first deal with negation as the first question to be examined in order to correctly stating the difference of intuitionist logic from classical logic—as if this difference was a mere variation—rather than to rely on the law of double negation, which represents a clear dichotomy between the two kinds of logic. This attitude seems correspond to the dominant appraisal that intuitionist logic has still to prove to be a self-reliant logic. It is not unlikely that the based inheritance of Hegel's dialectics refrained formal logicians to investigate double negations as reminding the inaccurately defined Hegel's process of a "negation of negation".

4. The Difficult Birth of an Alternative to the New Formal Logic

In the year 1908 Brouwer started a reflection on logic by a criticism to classical

logic. It continued as a fierce rejection of idealistic notions of modern logical arguing and fiercely was fought by Hilbert and many logicians. However, from Brouwer started a school which achieved many important results, including an axiomatic formulation of intuitionist logic (Heyting, 1930). After it, most logicians shared a conciliatory, ecumenical attitude in the name of the same basic tenet, the axiomatization of all theories, although intuitionists insisted to verbally affirm that in no way an intuitive logical system can be captured by a formal axiomatic.

However, intuitionist logic was the first and the major instance of many formal non-classical logics. They have been a long time considered as "deviant logics" (Haak, 1978). In the year 1936 Garrett Birkhoff and Janos von Neumann showed that the logic of quantum mechanics apparently deviates from classical logic. They looked for an appropriate kind of logic, yet by deliberately excluding intuitionist logic. They replaced the modularity law for the distributivity law, which in QM fails (Birkhoff & von Neumann, 1936). Karl Popper criticized their result by suggesting that "the kind of change in classical logic which would fit what they suggest [about the appropriate logic of quantum mechanics] would be the rejection of the law of the excluded middle (or "excluded third"), as proposed by Brouwer, but rejected by Birkhoff and von Neumann" (Popper, 1968: p. 685). David Jammer (1974: p. 86) seems suggest a more deep reason for this rejection; they wanted to preserve classical physics from any logical problem of an alternative logic, as the introduction of intuitionist logic in quantum mechanics seemed to imply.

This troublesome path of the history of intuitionist logic obstructed recognition of this kind of logic on a par to classical logic.

In the meantime modal logic, although ever more elaborated along a century, did not achieve the status of a well-defined logical system (Garson, 2018).

5. Analysis of a Scientific Text through the Law of the Double Negation

However, in past century some Intuitionists (Dummett, 1977) and Constructivists (Bishop, 1967: pp. 1-10) claimed inside respectively Logic and Mathematics a requirement, i.e. a proposition has to be supported by factual evidence; when a proposition does not satisfy such a requirement and there is no evidence of its falsity, its doubly negated proposition may be assumed.

Moreover, Dag Prawitz and Peer Melmnaas (1968) stated that the validity or not of the law of double negation separates classical logic (CL) from almost all non-classical kinds of logic, first of all the intuitionist one (IL), and then the minimal one, the positive one, etc. This shift in considering as the borderline between CL and the main kinds of non-CL the law of double negation (DNL) rather the law of LEM implies consequences of a great importance.

First, in order to analyze a proposition within a general setting its truth or falsity cannot be presupposed a priori, since this assumption means to choose clas-

sical logic only. By allowing also non-CL, first of all one has to recognize the validity or not of the law of the double negation of this assertion, since this fact constitutes the borderline between CL and non-CL. The logical value of a proposition can be correctly established by a second investigation belonging to a specific kind of logic.

Let us consider another very important consequence. Within a text the LEM is rarely manifested (by dilemmas); hence, through LEM almost never we can reveal which kind of logic is followed by the author of a text, in particular whether his logic is different from the classical one. Instead, we can easily manifest author's use of non-CL by inspecting the text by first recognizing all propositions which include two negations and after by examining whether the meaning of one of these doubly negated propositions is different from that of the corresponding affirmative proposition (DNP). For instances of DNPs, let us recall Hyppocrates' maxim: "First, do not harm" (≠"Do the good"); or the Decalog's commandment "Thou do not kill" (≠"Thou save life").

Thirty years experience of recognizing DNPs within original texts of logic, mathematics, physics, chemistry and also other subjects suggested the following rules:

1) By definition of a DNP one has to exclude the rhetoric use of the doubly negated propositions concerning facts of which we have evidence (e.g. "I have nothing else than 10$", which is easily verified as the same of the corresponding proposition: "I have 10$", which is obtained also by making use of the classical law of double negation)[4].

2) A single word, e.g. in-nocent[5] (≠righteous) may be composed by two negations and hence works as a DNP;

3) The word "only", being equivalent to nothing else, works as a DNP.

4) Modal words "must", "possible", "necessary", etc. work as DNPs, owing to the S4 translation between modal logic and IL (Hughes & Creswell, 1996: p. 224ff.).

5) A comparison as "more…than…", "less…than…" works as a DNP, because these words are not equivalent to the word "equal".

6) An interrogative negative proposition, whose answer is implicitly intended as a negative works as a DNP.

7) One has to notice that a proposition may understood a second negation; which may be recognized through the understanding the entire discourse: Jesus' teaching: "Do not oppose [evil] to evil"; Court's judgment: "Acquitted for insufficient evidence [of guilty]"; Popper: "Science is fallible [owing to negative expe-

[4]A longstanding tradition of English linguists suggests what Lawrence Horn called a "dogma" about double negations (Horn, 2001: p. 79ff.; Horn, 2008). This linguistic dogma asserts the absolute validity of the double negation law: whenever a doubly negated proposition is found in a text, it has to be changed into the corresponding affirmative proposition, because those who speak by means of doubly negated propositions want to be, for instance, unclear. In other words, all use of doubly negated proposition is of a rhetorical nature. Evidence for this "dogma" is the small number of studies on double negation in comparison with the innumerable studies on the single negation.

[5]Notice that here and in the following I separate the two negations included by a word by means of a hyphen.

rimental results]"; Jonas: "The ethics of the fear [of the mankind's suicide]"; hence, in order to discover the complete meaning of a negated proposition sometimes one has to refer to the context which may add a hidden second negation.

8) The negative character of a word may depend on the context; for example, in science the words "change", "different", "variation", etc. all work as negative words, because each of them represents a problem to be explained by scientists.

6. Classical Logic as the Characteristic Mark of Western Intellectual Civilization

Let us consider anew the history of logic under the light of the above stated divide between the two kinds of logic, CL and IL. The words *alétheia* and *veritas* have been rightly distinguished and opposed by Heidegger; actually they have to be considered as distinct according to not only their informal meanings but also an exact logical dichotomy: being a double negation, the former one belongs to non-CL, whereas the latter one belongs to CL. Therefore, in the light of the two meanings attributed by Greek civilization to the former word, *alétheia*, we see that Greek philosophy, when substituted, through Plato and Aristotle, the idealistic meaning of truth for the original meaning of *alétheia*, decisively chose CL. Afterwards, Roman civilization, by relying truth on the affirmative word *veritas* whose meaning was idealistic, re-affirmed CL.

However, in the Middle age Thomas Aquinates had suggested a new definition of truth: "*Veritas est adaequatio rei et intellectus*" (The truth is to conform mind to reality). This definition leaves to intend that truth is reachable by means of an approximation process, which at last ought to achieve the affirmative term, "equality". Instead, *adaequatio* is a modal word; it means "to become similar". Being the modal logic translatable into IL through the S4 model (Hughes & Cresswell, 1996: p. 224ff.), this new definition actually generated a compromise between CL and non-NCL, which represented a logically ambiguous way of intending the truth.

In retrospect, we see that the culture of Western civilization was almost entirely linked to the word *veritas* and hence in formal terms to CL.

7. A Dichotomy between Affirmative Theology and "Negative Theology"

Under the light of the above-mentioned innovation concerning the double negation law, let us inspect in a deeper way the history the intellectual thinking on the subject of the evidence supporting words and propositions.

Surely, both Plato and Aristotle had placed an embankment to Sophisms' peril, but they did not have recognized the difference between the original meaning of the word *alétheia* and its idealistic meaning.

The birth of Christianity has introduced de-establishing novelties as consequences of a God descending from the heaven to the Earth, i.e. the incarnation

dogma. No more was possible to tell a mythological story for explaining qualities of a heavenly God. No imaginative inference has been accepted to define the unprecedented novelties, also because Gospel language, focusing on the historical facts of a God, obliged the theological language to rely on personal experiences.

As first problem, the same name of God was a question. Being God's definition a priori unknown one had to arguing about an unknown Being, that however was incarnated in our world. Therefore no Platonist attitude was possible, no pre-established ideas could be exploited. The risk of paradoxical reasoning, like Sofists' ones, threatened this new intellectual exploration. On the other hand, the new faith required some solid results since the heresy peril was always present. Hence, the intellectual work concerned the theological problems of unknown beings about which no a priori resolutions there existed.

In particular, it was impossible circumscribe the nature of the Christian God through affirmative terms; the most common definition, God is truth, reiterated the intrinsic ambiguity of the word *alétheia*. Then, how to speak through non-affirmative words about an unknown Being? Truly, since long time there existed precursors of reflections on God (e.g. Philo of Alexandria, Parmenides), but Christianity moved this question from the speculations of some philosophers to an elaboration performed by the people of an entire Church. A definition of a new dogma about this new kind of God had to receive a widely shared intellectual agreement by a popular reflection about God's features. For these reasons the Christian theological debate was deeply innovative at the intellectual level, and at logical level too.

Christian theology generated two schools of theology: positive theology—based on only affirmative propositions on God—and negative theology—defining God through negative properties (e.g. infinite, ineffable, etc.). However, there exists an incorrect custom of coalescing inside the adjective "negative" also the case of two negations. This custom is so rooted that even in modern times a field of knowledge is called "negative" without making attention whether the proposition includes one or more negations. Such is the case of Popper's philosophy, which Quine (1974) qualified as a "negative ontology", since, at first glance, negated words (e.g., "fallibilism", "a never ended process", etc.) appear in Popper's crucial sentences. But the paper (Drago & Venezia, 2002) proved that his philosophy relies on DNPs. Also Jonas' ethical philosophy is commonly called "negative ethics", yet it is entirely illustrated by means of DNPs (Drago, 2002). Even in mathematical logic the translation from classical predicate logic to intuitionist predicate logic, obtained by inserting in a suitable way *two* negations, is called "the negative translation" (Troelstra & van Dalen, 1988: p. 49ff.).

Let us inspect the use of negative theology, which was not so minoritarian as many scholars believe; indeed, many theologians may be listed as belonging to "negative theology". Since negative theology is so full of negations, a widely shared opinion considered it a mystical, and even senseless way of thinking. In-

stead, it was decisive for defining Christian dogmas in a stable way along the centuries.

An impressive text of this kind of theology is 5th Century "Athanasian creed" about the Christian notion of Trinity. Let us quote some propositions:

> […] The Father un-created;[6] the Son un-created; and the Holy Ghost un-created. The Father un-limited; the Son un-limited; and the Holy Ghost un-limited. The Father non-mortal; the Son non-mortal; and the Holy Ghost non-mortal. And yet they are not three non-mortals; but one non-mortal. As also there are not three un-created; nor three in-finites, but one un-created; and one in-finite. […] And in this Trinity none is before, or after another; none is greater, or less than another. But the whole three Persons are co-non-mortal […]. (Anonymous, 2017)[7]

All in the above shows that at the time of Athanasius creed the tradition of negative theology on the very difficult notion of Christian Trinity closely approximated a consistent use of DNPs, belonging to IL.

The great relevance of the negative theology is proved also by the following impressive text, which was approved after a long and cumbersome debate by the theologians of the Calcedonian Council (451 a.C.). In order to define the dogma on the dual nature of Christ the definition stated nothing less four double negations: without mutation and without confusion […] without separation and without division". Clearly, no affirmative proposition is appropriate to circumscribe the content of this dogma.

8. The New Logic of Cusanus' Theology.

Nicholas of Cues, called Cusanus (1401-1464) is credited as a top exponent of "negative theology". Almost a century ago Ernst Cassirer (1927: pp. 15 and 31) suggested (without proof) that Cusanus had actively looked for a new kind of logic.

As a fact, Cusanus was in opposition to the "*Aristotelis secta [logica]*" (Cusanus, 1449: p. 463, n. 6)[8]. Cusanus distinguished in our mind two faculties, the *ratio*, i.e. the reasoning through positive truth and hence CL, and the *intellectus*, an arguing by means of *conjecturae*. He maintained that there exist three kinds of theology, i.e. beyond the positive and the negative theologies he invented a *theologia copulativa*, which we recognize to make a systematic use of DNPs. For example, "*Nihil certi habemus nisi nostram mathematicam*" (Cusanus, 1462: p. 936, No. 44) (We have nothing certain apart from our mathematics ≠ mathematics is certain). And also the definition of his basic notion: "*Coniectura est positiva assertio in alteritate veritatem uti est participans* [=*non in sua totali-*

[6]The word "created", if attributed to a god, disqualifies it to be an idol: hence, in this case the word is negative.

[7]By means of few, plain emendations of the original text I obtained a sequence of only DNPs; it constitutes a full consistent version of thetext according to non-CL (Drago, 2019: App. 2).

[8]I quote here and in the following from the translation into English language by Jasper Hopkins. His site, http://jasper-hopkins.info/ includes almost all Cusanus' philosophical books.

tate]" (Cusanus, 1440: p. 190, no. 57) (A conjecture is a productive sentence about the otherness to the extent that it shares a part of its truth" ≠ "a conjecture is a productive proposition about the truth"). Being his definition of *coniectura* a DNP, his arguing through conjectural arguments means, in modern terms, that he argued according to non-CL (Drago, 2010).

It is well-known that Cusanus introduced the surprising notion of the "coincidence of opposites". This notion is at risk of denying the logic principle of non-contradiction. Let us consider two instances of this notion. The title of Cusanus's first important book, *De Docta Ignorantia* (1440), is not a mere negation; to the negative word *ignorantia* he added a qualification (*docta*) which seems to give an oxymoron. This title leads the reader to inquire into another oxymoron: *Quomodo scire est ignorare* (How to know means to ignore).

But this logical invention played the role to prepare next inventions, first of all the discovery of new names of God, obtained by overcoming "the wall of the contradiction" which separates us humans from God. Eventually, he has invented several names which actually belong to modal and intuitionist logic. In particular, his more important book *Non Aliud* (1462) suggested this new name for God. Being *Aliud* a negative word (=not the same), this new name is a double negation. He stressed that it is not equivalent to the corresponding affirmative word, *Idem*, since about God we have always an incomplete evidence. Hence, this new name represents a failure of the DNL and therefore it pertains to IL[9] (Drago, 2019).

Actually, he had made use of DNPs since his first important book, the above mentioned *De Docta Ignorantia*. An accurate inspection of this text shows that the 53 lines of the first section include 20 DNPs. I verified that in this section he argued so accurately by means of cogent DNPs that their mere sequence is enough to preserve the logical thread of the text, provided that one implements them adding few connecting sentences. In other words, the sequence of all DNPs substantially represents the logical contents of this text (Drago, 2009).

As a consequence, Cusanus's theology was not a theology based on negative sentences, but a theology developed through DNPs and hence governed by non-CL. Notice that no relevant role is played by a negative proposition with respect to a corresponding DNP; Cusanus (1462: p. 1304, n. 41) stated: "the negation does not oppose to the affirmation"; that means: when a DNP is true, the corresponding negative proposition is essentially ambivalent.

Unfortunately, Cusanus did not have disciples. His crucial book *Non Aliud* was edited not before the end of 19th Century and the following scholars of his thinking have been attracted mainly by other subjects: 1) his notion of *coinci-*

[9](The following facts will be relevant to what will be presented in Sect. 11). Moreover, the DNPs of this text are linked together in such a way as to present three *ad absurdum* arguments (AAAs); the last DNP of each unit of argument works as a methodological principle for developing the next unit. The third unit concludes the argument by means of a DNP of a universal nature; that is, it covers all the possible cases considered by the initial problem; *desideramus scire* [*ex infinitis rebus*] *nos ignorare* (we desire to know [by infinite subjects] that we do not know); of course, this DNP is of a universal nature, because its subject is the entire universe of knowledge.

dentia oppositorum, which, being a single negation instead of a DNP, is a merely preparatory notion in order to exit out the classical logic; 2) his unsuccessful mathematical attempt, by exploiting his notion of infinity, to square a circle; and 3) the reconstruction of the cumbersome evolution of his thinking along the around 20 philosophical books written on numerous subjects (science too) according to different methodologies. A comprehensive and accurate synthesis of his thinking was achieved not before Kurt Flasch's (2001) book. No surprise if in past centuries no philosopher recalled Cusanus's consistent way of arguing through DNPs, not even Hegel (Stanfield, 2017: section 13.3). Moreover, no surprise if after the birth of modern science which is based on a heavy use of mathematics, none has saw a new logical arguing within theology, considered by most scholars a non-scientific, if not a unavoidably "irrational" thinking, because it deals with metaphysical subjects.

9. Galilei's Basic Requirement. A Missed Logical Revolution in Linguistic

In Aristotelian logic the constitutive terms of a discourse (and even the syllogisms) did not have an assured meaning, intended as a manifest correspondence with reality.

The question of this correspondence with reality became a crucial one when modern science born. In opposition to the Aristotle's speculative sciences, Galilei has required that a scientific proposition has to be equipped by an objective, experimental support. In the case an affirmative proposition lacks of evidence, a DNP is appropriate to designate its content as a hypothesis, just for advising that the positive proposition is not still achieved.

Galilei applied his requirement in a rigorous linguistic way. In the two last works (his major works, *Dialoghi* and *Discorsi*) he wrote in two different languages, just for distinguishing the apodictic (in particular the mathematical) truths, to be expressed by affirmative propositions, and the empirical, investigative truths, to be expressed through DNPs: in Latin the affirmative expositions; in vulgar Italian the debate on new hypotheses. The application of the new method for analysing in a logical way a scientific text to Galilei's two books shows a substantial use of non-classical logic.

For brevity's sake I bound myself to illustrate Galilei's scientific text about the accelerated motion in the *Discourse* (3rd day of Galilei, 1638; De Luise, 2009; Drago, 2009) The first easy result is that the theorems, declared as such by Galilei, do not contain DNP. Instead, in the first fifty pages of the text written in vulgar, there exist around 100 DNPs. This high frequency of DNPs indicates that they play a very relevant role in his arguing. Furthermore, the DNPs, once extracted from the text and read in sequence, are sufficient to restore almost the entire logical thread. Moreover, these DNPs can be grouped into cycles of reasoning. In its ideal model each cycle first poses a problem; then, to begin the argument, it introduces methodological principles; then, an *ad absurdum* argu-

ment (AAA) leads to a conclusion (also it is a DNP).

A further, surprising result concerns the first part about the uniform motion, to which Galilei dedicates a specific section of the text, the first one (pp. 167-177). Here Galilei exposes a logical construction of deductive kind (it is composed by one definition, four axioms and six theorems; pp. 169-176); here the DNPs are few and most of them are of a discursive nature. Instead, next Galilei's dealing with the subject of a uniformly accelerated motion he makes use of numerous (about 90) DNPs within 25 pages (178-203). Of course, the author was unaware of his specific kind of arguing in IL; to this unawareness is due the inaccuracies of previous data on DNPs. The high density of the DNPs (in mean almost 4 per page) indicates that they play a very important role in the arguments offered about the solution of the problem; here the DNPs are aimed at inductively solving this difficult problem of theoretical physics m. Indeed, these DNPs compose eight cycles of reasoning, which as a fact circumscribe Galilei's results on this difficult subject.

The results of this kind of logical analysis of a scientific text proves that DNPs do not concern author's merely linguistic habits, nor a specific rhetoric way of presenting the subject, but rather specific logical characteristics; that is, this kind of analysis manifests deep logical features of a Galilei's text.

This Galilei's well-known revolution in the methodology of science was later translated by Leibniz as a distinction between a "necessary" proposition, whose contrary proposition is contradictory to it, and a "contingent" proposition whose contrary one is not contradictory, as all experimental propositions owing to the approximated values of the measurements verifying its validity. (Leibniz, 1714: pp. 235-236)[10] But this difference between the two kinds of proposition passed unnoticed.

At last Galilei's evolution could have also been a linguistic revolution if it had been generalized to a request on all kinds of proposition. But as first his disciples Cavalieri and Torricelli made use of non-experimental propositions in order to state the inertia principle and moreover the metaphysical infinitesimals within mathematics (Drago, 1997). Later in this "exact" science also idealistic axioms (as e.g. Zermelo's) had place and they appeared as unavoidable to most scholars. Since Galilei's requirement has been ignored by the scientists themselves, linguists had a good reason for ignoring it.

10. A List of Scientific Theories Whose Developments Are Based on DNPs

The content of previous Section meets an objection: Galilei was unable to build an entire physical theory; his thinking belongs to an *in fieri* scientific theory, which is very different from a complete system. In the following in order to answer to this objection only scientific theories will be considered.

[10]From this distinction one draw that an isolated negation plays a subordinate role with respect to the double negation, which plays the role of distinguishing two main kinds of logic, hence it works at a level higher than a single kind of logic.

Since the year 1989 some investigations on past scientific theories discovered that there exist several scientific theories that have been presented by their respective authors through texts containing numerous DNPs. In other words, s*everal theories presented their contents by relying in an essential way on DNPs of non-CL.* They are the following scientific theories: classical chemistry, S. Carnot thermodynamics, L. Carnot's mechanics, Lagrange's mechanics, Avogadro's atomic theory, S. Carnot's thermodynamics, Lobachevsky's theory of parallel lines, Galois' theory, Einstein's 1905 paper on special relativity, Einstein's 1905 paper on quanta, Kolmogorov's minimal logic, Markov's theory of constructive numbers, Computer science (Church's thesis) (Drago, 2012). Let us examine the most representative theories.

10.1. DNPs in Lavoisier's Chemistry (1789)

At the end of the eighteenth Century a number of new scientific theories born. Let us consider the first theory which exited out Newtonian paradigm, Lavoisier's chemistry (Lavoisier, 1789). He put aside Newton's suggestion of gravitational force as the universal force for all kinds of interactions, and hence for the intermolecular interactions.

Already at the early times of chemistry some its characteristic propositions were DNPs (Drago & Oliva, 1999). The reason is manifest: chemists had to speak about unknown objects, atoms and molecules, whose existence lacked of experimental proofs, hence in hypothetical way. In the beginning of his well-known dialogue Boyle suggests four basic propositions; whose three propositions are DNPs: "It not seems absurd…"; "It is not even impossible…"; "I will not deny peremptorily…"

Afterwards, Lavoisier instead of referring his ideas to mainly real entities, like air, water, earth, and fire put just at the starting point of his theory a great problem: is matter divisible at a finite end or not? He applied the following methodological principle: "A mathematical divisibility *ad infinitum* does not apply to the matter of which the world is made". Let us remark that the proposition: "A mathematical division *ad finitum* applies to the matter" did not hold true because at that time no one could decide by experimental means at which finite extent matter is divisible.

Moreover, let us consider the famous definition of a chemical element given by Lavoisier: "We will call an element…what we cannot yet decompose"; it is a DNP, since a chemical element cannot be defined as a "simple" component of matter, whose terminating end of its decomposition was unknown.

In addition, Lavoisier suggested a new method for chemical research. This method relied upon certain experimental rules about the weights of substances involved in a chemical reaction. He stated it not as an absolute truth, but as a possibility (which means: "It is not the case that it is not…"): "Nothing is created, […] and a possible principle is that in each operation the quantity of matter before and after the operation is the same". Here the DNP is stated first

and then the positive proposition is achieved as a possible derivation, whose content is yet not assured, because there was not enough experimental evidence for stating it; being aware of that, Lavoisier correctly qualifies the statement as a "possible" principle.

It is well-known that for a long time this way of theorize chemistry was fiercely opposed by those chemists that considered chemistry as depending from the "assured" principles of Newtonian mechanics. A century elapsed before Mendeleev eventually presented a complete theory of chemistry by suggesting his periodic table, just by reasoning through DNPs (Drago, 2014).

But this extraordinary history of theoretical chemistry, essentially based on logic and in particular on DNPs, was ineffective on the general conception of science. In particular, the reaction of physicists was to consider chemistry in a separate way from theoretical physics (mainly Newtonian mechanics), because its main features were too disturbing the Newtonian model of a theory.

10.2. DNP within the Inertia Principle of Lazare Carnot's Mechanics (1803)

Let us consider the inertia principle, i.e. the principle differentiating modern science from ancient science.[11] In Lazare Carnot's mechanics it is the first of seven "hypotheses" (notice that he correctly attributes a dubitative meaning to these principles, which not ever have full experimental meanings).

> First hypothesis: A body, once put in a state of rest, would not be able by itself to leave that state, and, once set in motion, would not be able by itself to change either its speed or its direction. (Carnot, 1803: p. 49)

where the words "leave" and "change", by requiring an explanation on the part of the physicist with respect to the state of rest, are to be considered negative.

These double negations are not merely a figure of speech, without importance for physics, because, unlike figures of speech, it does not possess an affirmative proposition having a physical meaning. Indeed if the two negations are removed we would have an abstract word, with no physical meaning; we obtain precisely the affirmative sentence of Newton's version: the verb "perseveres" or "continues"; both words have animistic (or ethical) meaning, not an experimental one. Here we see the full logical distance between the two types of theorization that modern physicists made use for their theories.

But later L. Carnot's theory was ignored by misinterpreting it as the foundation of no more than technical physics, a subject concerning engineers only.

10.3. DNPs in Sadi Carnot's Thermodynamics (1824)

In order to found thermodynamic theory Sadi Carnot put the problem of the maximum efficiency in the heat/work conversions. In order to solve it, he looked

[11]As Norwood Hanson (1963: p. 103) put it: "…the First Law of Motion--the 'Law of Inertia'--this has everything a logician of science could look for. Understanding the complexities and perplexities of this fundamental mechanical statement is in itself to gain insight into what theoretical physics in general really is."

for a new scientific method; his method is very surprising because leads to argue *via* absurd facts, which however generated concrete knowledge on the behaviour of engineers' machines; notice that this way of arguing is reiterated by almost all thermodynamic textbooks. The theory argues through an *ad absurdum* theorem composed by DNPs about his celebrated cycle of four transformations; this is a manifestation of the power of the logical arguing, surely different from a deductive way from abstract, a priori principles.

The first part of the text of S. Carnot's book was analyzed through its DNPs by the paper (Drago & Pisano, 2000). The numerous DNPs of the first part of the book compose seven cycles of arguing; among them there is his well-known *ad absurdum* theorem about the maximum efficiency in *all* heat/work conversions. He concludes its proof by stating the requirement for obtaining a maximum efficiency:

> no change of temperature inside the bodies employed for obtaining the motrice power of heat occurs without a change in the volume". (Carnot, 1924: p. 23)

In the subsequent part of the theory the author makes use of the corresponding affirmative proposition, as a hypothesis from which he drew all consequences in a deductive way.

His work was considered "too difficult" by his scientific friends (Robelin, 1832), then it was ignored along 25 years and later Clausius and Kelvin, without understanding the logical novelty of S. Carnot's entire way of reasoning, reduced his theory to some suggestions useful for their formulation of the so-called modern thermodynamics.[12]

10.4. DNPs in Lobachevsky's Theory of Parallel lines (1840)

Nikolai Lobachevsky's main text puts the problem of how much parallel lines through a point outside a straight line exist. The fundamental problem of his theory is expressed with the following proposition: "…in the ignorance whether the parallel line is not only one…" (Lobachevsky, 1840: prop. 16) (≠"we know that the parallel line is one"). The numerous DNPs of this book have been analyzed by the paper (Bazhanov & Drago, 2010). In order to obtain evidence for his guess i.e. two parallel lines—he proves through DNPs five theorems, which, except one, are AAAs. In the end of prop. 22, which is proved by an AAA, he concludes that his supposition receives equivalent evidence to Euclid's hypothesis in both *all* points and in *all* figures. He concludes that he is allowed to assume the hypothesis of two parallel lines "…without leading to any contradiction in the results" (≠"is consistent"). After this conclusion he deductively develops his hyperbolic geometry from this hypothesis now playing the role of an axiom.

Not before the death of Lobachevski this new geometry was slowly taken in

[12]In 1909 Constantin Carathéodory suggested a new formulation of thermodynamics according to a first instance of an axiomatic in theoretical physics. But his main axiom (the second one) is so idealistic in nature that the author himself admitted that it cannot be of an experimental nature. (Carathéodory, 1909)

account, although qualified by most as "metageometry", which in that time was a similar to the word "metaphysics". At last, only the deductive parts of Lobachevsky's works have been considered, i.e. the part of CL occurring after his AAAs.

10.5. DNPs in Kolmogorov's 1824/25 Paper

In this paper Kolmogorov wanted to solve a basic problem within the debate between intuitionists and formalists:

> "Brouwer's writings have revealed that it is illegitimate to use the principle of the excluded middle [≠it is legitimate the principle of included middle] in the domain of transfinite arguments. Our task here will be to explain why this illegitimate use [of the principle of the excluded middle] has not yet led to contradictions [≠has led to compatible conclusions] and also why the above illegitimacy will often gone unnoticed [≠legitimacy is noticed].
>
> Only [=Nothing else] the finitary conclusions of mathematics can have significance in applications. But the transfinite arguments are often used to provide a foundation for finitary conclusions. Brouwer considers, therefore, that even those who are interested only [=nothing else] in the finitary results of mathematics cannot ignore [≠know] the intuitionistic critique [=rejection] of the principle of excluded middle [≠acceptation of the included middle].
>
> We shall prove that all the finitary conclusions obtained by means of a transfinite use of the principle of excluded middle [≠the finite use of the principle of included middle] are correct and can be proved even without its help.
>
> The natural question [It is unknown] is whether the transfinite premisses that are used to obtain correct finitary conclusions have any meaning [≠it is known that finite premisses… have meaning] (Kolmogorov, 1924/25: p. 416)

One easily verifies that the thread of the logical arguing is preserved by the mere sequence of DNS's. (One obtains the same result in the last part of the paper: the mere sequence of DNS's is enough for representing the logical core of the paper).

He made use of a rigorous way of arguing by defining a language composed by "pseudo-truths"; the meaning of each of them is "a judgment asserting its double negation statement", (Kolmogorov, 1925: p. 416) i.e. "It not true that it is not…", that is exactly a DNP (Drago, 2005).

Subsequently, Kolmogorov in an essential way argues through pseudotruths. Thus, the entire paper proceeds just through DNPs. This fact proves that, once he had ingenuously grasped the formal way of arguing in IL, Kolmogorov consistently reformulated the entire way of formally arguing according to IL.

In such a way he obtained a theorem of universal validity, for *all* possible implications. There exists no inconsistency in the use of LEM in finite sets.

Yet, for years this objective way to deal with the foundations of logic has remained largely unknown outside the USSR, even to Heyting.

10.6. The Introduction of Non-Classical Logic in Quantum Mechanics: Which Kind of Logic for Theoretical Physics?

The great novelties introduced by 20th century theoretical physics concerned also the kind of logic. After some unsuccessful attempts, a non-CL was recognized in quantum mechanics by two authoritative scholars, Garrett Birkhoff and Janos von Neumann (1936), on the ground of Hilbert space. From physical evidences the distributive law was taken by Birkhoff and von Neumann as the borderline between quantum logic and CL. This discovery proves that in scientific theories the non-CL is not a side-effect of just linguistic relevance, but also pertains to the core of the most advanced physical theory of the first half of 20th Century. Yet, this discovery of non-CL within quantum mechanics by Birkhoff and von Neumann was almost ignored by theoretical physicists for some decades.

Actually, their discovery was incomplete. They obtained an algebraic structure, which, in order to obtain a well-defined kind of logic, has to be suitably improved with a certain margin of discretion. After the 60s a lot of studies on the subject have been accumulated. But they have not led to a decisive progress (this field of study was characterized as a "labyrinth"; van Frassen, 1974: p. 224).

In contrast to this approach to quantum logic deriving an algebraic structure from the given mathematics of Hilbert space, one may follow the inverse path, in which logic precedes mathematics. Truly, quantum mechanics is aimed at solving the main problem of quantum measurement: the uncertainty principle. This methodological principle concerns the relationship between the measurements of both variables defining the state of a system. It is adequately expressed by a DNP, belonging to non-CL. As illustrated by a paper (Venezia, 2006), Heisenberg's statement for this principle may be formulated in the following way: "The more precisely the position is determined, the less precisely the momentum is known in the same time and vice-versa" (≠the position is determined with the same precision of the momentum. Notice that according to Heisenberg's correct insistence of referring the physical theory only to operational quantities, we have to consider no more than the relative precision instead of the idealistic notion of an "absolute precision").

Notice that the definition of a state of the system neither can be confined to play a lateral role, outside the principles of the theory (as for instance in thermodynamics the DNP expressing the relation between heat and work: (Q ≠ W); nor can be obscured by translating it by means of misleading words, e.g. in mechanics "perseveres". That is, it is inherent in the theory's interpretation of reality and hence the DNP is unavoidable and hence non-CL too. Under this light, QM results to have the same logic of a classical theory organized on DNPs. Therefore the remaining problem is not to find out a suitable kind of logic for

quantum mechanics, but a new formulation of quantum mechanics based on a sequence of DNPs in a similar way to the above illustrated theories.

11. The Difficult Discovery of an Alternative Kind of Organization of a Theory

In the time of its monopoly of mathematical logic, CL was put as the base for an axiomatic mathematics (Frege); but in early years of the 20th century this proposal was stopped by the crisis of the antinomies. Moreover, Hilbert (1925) declared that CL is indispensable to a mathematician as to a boxer are his fists, and even more it is indispensable for achieving the decisive goal of his program, i.e. to assure the foundations of science through the axiomatization of all scientific theories.

Yet, Goedel's theorems stopped this program. From these results many scholars (for instance, van Heijenoort, 1967, vol. 3, p. 356; Hintikka, 1989) derived the lesson that an axiomatic can be no longer considered as the only representative formulation of a mathematical (or a scientific) theory. One may conclude also that no longer CL alone is enough for representing an entire mathematical theory. Brouwer had already presented a similar thesis; but by having an exclusive attitude, wanted to reject LEM when applied to infinite sets and then the entire CL. In opposition to him, Hilbert a classical scholars rejected Brouwer's attitude on mathematics and logic. A great debate followed. Hundred years later the beginning of this debate, Martin-Loef (2008) recognized its inconclusiveness.

Let us consider this problem under the light of the DNPs which occur within a theory aimed at inductively solving a fundamental problem. This problem is expressed through a DNP (e.g. in thermodynamics "it is not true that heat is not work". and therefore one has to find how heat is converted into work). One cannot idealize the problem A into a proposition A ("heat is work") because this move would eliminate the problem.

Then it is clear that, while within a deductive theory we deduce from the vertex of few assured principles a pyramid of theorems, which infinitely succeed one after another, without ever closing their series, in a theory based on a problem the reasoning is essentially cyclic, in the sense that, given the problem posed as A and the direction indicated by a methodological principle to solve it, the theory is aimed at revealing as many contents as possible of that proposition A which since the beginning could not be considered as true through the simple idealistic operation of suppressing the two negations of the problem proposition.

A comparative analysis on the above mentioned texts shows the main features characterising their common kind of arguing, in fact of an inductive kind. Each of these texts is aimed at inventing a new method for solving the previously given problem; e.g., in the above we saw that Kolmogorov's new method was to define intuitionist propositions as "pseudo-truths" of the classical assertions (Drago, 2005). In order to solve the given problem the DNPs are linked together to constitute *ad absurdum* proofs (e.g. "otherwise…not…"), which in chain con-

clude *a universal predicate*, again a DNP, suggesting a possible resolution of the starting problem for all cases.

Being assured to have collected all possible logical evidence, at this point an author of such a theory translates the above predicate into the corresponding affirmative predicate; which is then assumed as a new axiom from which to draw in a deductive way, i.e. by means of CL, all possible consequences, to be tested against reality (Drago, 2012).

From the table of (Dummett, 1977: p. 29) it is easy to show that this translation of the concluding DNP corresponds to change the main thesis A of the square of opposition, from its non-classical version to the classical one and moreover to change the entire kind of logic. The change of the kind of logic apparently results from an implicit application of Leibniz' principle of sufficient reason—whose antecedent is itself a doubly negated sentence ("Nothing is without reason") and the consequent is the corresponding affirmative proposition ("Everything has a reason").[13]

This kind of translation is verbally manifested by three eminent scientists. Galilei:

> Let us take it as a *postulate*, whose absolute truth will be stated by seeing that further conclusions, built on this *hypothesis*, perfectly fit the [entire field of] experience. Having admitted [by applying PSR] this…*principle*, the Author moves to consider the propositions, which are deductively concluded [from it]…" (Galilei, 1638: pp. 167-290, 191; my emphasis).

Let's put aside that we would not have expected Galilei to use the words "absolute truth". He is dealing with a postulate which, if considered as a hypothesis, may give valid conclusions; that is, it is a methodological principle. In the final sentence we find this concept transformed into the concept of "principle"; which, as understood by Galilei, is clearly an axiom principle. So this period indicates a progressive dynamic through the sequence of three concepts: hypothesis, postulate (or methodological principle) and axiom principle. This dynamic corresponds to what a physicist usually does; he questions the experience on the basis of a hypothesis-intuition, then exploits it as a methodological principle to direct his research to find out a systematic theory of the field of phenomena that he is studying, up to the point of corroborating it as a certain principle of a theory that can be deductively organized.

Also Lobachevsky verbally manifests this translation:

> [My supposition of two parallel lines] can likewise be *admitted* [as a principle-axiom for the following, *deductive* part of my theory] without leading to any contradiction in the results and [deductively] *founds* a new geome-

[13]Notice that the translation performed by this principle merely constitutes the inverse translation of the "negative translation", illustrated by (Troelstra & van Dalen, 1988, p. 56ff.). Whereas the latter translation (being from the real world into a hypothetical world) may be always performed, the former one (gaining reality from a hypothetical world) is possible only under some requirements; Markov stated that doubly negated predicate on which it is applied has to fulfill some requirements: to be both decidable and derived from an AAA. (Drago, 2012)

try…" (Lobachevsky, 1840: prop. 22; my emphasis).

Albert Einstein reiterates similar words when, in the second page of his celebrated paper founding special relativity, he anticipates the reader of what will be established in the following.

> We will raise the [informal] *conjecture* (the substance of which will be hereafter called [by applying PSR] the "[axiom-]*principle* of relativity") to the state of a [plausible] *postulate*". (Einstein, 1905: p. 891; my emphasis)

I conclude that there exists evidence of a scientific nature, for interpreting the conclusion of the sequence of AAAs of this kind of theory as an application of the PSR.

All together, the above features constitute an alternative model of organizing a theory, which is called a problem-based organization (PO). Let us recall also that in the past Willem Beth stressed that present science is biased by a unique model to organize a scientific theory, i.e. the deductive one theorized by Aristotle. He suggested exiting out it by following a non-deductive method. However, he did not present an alternative model of organization (Beth, 1959). The above analysis obtained in a formal way Beth's goal.

As a consequence of the two models of the organization of a theory—the axiomatic one and the problem-based one—a single proposition plays very different roles: 1) an affirmative proposition of CL, 2) a doubly negated proposition of either CL or non-CL, 3) the conclusion of an AAA, 4) the assertion of PSR, to be intended as a translation between two kinds of different logic. Hence, no exhaustive analysis of a single proposition is possible without considering its theoretical context.

I conclude that not only negative theology, but also well-founded and crucial scientific theories present the above-illustrated way of arguing, the intuitionist one, which therefore has to be put on the same par of the old classical way of arguing. A dichotomy results from both the two kinds of logic and, in a parallel way, the two kinds of the organization of a theory.

12. Two Dichotomies as the Foundations of Science

A century ago Brouwer started a harsh debate with Hilbert on which kind of *infinity* has to be applied within mathematics—either the potential infinity (PI) only or also the actual infinity (AI). In the '60s the original version of Brouwer's program of re-constructing great part of current mathematics by making use of (almost) only PI has received two formalizations by respectively Andrei Markov (1962) and Errett Bishop (1967) (constructive mathematics). Thus, one more dichotomy is to be recognized; it is represented by *the two kinds of infinity*, either PI or AI, and equivalently but *in formal terms*, *by the two kinds of mathematics*, either the *constructive* mathematics or the *classical* mathematics.

This dichotomy adds to the previous one on the kind of logic because it is independent of the latter one. As a result, the foundations of science include two

dichotomies, each one being expressed in both formal mathematical terms and philosophical terms.

They are scientific in nature because they are unavoidable by a scientist founding a theoretical system; for consistency reasons, (s)he has to reject or not both double negation law (DNL) and idealistic mathematics. But, being these decisions undecidable by respectively constructive or empirical means, they are also philosophical in nature.

As a consequence, a definition of what is a scientific theory is achieved; it is defined as theory based on both scientific evidence and the choices on the two dichotomies.[14]

13. Kolmogorov's 1932 Paper as a Structural Foundation of IL

Let us recall that Hilbert founded CL according to his program, illustrated by his paper (Hilbert, 1925). Under the light of the two dichotomies we see that the basic choices of this program were AO (an axiomatic) based on idealistic axioms including AI. The original Brouwer program was based on the opposite choices, i.e. PI and his mistrust on both CL and all axiomatic systems; the latter features allude to both the foundation of non-CL and the model of a PO theory. In sum, one may attribute to Intuitionism the choices PI & PO. As a consequence, the differences between Formalism and Intuitionism are of a radical nature; they cannot be conciliated, but only recognized in their philosophical and formal terms; hence, Heyting's axiomatic of IL (Heyting, 1930) has to be classed as a compromise between the two programs, but it was considered the prominent novelty concerning IL.

Then the question arise of which is the correct foundation of IL according to the intuitionist choices, PI&PO. After the subjective foundation of IL by Brouwer, Kolmogorov's 1932 paper (Kolmogorov, 1932; for a detailed analysis see Drago, 2021) suggested an objective foundation. In the following it will be proven that the logical theory of Kolmogorov is a foundation of IL closely corresponding to the couple of choices of the intuitionist program.

Kolmogorov's choice for PI (*i.e. potential infinity or equivalently constructive mathematics*) is manifested when he state to be addressing his search to "the *concrete* areas of mathematics". (p. 329; my Italic) The choice is reiterated by the following words: "*Any proposition that is not without content should refer to one or more* ***completely determinate states of affairs accessible to our experience***…" (p. 332; Italic of the author; my emphasis in bold). Hence, no mathe-

[14]Notice that Leibniz suggested that our reason meets two labyrinths, the labyrinth of two kinds of infinity, either actual or potential, and the labyrinth of "either freedom or law" (Leibniz, 1710, "Preface"). They have been called by him "labyrinths" because he hoped to solve them by the reason itself; but, after three centuries of unsuccessful research, it is justified to conclude that human reason cannot solve them and hence they have to be rather called dichotomies of human reason. Actually, the two labyrinths correspond to the above illustrated two dichotomies, provided that the latter labyrinth is viewed as a subjective description of the dichotomy about the two kinds of organization: either a *free* search for discovering a new method aimed at solving a given problem, or the obedience to a compulsory *law* derived from few principles.

matical or logical notion of an idealistic nature is allowed by him, but only those corresponding to PI.

Kolmogorov's choice for the alternative organization (*i.e. PO or equivalently the intuitionist logical way of arguing*) is recognized through two facts:

1) he neatly distinguishes his objective calculus, governed by not axioms, but "rules" (pp. 328, 330, 331, fn. 6a) from Heyting's axiomatic system; he underlines this point in fn.s no.s 6a-8.

2) a detailed analysis of the development of his paper shows the following features, which are the typical ones of a PO theory.

i) At the beginnings of his paper he presents a clear *crucial problem*: a) to "systematize the schemata of the solution of problems, for example, of geometrical construction problems." (p. 328); in other words, to introduce "…a new *calculus of problems*." (p. 328). This proposition means that his problem was to give one more formalization of this logic by means of an objective calculus.

> One can introduce a corresponding symbolism [to the rules] and give the formal computational rules for the symbolical construction of the system of such schemata for the solution of problems. Thus in addition to theoretical logic one obtains a new *calculus of problems*. (p. 328)

In its turn, the resolution of this problem is aimed by him to solve the theoretical problem: b) to show that "*The calculus of problems is formally identical with the Brouwerian intuitionist logic…recently…formalized by Mr. Heyting…*" (p. 328) Then he anticipates the conclusion: c) "It will be shown that intuitionist logic should be replaced by the calculus of problems, for its objects are in reality not theoretical [read: idealistic] propositions, but rather problems." (p. 328) That means that his chief problem was to equip IL with an objective semantics.

ii) As in all PO theories, the background knowledge demanded to a reader of the paper is *the common knowledge*: "We do not define what a *problem* is; rather we explain this by simple examples." (p. 328) Moreover, "we must assume that we have already solved the following two groups of problems *A* and *B*." (p. 330) (which usually are instead intended as axioms).

iii) He states a *methodological principle*, i.e. a principle orienting the new search. It is manifested in the second section, where he presupposes that the reader will accept "the intuitionistic epistemological assumptions…" (p. 328) According to Kolmogorov "The basic methodological principle of the intuitionist critique [and foundations] of logical and mathematical theories is the following: "*Any proposition that is not without content should refer to one or more completely determinate states of affairs accessible to our experience…*" (p. 332).

iv) The text includes twenty *DNP*s.

In retrospect, we see that intuitionists' persistence in supporting a merely subjective viewpoint left the notion of the organization of an entire scientific theory in the far horizon or, in alternative, led Heyting to search a structural foundation by joining the Formalist kind of organization, AO. One may conclude that the main missing of intuitionists' search for founding IL was the defi-

nition of an alternative model of the organization of a theory. Unfortunately, they did not follow Brouwer's wise suggestion to reflect on all previous mathematical (and scientific) theories which implicitly suggest logical rules:

> in arguments concerning empirical facts spanned upon mathematical systems, the logical principles are not directories, but regularities discovered afterwards in the accompanying language. (Brouwer, 1975: p. 108)

This investigation could have suggested what structures could be introduced into intuitionism (as it was proven by the above comparison of some scientific theories).

In conclusion, by a rational reconstruction of Kolmogorov's system according to both the choice for the PO model and the choice for PI, we have obtained a complete and objective foundation of intuitionist logic, to be put on the same par of an axiomatic theory of it (i.e. Hilbert's). In such a way we have founded logic in a structural way, going beyond the subjectivism of personal intuitions, as well as the analyses on the objective meanings of a crucial logical notion (e.g. inference, negation, etc.).

First was Hilbert to found a kind of logic by making use of one couple of choices, AI and AO; yet he was unaware that it was only one couple of choices out four. By means of the opposite couple of choices Kolmogorov's 1932 paper has obtained a foundation of IL which is radically different from Hilbert's CL and also Heyting's IL. Thus, there exist two separate formal formulations of logic, a formulation as CL and another as IL. In other words, the work of founding logic is essentially of a pluralist nature. That does not refrain that OP formulations based on non-CL cannot be translated in AO formulations, and *viceversa* (e.g. the Kolmogorov's translation of Heyiting's AO formulation into his PI and PO formulation; or in mathematics the translation of the AO and AI infinitesimal analysis into the PO and PI Cauchy-Weierstrass calculus).

14. A Quick Summary of the History of Logic as a Conflictual History: A Couple of Two à la Koyré' Propositions as Its Interpretative Categories

In last century the historiography of science became interpretative in nature. The most celebrated interpretative account was Kuhn's book (Kuhn, 1969) on the history of science—actually, a quick view on classical physics only. His categories apply to the history of logic only in a partial way. The most famous Kuhn's category (and also the most ambiguous one in his book) is that of a "paradigm". Surely, history of logic presents a long term-paradigm, that of CL; it began after the change in the meaning of the word *alétheia* and after the work of Aristotle; which had no rival till up to the last century: it dominated logical thinking along more than two millennia. In no other science a paradigm persisted a longer period of time. Thus, history of logic offers the more strong confirmation to a basic category of Kuhn's historiography.

However, the other Kuhn's categories (anomaly, crisis, Gestalt, revolution,

paradigm shift) do not seem occur in history of logic, since its two main historical changes—one from the word "alétheia" to "truth" in ancient times and the birth of non-CL in modern times—did not occur through a crisis, overcome by an abrupt Gestalt phenomenon in logicians community. Actually, Kuhn's illustration of the history of classical physics by means of a succession of paradigms one exactly after the previous one—through a *Gestalt*, i.e. an instantaneous turning point of this succession, cannot represent a conflict between divergent views on the foundations.

Instead, interpretative historiographer, Alexander Koyré, represented as a conflict between the new science with the old science in the study-case of the birth of modern science. In that time there was a dramatic conflict between the Aristotelians and Galilei, and more in general the refractory summit of Catholic Church and the new science. Almost each of the several Koyré's texts illustrating this period summarizes his interpretation through two propositions, which actually constituted his interpretative categories: "Dissolution of the ancient Cosmos and geometrization of space". It was suggested that Koyré actually interpreted the entire period of time as a theoretical effort to rely on the basic choices of Newtonian science, which are AI (through the use of infinitesimals) and AO (a deductive theory from three principles-axioms). Hence, one can associate Koyré's propositions with the basic choices of (the scientific theories preparing and at last achieving) Newton's mechanics:

"Dissolution of finite [not PI] Cosmos [not PO], and geometrization [AI] of the space [AO]".

At the end of the 18th Century several theories (Lavoisier's chemistry, Lazare Carnot's mechanics and then Sadi Carnot thermodynamics and Lobachevsky's theory of parallel lines) have been based on the opposite choices to Newton's, PI & PO. One may associate to these choices similar propositions:

"Evanescence of force-cause [not AI] also as playing the role of first principle of a theory [not AO], and discretization [PI] of matter [PO]." (Drago, 2018)

In the case of logic, by taking example from the above suggestions of the two couples of interpretative propositions according to the basic choices I suggest two similar propositions for characterizing the 20th Century conflict between formalism and intuitionism:

Formalism: "Dissolution [not PO] of intuition in mathematics and logic [not PI] and axiomatization [AO] of every infinitary system [AI]."

Intuitionism: "Mistrust in axioms [not AO] of idealistic nature [not AI], and constructively [PI] reiterate the entire mathematical world and [even a] logic [PO]."

Notice that I wrote "even" because neither Brouwer defined intuitionist logic. In the case of the entire history of logic, one may suggest the following propositions:

Classical logic: "Dissolution of informal logic (not PO) and Sophists' arguments (PI), and absolute decision (AI) on the truth of each proposition of a logical deductive system (AO)."

Alternative, *non-classical logic*: "Evanescence (not AO) of idealistic notions (AI), and evidence-based (PI) searching for solving a logical problem (PO)."

In retrospect it may be surprising that a long time (almost a century) is elapsed since birth of Lobachevsky's geometry which elicited the problem of the foundations of mathematics to the birth of a conflict between two alternative views on this problem; and then even more time elapsed since the birth of a philosophical alternative (Brouwer's) to the recognition of its parity with the dominant one. Yet, under the light of the above results, one sees that it was very difficult to transcend the dominant attitude of the Euclidean geometry in the foundations of the entire mathematics. A subsequently it was very difficult to induce from a single attitude (either a formalist one or an intuitionist one) which was affirmed in an exclusive way, the complexity of four choices in mutual conflict.

15. Conclusion on the Birth of a Logical Pluralism

In a paper concerning the difference between CL and other kinds of non-CL, Prawitz recalled:

> A basic tenet of intuitionism is that classical logic contains some invalid forms of reasoning and consequently has to be rejected and, at least within mathematics, replaced by intuitionistic logic. In discussions of intuitionist logic the question of the validity of this claim is often evaded, and instead intuitionist logic is justified as being of interest from some special point of view which does not necessarily repudiate the canons of classical logic but allows the peaceful coexistence of the two systems. However, for anybody concerned with logic, the question whether the intuitionist attack on classical logic is justified must be a vital issue. (Prawitz, 1977: p. 2)

The present paper suggests that the attack of intuitionism was inappropriate; only the choices on the two dichotomies, not the theories based on these choices, are exclusivist, owing to the nature of what one and half century ago occurred for the first time within science, the dichotomies. Therefore the solution of this attack could not be of a technical nature, i.e. to indifferently adopt all technical tools, but the solution is to accept a co-existence of essentially different theories, each based on a different couple of choices.

This solution implies to distinguish the theories according to the basic choices on the two dichotomies; in particular it implies to consider the radical variations of meaning generated by the two basic choices, for ex. the choices AO and PO, on the kind of organization of a theory; e.g. the meanings of principle (either methodological one or axiom), implication (either the intuitionist one which lacks of a truth table, and a classical one, enjoying such a table), proof (either indirect or direct), translation between two kinds of logic theory (either the principle of sufficient reason or the negative translation). All these radical differences could not be induced from gradual enlargements of CL; actually, already Kurt Goedel (1933) paper has proved that between CL and IL there exists an infinite abyss.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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